NEW LOWER BOUNDS ON THE MULTICOLOR RAMSEY NUMBERS $r_k(C_4)$

FELIX LAZEBNIK AND ANDREW J. WOLDAR

Abstract. The multicolor Ramsey number $r_k(C_4)$ is the smallest integer n for which any k-coloring of the edges of the complete graph K_n must produce a monochromatic 4-cycle. In [6] and [3] it was proved that $r_k(C_4) \geq k^2 - k + 2$ for $k - 1$ being a prime power. In this note we establish $r_k(C_4) \geq k^2 + 2$ for k being an odd prime power.

(Journal of Combinatorial Theory, Series **B 79**, 172–176 (2000))

1. Introduction

Let $k \geq 1$ be an integer, and let $G_1, \dots, G_k$ be graphs. The multicolor Ramsey number $r(G_1, \dots, G_k)$ is defined to be the smallest integer $n = n(k)$ with the property that any k-coloring of the edges of the complete graph K_n must result in a monochromatic subgraph of K_n isomorphic to G_i for some i. (Here, by "monochromatic subgraph" we mean a subgraph all of whose edges have the same color.) When all graphs G_i are identical, we usually abbreviate $r(G, \dots, G)$ by $r_k(G)$. Clearly, the notion of multicolor Ramsey number is a natural generalization of that of the classical Ramsey number $r(s,t) := r(K_s, K_t)$.

We focus our attention on the special case where each G_i is a 4-cycle C_4. Here it is known that

$$k^2 - k + 2 \leq r_k(C_4) \leq k^2 + k + 1,$$

with the upper bound (due independently to Chung [2] and Irving [6]) holding for all $k \geq 1$, and the lower bound (due inependently to Irving [6] and Chung and Graham [3]) holding for $k - 1$ being a prime power. In this note we show that

$$k^2 + 2 \leq r_k(C_4) \tag{1.1}$$

for k being an odd prime power.

In order to accomplish this, we first describe a special C_4-free graph Γ of order q^2 and size $q(q^2-1)/2$ where q is an odd prime power. (Recall that the order of a graph is its number of vertices, and the size is its number of edges.) We show how to color the edges of the complete graph K_{q^2} in q colors such that each maximal monochromatic subgraph is isomorphic to Γ. Such colorings can also be viewed as edge decompositions of K_{q^2} into isomorphic copies of Γ. Clearly, the existence of such colorings implies that $r_q(C_4) \geq q^2 + 1$. Then we show that the

---

Date: January 16, 2003.

Key words and phrases. Edge coloring, monochromatic graph, multicolor Ramsey number, edge decomposition, polarity graph.

This research was partially supported by NSF grant DMS-9622091.

edge coloring of K_{q^2} can be extended to a q-coloring of the edges of K_{q^2+1} such that each monochromatic subgraph is C_4-free. This will prove (1.1).

2. Construction and Proofs

Let q be a prime power and $\mathbb{F}_q$ be the finite field of q elements. Set $V = \mathbb{F}_q \times \mathbb{F}_q$. Consider a simple graph Γ with the vertex set V and the edge set defined in the following way: two distinct vertices (a_1, a_2) and (b_1, b_2) are adjacent if and only if

$$a_2 + b_2 = a_1 b_1. \tag{2.1}$$

Using this definition of adjacency, the degree of an arbitrary vertex (x, y) of Γ can be computed as follows. The neighbors of (x, y) are clearly those vertices $(u, xu - y)$ which are distinct from (x, y). Since $(u, xu - y) = (x, y)$ if and only if $u = x$ and $y = x^2/2$, each of the q vertices of the form $(x, x^2/2)$ has exactly $q - 1$ neighbors, while each of the remaining $q^2 - q$ vertices has exactly q neighbors. This implies that the number of edges of Γ is $\frac{1}{2}(q(q-1) + (q^2 - q)q) = q(q^2 - 1)/2$.

Proposition 1. *Let q be an odd prime power. Then Γ is C_4-free, and there exists a q-coloring of the edges of K_{q^2} such that each maximal monochromatic subgraph is isomorphic to Γ.*

Proof. Let $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$ be the four consecutive (distinct) vertices of a 4-cycle in Γ. Then

$$\begin{aligned} a_2 + b_2 &= a_1 b_1 \\ b_2 + c_2 &= b_1 c_1 \\ c_2 + d_2 &= c_1 d_1 \\ d_2 + a_2 &= d_1 a_1, \end{aligned} \tag{2.2}$$

which implies $(a_1 - c_1)(b_1 - d_1) = 0$. Thus one has either $a_1 = c_1$ or $b_1 = d_1$, which implies either that $(a_1, a_2) = (c_1, c_2)$ or $(b_1, b_2) = (d_1, d_2)$, a contradiction. Hence Γ is C_4-free.

To prove the second statement, we identify the vertex set of K_{q^2} with V. This makes Γ a spanning subgraph of K_{q^2}. For each $\alpha \in \mathbb{F}_q$, define the mapping $\phi_\alpha : V \to V$ by $(x, y) \mapsto (x, y + \alpha)$. Clearly, ϕ_α is a bijection. Now define $\phi_\alpha(\Gamma)$ to be the spanning subgraph of K_{q^2} with two distinct vertices $(a_1, a_2 + \alpha)$ and $(b_1, b_2 + \alpha)$ adjacent if and only if the vertices (a_1, a_2) and (b_1, b_2) are adjacent in Γ. It is immediate from this definition that subgraph $\phi_\alpha(\Gamma)$ is isomorphic to Γ for each $\alpha \in \mathbb{F}_q$; indeed, ϕ_α is an explicit isomorphism.

We claim that the subgraphs $\phi_\alpha(\Gamma)$, $\alpha \in \mathbb{F}_q$, are pairwise edge-disjoint. Indeed, suppose two distinct vertices (x, y) and (u, w) are adjacent in both $\phi_\alpha(\Gamma)$ and $\phi_\beta(\Gamma)$. Then the pairs of verices $\{(x, y - \alpha), (u, w - \alpha)\}$ and $\{(x, y - \beta), (u, w - \beta)\}$ are both edges in Γ. Using (2.1) we obtain

$$(y - \alpha) + (w - \alpha) = (y - \beta) + (w - \beta) = xu,$$

which implies $\alpha = \beta$. This proves that the q subgraphs $\phi_\alpha(\Gamma)$ are indeed pairwise edge-disjoint. Since the sizes of Γ and K_{q^2} are $q(q^2 - 1)/2$ and $q^2(q^2 - 1)/2$, respectively, we obtain an edge partition of K_{q^2} into q isomorphic copies of Γ. To complete the proof, we merely choose $\mathbb{F}_q$ as our color set and we color all edges of $\phi_\alpha(\Gamma)$ with the color α. □

An immediate consequence of Proposition 1 is the following result.

Corollary 1. *For q an odd prime power, $r_q(C_4) \geq q^2 + 1$.*

But, in fact, this bound can be slightly improved.

Theorem 1. *For q an odd prime power, $r_q(C_4) \geq q^2 + 2$.*

Proof. It suffices to exhibit a q-coloring of the edges of K_{q^2+1} in which there is no monochromatic C_4. Fix a vertex v of K_{q^2+1} and denote the subgraph induced by its set of neighbors by N. As N is isomorphic to K_{q^2}, we can q-color its edges in accordance with Proposition 1, that is, by assigning color $\alpha \in \mathbb{F}_q$ to all edges of N of the form $\{(a_1, a_2), (b_1, a_1 b_1 - a_2 + 2\alpha)\}$. Thus it remains only to color the edges $\{v, u\}$ for $u \in V(N)$, and we do this by simply assigning color β to edge $\{v, u\}$ if β is the first coordinate of u. We claim that the constructed q-coloring of the edges of K_{q^2+1} contains no monochromatic 4-cycle.

Suppose, by way of contradiction, that a monochromatic 4-cycle exists, say of color α. Clearly v must be one of its vertices, so denote the consecutive vertices of this cycle by v, w, x, y. Then w and y have common first coordinate α, say $w = (\alpha, w_2)$ and $y = (\alpha, y_2)$. By the above, one now has $x = (x_1, \alpha x_1 - w_2 + 2\alpha)$ (by adjacency of x to w) and $x = (x_1, \alpha x_1 - y_2 + 2\alpha)$ (by adjacency of x to y). Thus $w_2 = y_2$ and we obtain the contradiction $w = y$. □

It is easy to argue that it is impossible to extend the obtained q-coloring of K_{q^2+1} to a q-coloring of K_{q^2+2} without creating a monochromatic 4-cycle.

3. Generalizations and Concluding remarks

In [1] it is proved that $r_3(C_4) = 11$ (see also [4]), but this is the only case in which $r_k(C_4)$ is explicitly known. There are, however, lower bounds for very small k which improve the general lower bound from [6] and [3], namely $r_4(C_4) \geq 18$ (improving $r_4(C_4) \geq 14$) and $r_5(C_4) \geq 25$ (improving $r_5(C_4) \geq 22$), see [5]. Of these two cases, our theorem applies only to the latter, and the resulting bound is $r_5(C_4) \geq 27$. In fact, one actually has $r_5(C_4) \in \{27, 28, 29\}$, which is an easy consequence of a result in [9] which states that any graph of order 29 and size at least 81 must contain a 4-cycle.

The construction of graph Γ and the results of this note allow the following generalization.

Let R be a commutative ring. For each integer i, $2 \leq i \leq n$, let $f_i : R^{2i-2} \to R$ be a function which satisfies

$$f_i(x_1, y_1, \ldots, x_{i-1}, y_{i-1}) = f_i(y_1, x_1, \ldots, y_{i-1}, x_{i-1}),$$

for all $x_i, y_j \in R$. Define $\Gamma_n = \Gamma(R; f_2, \ldots, f_n)$ to be the graph with vertex set $V(\Gamma_n) = R^n$, where distinct vertices $(a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ are adjacent if and only if the following $n - 1$ relations on their coordinates hold:

$$\begin{aligned} a_2 + b_2 &= f_2(a_1, b_1) \\ a_3 + b_3 &= f_3(a_1, b_1, a_2, b_2) \\ \cdots \quad &\cdots \\ a_n + b_n &= f_n(a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}) \end{aligned} \tag{3.1}$$

Graph Γ of this note is a particular case of this construction, namely when $n = 2$, $R = \mathbb{F}_q$, and $f_2(x_1, y_1) = x_1 y_1$.

In [7] the authors show that when R is finite of cardinality $r \geq 3$ and of characteristic not equal to 2, then the complete graph K_{r^n} can be decomposed into r^{n-1} edge-disjoint copies of Γ_n. Clearly, this decomposition provides a lower bound on the multicolor Ramsey number $r(G_1, \ldots, G_{r^{n-1}})$, specifically

$$r(G_1, \ldots, G_{r^{n-1}}) \geq r^n + 1,$$

when $G_1, \ldots, G_{r^{n-1}}$ are any graphs not contained in Γ_n.

Acknowledgments

The authors are grateful to Linyuan Lu, for pointing to the relation between the edge decomposition properties of our graphs from [7] and multicolor Ramsey numbers, and for suggesting reference [3].

References

[1] A. Bialostocki and J. Schönheim, On some Turán and Ramsey numbers for C_4, *Graph Theory and Combin.* (Ed. B. Bollobás), Academic Press, London (1984), 29-33.

[2] F. R. K. Chung, On triangular and cyclic Ramsey numbers with k colors, in *Graphs and Combinatorics* (Ed. R. Bari and F. Harari), Springer LNM 406, Berlin (1974), 236-242.

[3] F. R. K. Chung and R. L. Graham, On multicolor Ramsey numbers for complete bipartite graphs, *J. Combin. Theory* Ser. B 18 (1975), 164-169.

[4] C. Clapham, The Ramsey number $r(C_4, C_4, C_4)$, *Periodica Math. Hungarica* 18 (1987), 317-318.

[5] G. Exoo, Constructing Ramsey graphs with a computer, *Congr. Numerant.* 59 (1987), 31-36.

[6] R. W. Irving, Generalized Ramsey numbers for small graphs, *Discrete Mathematics* 9 (1974), 251 – 264.

[7] F. Lazebnik and A. J. Woldar, General properties of some families of graphs defined by systems of equations, submitted.

[8] S. P. Radziszowski, *Small Ramsey numbers*, Electronic J. Combin. 1 DS1 (1994), 1–30.

[9] Y. Yuansheng and P. Rowlinson, On extremal graphs without four-cycles, *Utilitas Math.* 41 (1992), 204-210.

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

E-mail address: `lazebnik@math.udel.edu`

Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA

E-mail address: `woldar@ucis.vill.edu`