

## On the Number of Maximal Independent Sets in Some $(v,e)$ -Graphs

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### ABSTRACT

*Let  $V(G)$  be the set of vertices of a simple undirected graph  $G$  and  $S$  be a subset of  $V(G)$ .  $S$  is an independent set in  $G$  if no two vertices of  $S$  are joined by an edge of  $G$ .  $S$  is a maximal independent set (m.i.s.) in  $G$  if  $S$  is independent and is not a subset of any other independent set. Let  $\mu(G)$  denote the number of m.i.s. of a graph  $G$ , and  $\mu(v, e) = \max \{\mu(G) \mid G \text{ has } v \text{ vertices and } e \text{ edges}\}$ . For  $0 \leq e \leq v$ , new bounds for  $\mu(v, e)$  are found. For some subranges of the parameters,  $\mu(v, e)$  is determined and extremal graphs are described. The results refine some known theorems from extremal graph theory and an upper bound for the running time of an algorithm of E. L. Lawler for determining the chromatic number of a graph.*

### 1. Introduction

The definitions in this paper are based on [Bo76]. All graphs we consider are undirected labelled graphs without loops and multiple edges.  $V(G)$  and  $E(G)$  denote sets of vertices and edges of  $G$  respectively. The number of elements of a finite set  $A$  is denoted by  $|A|$ . We write  $v = v(G) = |V(G)|$  and  $e = e(G) = |E(G)|$  and call  $G$  a  $(v, e)$ -graph. Let  $\{x; y\}$  be an edge of  $G$ . Then by  $G - \{x; y\}$  we mean the graph obtained from  $G$  by deleting  $\{x; y\}$ . By  $K_v$ ,  $K_v^c$ ,  $T_v$  and  $K_{m,n}$  we denote correspondingly the complete graph on  $v$  vertices (any two vertices are joined by an edge), the completely disconnected graph on  $v$  vertices (no edges at all), a tree on  $v$  vertices, and the complete bipartite graph whose vertex classes contain  $m$  and  $n$  vertices. By  $G + H$  we denote the disjoint union of graphs  $G$  and  $H$ . For a given  $x \in V(G)$ , by  $N_G(x)$  we denote the set of all neighbors of  $x$  in  $G$ , i. e. the set of all  $y \in$

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<sup>1</sup> This paper is based on a part of a Ph. D. Thesis written by the author under the supervision of Prof. H. S. Wilf at the University of Pennsylvania.

$V(G)$  such that  $\{x,y\} \in E(G)$ . A set  $S$ ,  $S \subseteq V(G)$ , is an *independent set* in  $G$  if no two vertices of  $S$  are joined by an edge of  $G$ .  $S$  is a *maximal independent set* (m.i.s.) in  $G$  if  $S$  is independent and is not a subset of any other independent set of  $G$ . Let  $\mathcal{M}(G)$  denote the set of all m.i.s. of vertices in  $G$ ,  $\mu(G) = |\mathcal{M}(G)|$ ,  $\mu(v) = \max\{\mu(G): |V(G)| = v\}$ ,  $\mu(v, e) = \max\{\mu(G): G \text{ is a } (v, e)\text{-graph}\}$ . Obviously, for any  $(v, e)$ -graph  $G$ ,  $\mu(G) \leq \mu(v, e) \leq \mu(v)$ . A *clique* in  $G$  is a maximal complete subgraph of  $G$ . Let  $cl(G)$  denote the number of cliques of graph  $G$ . Let  $cl(v) = \max\{cl(G): |V(G)| = v\}$ ,  $cl(v, e) = \max\{cl(G): G \text{ is a } (v, e)\text{-graph}\}$ . Let  $G^c$  denote the complement of graph  $G$ . It is easy to see that  $\mu(G) = cl(G^c)$ ,  $\mu(v) = c(v)$  and  $\mu(v, e) = cl(v, v(v-1)/2 - e)$ . The following problem was formulated by H. S. Wilf:

For the given pair of positive integers  $(v, e)$ , find  $\mu(v, e)$  or give a non-trivial upper bound of  $\mu(v, e)$ .

In this paper we present some partial results. Problems similar to this, but for different families of graphs, were considered by several authors. The value of  $cl(v)$  was determined by Miller and Muller [MiMu60] and independently by a different method by Moon and Mosser [MoMo65] in which they characterized the extremal graphs. They found that

$$\mu(v) = cl(v) = \begin{cases} 3^t & \text{if } v = 3t \geq 3; \\ 4 \cdot 3^{t-1} & \text{if } v = 3t+1 \geq 4; \\ 2 \cdot 3^t & \text{if } v = 3t+2 \geq 2 \end{cases} \quad (1.1)$$

and that the extremal graphs (for the number of cliques) are Turan graphs  $T_t(v)$ .

It turned out that the problem of finding  $cl(v)$  can be shown to be equivalent (Yao [Y76] attributes this result to D. E. Muller) to the problem of Katona on minimal separating systems:

Given the set  $[n] = \{1, 2, \dots, n\}$ . Find the smallest number  $f(n)$  for which there exists a family of subsets of  $[n]$   $\{A_1, A_2, \dots, A_{f(n)}\}$  with the following property: given any two elements  $x, y \in [n]$  ( $x \neq y$ ), there exist  $k, \ell$  such that  $A_k \cap A_\ell = \emptyset$ , and  $x \in A_k, y \in A_\ell$ .

Katona's problem was solved by Yao [Y76] and independently by a similar method by Cai [Ca83].

There were several papers in which the authors restricted their attention to a subset  $\mathcal{F}$  of all the graphs on  $v$  vertices and determined either  $cl(v, \mathcal{F}) = \max\{cl(G): G \in \mathcal{F}\}$  or  $\mu(v, \mathcal{F}) = \max\{\mu(G): G \in \mathcal{F}\}$ .

Hedman [He85,He] found the maximal number of cliques for the family  $\mathcal{F}$  of all graphs on  $v$  vertices with the given clique number  $\omega$  (the clique number  $\omega$  of a graph  $G$  is the greatest number of vertices in a clique of  $G$  among all cliques of  $G$ ).

Wilf [W86] found the largest number of m.i.s. vertices that any tree of  $v$  vertices can have. The same results were obtained by Cohen [Co84] and Sagan [Sa88] by different methods. Sagan's paper completely describes all extremal graphs. If we denoted this number by  $\mu(v, \text{Tree})$ , then the result is

$$\mu(v, \text{Tree}) = \begin{cases} 2^{t-1} + 1 & \text{if } v = 2t > 0; \\ 2^t & \text{if } v = 2t+1; \\ 1 & \text{if } v = 0 \end{cases} \quad (1.2)$$

Furedi [Fu87] gave a new proof of (1.1) and established an exact upper bound for  $\mu(G)$  for a non-extremal graph  $G$ . In the same paper he found  $\mu(v, \text{Conn})$  = the maximum number of m.i.s. that a connected graph on  $v$  vertices can have (for  $v > 50$ ) and described all extremal graphs. Independently, Griggs, Grinstead and Guichard [GGG88] determined  $\mu(v, \text{Conn})$  for all  $v \geq 6$  and described all the extremal graphs. Their result is

$$\mu(v, \text{Conn}) = \begin{cases} 2 \cdot 3^{t-1} + 2^{t-1} & \text{if } v = 3t > 6; \\ 3^t + 2^{t-1} & \text{if } v = 3t+1 > 6; \\ 4 \cdot 3^{t-1} + 3 \cdot 2^{t-2} & \text{if } v = 3t+2 > 6 \end{cases} \quad (1.3)$$

Harary and Lempel [HL74] studied the extremal graphs for the family of all graphs on  $v$  vertices with  $e$  edges. They developed some standard forms for such graphs and suggested a transformation which brings an extremal graph into this form. Similar results were obtained independently by the author. Unfortunately they have not helped much in finding  $\mu(v, e)$ .

Another motivation for the present work was an article by E. Lawler [La76] in which an algorithm for determining the chromatic number of a graph is discussed, and it is shown that its run time, in the worst case, is  $O[ev(1+3^{1/3})^v]$  for graphs of  $e$  edges and  $v$  vertices. The appearance of  $3^{1/3}$  derives from (1.1) because of the fact that a graph on  $v$  vertices has at most  $3^{v/3}$  maximal independent sets. The fact that the graph has  $e$  edges is not used when the greatest number of maximal independent sets is estimated.

In Section 2 we give new bounds for  $\mu(v, e)$  and determine  $\mu(v, e)$  exactly for some ranges of  $v$  and  $e$ . The main results are in Theorems 2.1, 2.6 and Corollary 2.3.

## 2. Results

In this section we determine explicitly or find bounds for  $\mu(v, e)$ , for  $0 \leq e \leq v$ . We start with the following.

**Theorem 2.1** *Let  $v, e$  be non-negative integers,  $0 \leq e \leq v$ , and  $m(v, e) = 2^{v-e} \cdot 3^{(2e-v)/3}$ . Then*

$$\mu(v, e) \leq m(v, e) \quad (2.1)$$

*The equality in (2.1) occurs if and only if  $v/2 \leq e \leq v$  and  $2e - v = 3t$  for some non-negative integer  $t$ . The only graph  $G$ , for which  $\mu(G) = \mu(v, e) = m(v, e)$  is  $G = (v - e)K_2 + tK_3$ .*

**Proof** We notice that for  $e > v$ , our upper bound  $m(e, v)$  is worse than  $\mu(v)$  given by (1.1). This explains the restriction  $0 \leq e \leq v$ . If  $v = e = 0$ , then  $\mu(\emptyset) = m(0, 0) = 1$ . If  $v = 1, e = 0$ , then  $\mu(K_1) = 1 < m(1, 0) = 2/(3^{1/3})$ . Let  $G$  be an extremal graph and  $G_1, G_2, \dots, G_n$  be connected components of  $G$ . Suppose  $G_i$  is a  $(v_i, e_i)$ -graph,  $1 \leq i \leq n$ . Then  $\sum v_i = v$  and  $\sum e_i = e$ . Since  $\mu(v, e) = \mu(G) = \prod \mu(G_i)$  and  $m(v, e) = \prod m(v_i, e_i)$ , Then in order to prove the theorem it is sufficient to show that for all  $i, 1 \leq i \leq n, \mu(G_i) \leq m(v_i, e_i)$ .

**Lemma 2.2** *For any connected  $(v, e)$ -graph  $G, \mu(G) \leq m(v, e)$ .  $\mu(G) = m(v, e)$  if and only if  $G = \emptyset, G = K_2$  or  $G = K_3$ .*

**Proof** It is enough to show that

$$\mu(v) \leq m(v, e). \quad (2.3)$$

Then the first statement will be proved. Let  $e = v + p$ . Then

$$m(v, v+p) = 2^{-p} \cdot 3^{(v+2p)/3} = (9/8)^{p/3} 3^{v/3},$$

and (2.3) can be easily checked by using the table from Figure 1. Entries in the  $\mu(v)$  column come from (1.1) and (1.2) (the only connected  $(v, e)$ -graphs with  $e \leq v - 1$  are trees, and  $e = v - 1$ ).

e	v	$\mu(v)$	$m(v,e)$
$e = v + p, p \geq 0$	$v = 3t, t \geq 1$	$3^t$	$(9/8)^p 3^t$
	$v = 3t+1, t \geq 1$	$4 \cdot 3^{t-1}$	$3^{1/3} (9/8)^p 3^t$
	$v = 3t+2, t \geq 0$	$2 \cdot 3^t$	$3^{2/3} (9/8)^p 3^t$
$e = v - 1$	$v = 2t, t \geq 1$	$2^{t-1} + 1$	$(8/9)^{1/3} 3^{2t/3}$
	$v = 2t + 1, t \geq 0$	$2^t$	$(2/3)^{1/3} 3^{2t/3}$

Figure 1

Comparing the entries in the table on Figure 1, we conclude that equality occurs if and only if  $v = 3t, p = 0$  or  $v = 2, p = -1$ . For  $v = 3t, p = 0$ , we have  $e = 3t, \mu(3t) = 3^t$ . As it follows from [MiMu60] and [MoMo65], the only extremal graph in this case is  $tK_3$ . This graph is connected for  $t = 1$ . Therefore  $G = K_3$ . For  $v = 2, p = -1$ , we have  $e = 1$  and  $G = K_2$ . This proves the lemma.  $\square$

Thus (2.2) is true for each connected component of  $G$  and the bound (2.1) is proved. In order to get an equality in (2.1), each connected component of  $G$  has to be either  $K_3$  or  $K_2$ . Suppose  $G = tK_3 + sK_2$ , for some non-negative integers  $t, s$ . then

$$3t + 2s = v \text{ and } 3t + s = e \tag{2.4}$$

The only solution of (2.4) is  $s = v - e, t = (2e - v)/3$ , and this concludes the proof of the theorem.

**Corollary 2.3** *Let  $v, e$  be non-negative integers,  $v/2 \leq e \leq v$ . If  $2e - v = 3t + 1$ , then  $3^{-1/3}m(v, e) \leq \mu(v, e) < m(v, e)$ . If  $2e - v = 3t + 2$ , then  $\mu(v, e) = (1/2)(3^{1/3})m(v, e)$  for  $v = 4, 6$ ;  $(5/3^{5/3})m(v, e) \leq \mu(v, e) < m(v, e)$  for  $v = 5, v \geq 7$ .*

**Proof** The upper bounds follow from Theorem 2.1. In the case  $2e - v = 3t + 1$ , the lower bound comes from the graph  $tK_3 + (v - e - 1)K_2 + P_2$ , where  $P_2$  is a path with two edges. If  $2e - v = 3t + 2$ , then for  $v = 4$ , the only possible value of  $e$  is 4. In

both cases  $\mu(v, e) = (1/2)(3^{1/3})m(v, e)$ . For  $v = 5$  or  $\geq 7$ , the lower bound comes from the graph  $(t-1)K_3 + (v-e)K_2 + H$ , where  $H$  is  $(5, 5)$ -graph shown on Figure 2. The lower bounds seem to be the best possible, but the author has been unable to prove it.

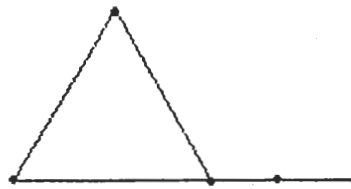


Figure 2

It turns out that for  $e$ ,  $0 \leq e \leq v/2$ , the result of Theorem 2.1 can be substantially improved. The following lemma is the main step in this direction. It is also interesting on its own.

**Lemma 2.4** *Let  $G$  be a graph and  $\{x; y\}$  be an edge of  $G$ . Then*

$$\mu(G) \leq 2\mu(G - \{x; y\}). \quad (2.5)$$

The equality in (2.5) occurs if and only if  $\{x; y\}$  is a connected component of  $G$ .

**Proof** The idea of the proof is to partition both  $\mathcal{M}(G)$  and  $\mathcal{M}(G - \{x; y\})$  into several classes and to compare numbers of elements in the corresponding classes. The description of the partitions is rather long, but the comparison will be easy. Figure 3 illustrates both stages of the proof. We divide  $\mathcal{M}(G)$  into the following 7 classes some of which can be empty ( $\dot{\cup}$  stands for the disjoint union of sets):

$$\mathcal{M}_{x,1}(G) = \{M \in \mathcal{M}(G): M = \{x\} \dot{\cup} M', M' \neq \emptyset, M' \cap N_G(y) = \emptyset\};$$

$$\mathcal{M}_{y,1}(G) = \{M \in \mathcal{M}(G): M = \{x\} \dot{\cup} M', M' \neq \emptyset, M' \cap N_G(x) = \emptyset\};$$

$$\mathcal{M}_{x,2}(G) = \{M \in \mathcal{M}(G): M = \{x\} \dot{\cup} M', M' \cap N_G(y) \neq \emptyset\};$$

$$\mathcal{M}_{y,2}(G) = \{M \in \mathcal{M}(G): M = \{x\} \dot{\cup} M', M' \cap N_G(x) \neq \emptyset\};$$

$$\mathcal{M}_3(G) = \{M \in \mathcal{M}(G): M \neq \emptyset, x \notin M, y \in M\};$$

$$\mathcal{M}_x(G) = \begin{cases} \{x\}, & \text{if } \{x\} \in \mathcal{M}(G) \\ \emptyset, & \text{otherwise} \end{cases}$$

$$\mathcal{M}_y(G) = \begin{cases} \{y\}, & \text{if } \{y\} \in \mathcal{M}(G) \\ \emptyset, & \text{otherwise} \end{cases}$$

It is easy to check that  $\mathcal{M}(G)$  is the disjoint union of these classes. Similarly we divide  $\mathcal{M}(G - \{x; y\})$  into the following 5 classes:

$$\mathcal{M}_{x,y,1}(G - \{x; y\}) = \{M \in \mathcal{M}(G - \{x; y\}) : M = \{x\} \dot{\cup} \{y\} \dot{\cup} M', M' \neq \emptyset\};$$

$$\mathcal{M}_{x,2}(G - \{x; y\}) = \{M \in \mathcal{M}(G - \{x; y\}) : x \in M, M \cap N_{G-\{x; y\}}(y) \neq \emptyset\};$$

$$\mathcal{M}_{y,2}(G - \{x; y\}) = \{M \in \mathcal{M}(G - \{x; y\}) : y \in M, M \cap N_{G-\{x; y\}}(x) \neq \emptyset\};$$

$$\mathcal{M}_3(G - \{x; y\}) = \{M \in \mathcal{M}(G - \{x; y\}) : M \neq \emptyset, x \notin M, y \notin M\};$$

$$\mathcal{M}_{x,y}(G - \{x; y\}) = \begin{cases} \{x, y\}, & \text{if } \{x, y\} \in \mathcal{M}(G - \{x; y\}) \\ \emptyset, & \text{otherwise} \end{cases}$$

It is easy to check that  $\mathcal{M}(G - \{x; y\})$  is the disjoint union of these classes.

The following bijections between some of these classes are obvious:

$$f_{x,1} : \mathcal{M}_{x,1}(G) \rightarrow \mathcal{M}_{x,y,1}(G - \{x; y\}), f_{x,1}(\{x\} \dot{\cup} M') = \{x, y\} \dot{\cup} M';$$

$$f_{y,1} : \mathcal{M}_{y,1}(G) \rightarrow \mathcal{M}_{x,y,1}(G - \{x; y\}), f_{y,1}(\{y\} \dot{\cup} M') = \{x, y\} \dot{\cup} M';$$

$$f_{x,2} : \mathcal{M}_{x,2}(G) \rightarrow \mathcal{M}_{x,2}(G - \{x; y\}), f_{x,2}(M) = M;$$

$$f_{y,2} : \mathcal{M}_{y,2}(G) \rightarrow \mathcal{M}_{y,2}(G - \{x; y\}), f_{y,2}(M) = M;$$

$$f_3 : \mathcal{M}_3(G) \rightarrow \mathcal{M}_3(G - \{x; y\}), f_3(M) = M;$$

$$f_x : \mathcal{M}_x(G) \rightarrow \mathcal{M}_{x,y}(G - \{x; y\}), f_x(\{x\}) = \{x, y\}, \text{ if } \{x\} \text{ is m.i.s.};$$

$$f_x(\{\emptyset\}) = \emptyset, \text{ if } \{x\} \text{ is not m.i.s.};$$

$$f_y : \mathcal{M}_y(G) \rightarrow \mathcal{M}_{x,y}(G - \{x; y\}), f_y(\{y\}) = \{x, y\}, \text{ if } \{y\} \text{ is m.i.s.};$$

$$f_y(\{\emptyset\}) = \emptyset, \text{ if } \{y\} \text{ is not m.i.s.}$$

(Notice that for each of these mappings the domain and the range are non-empty simultaneously.)

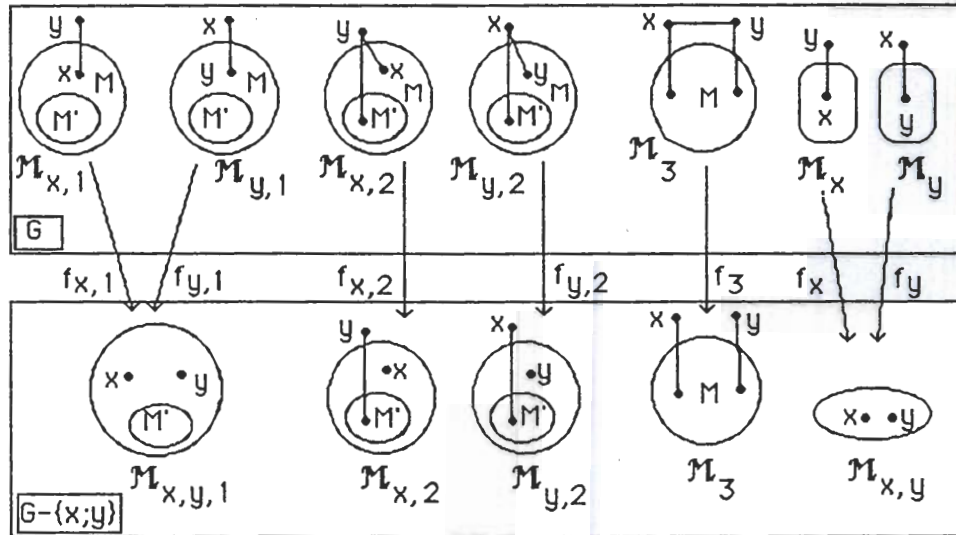


Figure 3

Finally we denote  $|\mathcal{M}_{x,1}(G)| = |\mathcal{M}_{y,1}(G)| = |\mathcal{M}_{x,y,1}(G-\{x;y\})| = i_1$ ,

$|\mathcal{M}_{x,2}(G)| = |\mathcal{M}_{x,2}(G-\{x;y\})| = i_{x,2}$ ,  $|\mathcal{M}_{y,2}(G)| = |\mathcal{M}_{y,2}(G-\{x;y\})| = i_{y,2}$ ,

$|\mathcal{M}_3(G)| = |\mathcal{M}_3(G-\{x;y\})| = i_3$ ,  $|\mathcal{M}_x(G)| = i_x (= 1 \text{ or } 0)$ ,

$|\mathcal{M}_y(G)| = i_y (= 1 \text{ or } 0)$ ,  $|\mathcal{M}_{x,y}(G-\{x;y\})| = i_{x,y} (= 1 \text{ or } 0)$ .

Then

$$2\mu(G-\{x;y\}) = 2(i_1 + i_{x,2} + i_{y,2} + i_3 + i_{x,y}),$$

$$\mu(G) = 2i_1 + i_{x,2} + i_{y,2} + i_3 + i_x + i_y.$$

Hence,  $\mu(G) = 2\mu(G-\{x;y\}) - (i_{x,2} + i_{y,2} + i_3) - (2i_{x,y} - i_x - i_y)$ . Obviously  $i_{x,2} + i_{y,2} + i_3 \geq 0$ . If at least one of  $i_x$  or  $i_y = 1$ , then  $i_{x,y} = 1$ , and  $2i_{x,y} - i_x - i_y \geq 0$ . If  $i_x = i_y = 0$ , then again  $2i_{x,y} - i_x - i_y \geq 0$ . Therefore we get

$$\mu(G) \leq 2\mu(G-\{x;y\}) \tag{2.6}$$

The equality sign in (2.6) occurs if and only if  $i_{x,2} + i_{y,2} + i_3 = 0$  and  $2i_{x,y} - i_x - i_y = 0$ . The first of the equalities implies  $i_{x,2} = i_{y,2} = i_3 = 0$ . If  $i_{x,2} = i_{y,2} = 0$ , then



vertices  $x$  and  $y$  have the same set of neighbors in  $G$  (each independent set of a graph is a subset of at least one m.i.s.). But  $i_3 = 0$  implies that this set of neighbors is empty. Therefore the edge  $\{x;y\}$  is a connected component in  $G$ . If this is the only connected component of  $G$ , i. e.  $G = K_2$ , then  $i_{x,y} = i_x = i_y = 1$  and  $2i_{x,y} - i_x - i_y = 0$ . If  $G$  has more than one connected component, then  $i_{x,y} = i_x = i_y = 0$  and again  $2i_{x,y} - i_x - i_y = 0$ . The lemma is proved.  $\square$

**Corollary 2.5** *If an extremal  $(v, e)$ -graph  $G$  has two isolated vertices, then it is a disjoint union of edges and isolated vertices.*

**Proof** Let  $G$  have a connected component  $H$  with 2 or more edges and two isolated vertices  $a$  and  $b$ . By deleting an edge  $\{x;y\}$  in  $H$  and joining vertices  $a$  and  $b$  we obtain a  $(v, e)$ -graph  $G'$ . Since  $\mu(H + \{a\} + \{b\}) = \mu(H)$  and  $\mu(\{a;b\} + (H - \{x;y\})) = 2\mu(H - \{x;y\})$ , and by Lemma 2.4,  $\mu(H) < 2\mu(H - \{x;y\})$ , then  $\mu(H + \{a\} + \{b\}) < \mu(\{a;b\} + (H - \{x;y\}))$ . All other connected components (with the vertices in  $V(G) - \{a, b\} - V(H)$ ) of  $G$  and  $G'$  are the same. Therefore  $\mu(G) < \mu(G')$ , which contradicts the extremality of  $G$ .  $\square$

The following theorem gives the exact value of  $\mu(v,e)$  and describes the extremal graphs for  $0 \leq e \leq v/2$ .

**Theorem 2.6** Let  $0 \leq e \leq v/2$ . Then  $\mu(v,e) = 2^e$ , and the only extremal graph is  $eK_2 + (v - 2e)K_1$ .

**Proof** The greatest number of vertices in a graph which are incident to  $e$  edges is  $2e$  and this happens *only* if the graph is  $eK_2$ . Therefore if  $v - 2e \geq 1$ , then the statement of the theorem follows from Corollary 2.5. so we assume that  $v = 2e$ . Let  $G$  be extremal graph and  $G_1, G_2, \dots, G_n$  be connected components of  $G$ . Suppose  $v(G_i) = v_i$  and  $e(G_i) = e_i$ ,  $1 \leq i \leq n$ . Since for all  $i$ ,  $1 \leq e_i \leq v_i - 1$ , then

$$\sum e_i = e \geq v - n \text{ and } n \geq v - e = 2e - e = e.$$

If  $G$  had two isolated vertices, then, due to Corollary 2.5, it would have at least  $2e + 2$  vertices and this is not the case.

Suppose  $G$  has no isolated vertices. Then each component must have at least one edge and  $n \leq e$ . So  $n = e$ ,  $G = eK_2$  and the theorem is proved.

The only case left is when  $G$  has only one isolated vertex. Then each of the remaining  $n - 1$  components has at least one edge. If each of them has exactly one edge, then  $G = eK_2 + K_1$  and  $v(G) = 2e + 1$ , but the latter is false. So there should be a component with at least two edges. It cannot have three edges, since in this case  $e(G) \geq$

$3 + (n - 2) = n + 1 > e$ . Thus  $G = P_2 + (e - 2)K_2 + K_1$  ( $P_2$  is a path with two edges), and  $\mu(G) = 2 \cdot 2^{e-2} = 2^{e-1}$ . But this is less than  $2^e = \mu(eK_2)$  which contradicts the extremality of  $G$ . Therefore the theorem is proved.

**Acknowledgement** The author wishes to thank the referee for his numerous comments, suggestions and corrections which resulted in the improvement of the original version of this paper.

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