Extremal Graphs without Three-Cycles or Four-Cycles

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Abstract. We derive bounds for $f(v)$, the maximum number of edges in a graph on v vertices that contains neither three-cycles nor four-cycles. Also, we give the exact value of $f(v)$ for all v up to 24 and constructive lower bounds for all v up to 200.

1 Introduction

This paper investigates the values of $f(v)$, the maximum number of edges in a graph of order v and girth at least 5. For small values of v we also enumerate the set of extremal graphs. This problem has been mentioned several times by P. Erdős (for instance, in [6]) who conjectured that $f(v) = (1/2 + o(1))^{3/2}v^{3/2}$.

We begin with some basic definitions. Let G be a simple graph with *order* $v = |V(G)|$ and *size* $e = |E(G)|$. Let $d(x) = d_G(x)$ denote the degree of a vertex x of G, and let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees of vertices of G. Given $x \in V$, $N(x) = N_G(x)$ and $N[x] = N_G[x] = N(x) \cup \{x\}$ denote the (open) neighborhood and closed neighborhood, respectively, of x. For $V' \subset V$, define $N(V') = \bigcup_{x \in V'} N(x)$ and $\langle V' \rangle$ as the subgraph induced by V'. By $C_n, n \geq 3$, we denote an n-cycle, i.e. a cycle with n vertices. We will also refer to a 3-cycle as a *triangle* and to a 4-cycle as a *quadrilateral*. P_n denotes a path on n vertices. The *girth* of graph G, $g(G)$, is the length of the shortest cycle in G. The argument G will be left off when the intended graph is clear.

Given graphs $G_1, G_2, \ldots, G_k$, let $ex(v; G_1, G_2, \ldots, G_k)$ denote the greatest size of a graph of order v that contains no subgraph isomorphic to some $G_i, 1 \leq i \leq k$. One of the main classes of problems in extremal graph theory, known as Turán-type problems, is for given $v, G_1, G_2, \ldots, G_k$ to determine explicitly the function $ex(v; G_1, G_2, \ldots, G_k)$, or to find its asymptotic behavior. Thus, the problem we consider in this paper, that of finding the maximum size of a graph of girth at least 5, can be stated as finding the value of $ex(v; C_3, C_4)$.

Exact results for all values of v are known for only a few instances of Turán-type problems. Asymptotic results are quite satisfactory if each graph G_i, $1 \leq i \leq k$, has chromatic number $\chi(G_i)$ at least 3. Let this be the case, and let $\chi = \min\{\chi(G_i) : 1 \leq i \leq k\}$. Then

$$ex(v; G_1, G_2, \ldots, G_k) \sim \left(1 - \frac{1}{\chi - 1}\right)\binom{v}{2}, \qquad v \to \infty. \tag{1.1}$$

If at least one of the G_i's is bipartite, then $\chi = 2$, and there is no general result similar to (1.1). Excellent references on the subject are [1] and [19] (pp. 161–200).

It is well known that $ex(v; C_3) = \lfloor v^2/4 \rfloor$, and the extremal graph is the complete bipartite graph $K_{\lfloor v/2 \rfloor, \lceil v/2 \rceil}$. The exact value of $ex(v; C_4)$ is known for all values of v of the form $v = q^2 + q + 1$, where q is a power of 2 [8], or a prime power exceeding 13 [9], and it is equal to $q(q+1)^2/2$. The extremal graphs for these values of v are known, and were constructed in [3, 7]. For $1 \leq v \leq 21$, the values of $ex(v; C_4)$ and the corresponding extremal graphs can be found in [4]. It is well known that $ex(v; C_4) = (1/2 + o(1))v^{3/2}$ (see

[1, 7]). It is important to note that attempts to construct extremal graphs for $ex(v; C_3, C_4)$ by destroying all 4-cycles in the extremal graphs for $ex(v; C_3)$, or by destroying all 3-cycles in the extremal graphs for $ex(v; C_4)$, fail; neither method yields graphs of order v with $f(v)$ edges.

In Section 2 we derive upper and lower bounds on the value of $f(v)$, and provide some remarks on the structure of extremal graphs. In Section 3 we present the exact values of $f(v)$ for all v up to 24; we also enumerate all of the extremal graphs of order less than 11. In Section 4 we provide a table of constructive lower bounds for $f(v)$, $25 \leq v \leq 200$, that are better than the theoretical lower bounds.

2 Theoretical Results

In this section we present some theoretical results about $f(v)$ and the structure of the extremal graphs. Many of them will be used in the subsequent sections. We call a graph G of order v *extremal* if $g(G) \geq 5$ and $e = e(G) = f(v)$.

Proposition 2.1 *Let G be an extremal graph of order v. Then*

1. *The diameter of G is at most 3.*
2. *If $d(x) = \delta(G) = 1$, then the graph $G - \{x\}$ has diameter at most 2.*

Proof: Let x, y be two vertices of G of distance at least 4. Since xy is not an edge of G we can add this edge to $E(G)$. The obtained graph $G' = G + xy$ has girth $g(G') \geq 5$ and one more edge than G. This contradicts the assumption that G is extremal and proves part 1. Let a, b be two vertices of $G - \{x\}$ of distance at least 3. By deleting the only edge of G incident to x and introducing two new edges xa and xb, we obtain a graph G'' of order v having one more edge than G and girth at least 5. This contradicts the assumption that G is extremal and proves part 2. ■

It turns out that the extremal graphs of diameter 2 are very rare. In fact, it was claimed in [2] that the only graphs of order v with no 4-cycles and of diameter 2 are:

1. The star $K_{1,v-1}$;
2. *Moore* graphs: C_5, Petersen graph (the only 3-regular graph of order 10, diameter 2 and girth 5, see G_{10} in Figure 1), Hoffman-Singleton graph [14] (the only 7-regular graph of order 50, diameter 2 and girth 5), and a 57-regular graph of order 3250, diameter 2 and girth 5 if such exists (its existence is still an open problem [14]);
3. Polarity graphs (see [3, 7]).

Remark: The only graphs from the list above that in addition contain no triangles and are regular are the Moore graphs. It is also known that a graph of diameter $k \geq 1$ and girth $2k + 1$ must be regular [20].

We now derive an upper bound on $f(v)$. Let $\bar{d} = \bar{d}_G = \frac{1}{v}\sum_{x \in V(G)} d(x)$ be the average degree of G, and let $\sigma^2 = \sum_{x \in V(G)}[\bar{d} - d(x)]^2$. By $D_i = D_i(G)$ we denote the number of unordered pairs of vertices of G of distance i apart.

Theorem 2.2 *Let G be an extremal graph of order v and size e. Then*

$$f(v) = e = \frac{1}{2}\sqrt{v^2(v-1) - v\sigma^2 - 2vD_3} \leq \frac{1}{2}v\sqrt{v-1} \tag{2.1}$$

The inequality in (2.1) becomes an equality if and only if G is an isolated vertex or G is regular and of diameter 2, i.e., G is a Moore graph.

Proof: According to Proposition 2.1 G is connected and of diameter at most 3. Since G is $\{C_3, C_4\}$-free, we have $D_2 = \sum_{x \in V(G)} \binom{d(x)}{2}$. Therefore

$$\binom{v}{2} = D_1 + D_2 + D_3 = e + \sum_{x \in V(G)} \binom{d(x)}{2} + D_3. \tag{2.2}$$

Since $\bar{d} = 2e/v$ and $\sum_{x \in V(G)} d(x)^2 = \sigma^2 + 4e^2/v$, we can rewrite (2.2) as

$$\binom{v}{2} = \sigma^2/2 + 2e^2/v + D_3. \tag{2.3}$$

Solving (2.3) with respect to e we obtain (2.1). The statement about the equality sign in the inequality (2.1) follows immediately from the proof above and the Remark following Proposition 2.1. ■

The inequality (2.1) was found independently by Dutton and Brigham [5].

The following result can be obtained via more accurate estimates of σ^2 and D_3 in some particular cases. The proof appears in [12].

Corollary 2.3 *Let G be an extremal graph of order v, size e, diameter 3, and not regular; then*

$$f(v) = e \leq \frac{1}{2}\sqrt{v^2(v-1) - \frac{5}{2}v}.$$

If, in addition, the average degree of G is an integer, then

$$f(v) = e \leq \frac{1}{2}\sqrt{v^2(v-1) - 4v}.$$

Now we derive a lower bound for $f(v)$. Let q be a prime power, and let $v_q = q^2+q+1$, and $e_q = (q+1)v_q$. By B_q we denote the point-line incidence bipartite graph of the projective plane $PGL(2,q)$. More precisely, the partite sets of B_q represent the set of points and the set of lines of $PGL(2,q)$, and the edges of B_q correspond to the pairs of incident points and lines. Then B_q is a $(q+1)$-regular bipartite graph of order $2v_q$ and size e_q. It is easy to show that $g(B_q) = 6$. By adding vertices and edges to B_q, one can easily derive the following proposition; the proof is given in [12].

Proposition 2.4 *Let G be an extremal graph of order v and size e. Let q be the largest prime power such that $2v_q \leq v$. Then $f(v) = e \geq e_q + 2(v - 2v_q) = 2v + (q-3)v_q$.*

Combining the results of Theorem 2.2 and Proposition 2.4, we obtain the well known

Corollary 2.5 $$\frac{1}{2\sqrt{2}} \leq \liminf_{v\to\infty} \frac{f(v)}{v^{3/2}} \leq \limsup_{v\to\infty} \frac{f(v)}{v^{3/2}} \leq \frac{1}{2}.$$

We next define a restricted type of tree; many of the proofs in Section 3 rely on the presence of these trees in the extremal graphs. Consider a vertex x of maximum degree Δ in a $\{C_3, C_4\}$-free graph G. Let the neighborhood of x be $N(x) = \{x_1, x_2, \ldots, x_\Delta\}$. Clearly, $N(x)$ is an independent set of vertices. Furthermore, the sets of vertices $N(x_i) - \{x\}$, $1 \leq i \leq \Delta$, are pairwise disjoint; otherwise there would be a quadrilateral in G. This motivates the notion of an *(m,n)-star* $S_{m,n}$, that is defined to be the tree in which the root has m children, and each of the root's children has $n \geq 1$ children, all of which are leaves. The subtree containing a child of the root and all its n children is called a *branch* of $S_{m,n}$. We adopt the following notation. Given a rooted tree T of height 2, denote its root by r, the children of r by $r_1, r_2, \ldots$, and the children of r_i (that is, the leaves of T in the i-th branch) by $r_{i,1}, r_{i,2}, \ldots$. Denote by R_i the set of leaves in the i-th branch, that is, $R_i = N(r_i) - \{r\}$. Define $R = \bigcup_i R_i$ as the set of leaves in T. We will use the following easily established facts:

1. $|V(S_{m,n})| = 1 + m + mn$ and $|E(S_{m,n})| = m + mn$.
2. Every $\{C_3, C_4\}$-free graph G with at least 5 vertices contains $S_{\Delta,\delta-1}$.
3. For any nonadjacent vertices t and u in $S_{m,n}$ that are not both leaves, $S_{m,n} + tu$ contains a C_3 or C_4.
4. Suppose a $\{C_3, C_4\}$-free graph G contains an (m,n)-star S and a vertex $z \notin V(S)$. Then either (i) z is adjacent to only one vertex in S, or (ii) z is adjacent to two leaves from different branches in S. If $r_i z \in E(G)$, then we will also consider z as a vertex in S; that is, z is regarded as a leaf in R_i. Under this convention, every child of the root in any $S_{m,n}$ has *at least* n children of its own.

As an immediate consequence we have the following propositions. As simple as they are, we use them extensively to determine precise values of $f(v)$.

Proposition 2.6 *For all $\{C_3, C_4\}$-free graphs G, $v \geq 1 + \Delta\delta \geq 1 + \delta^2$.*

Proposition 2.7 *For all $\{C_3, C_4\}$-free graphs G on $v \geq 1$ vertices and e edges, $\delta \geq e - f(v-1)$ and $\Delta \geq \lceil 2e/v \rceil$.*

Proposition 2.8 *For any $\{C_3, C_4\}$-free graph G of order $v \geq 1$, we have $v \geq 1 + \lceil 2f(v)/v \rceil (f(v) - f(v-1))$.*

3 Values of $f(v)$ for $v \leq 24$ and $v = 50$

Let $\mathcal{F}_v$ denote the set of $\{C_3, C_4\}$-free graphs of order v, and let $\mathcal{F}_v^*$ denote the set of extremal graphs of order v. Define $F(v) = |\mathcal{F}_v^*|$. We now determine the values of $f(v)$ for $1 \leq v \leq 24$. We also determine the values of $F(v)$ for $1 \leq v \leq 10$ and $v = 19, 20, 50$. The authors of [13] determined the remaining values of $f(v)$ for $v \leq 30$ and $F(v)$ for $v \leq 21$; for the sake of completeness we include those values as well in the following theorem.

Theorem 3.1 *$f(v)$ and $F(v)$ have the following values:*

v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$f(v)$	0	1	2	3	5	6	8	10	12	15	16	18	21	23	26	28	31
$F(v)$	1	1	1	2	1	2	1	1	1	1	3	7	1	4	1	22	14

v	18	19	20	21	22	23	24	25	26	27	28	29	30	50
$f(v)$	34	38	41	44	47	50	54	57	61	65	68	72	76	175
$F(v)$	15	1	1	3										1

Proof: For $1 \leq v \leq 10$, we have constructed $\{C_3, C_4\}$-free graphs with $\lfloor v\sqrt{v-1}/2 \rfloor$ edges, which is the upper bound on $f(v)$ from Theorem 2.2. These graphs are shown in Figure 1. It is not difficult to verify that Figure 1 provides a complete enumeration of the extremal graphs up to order 10.

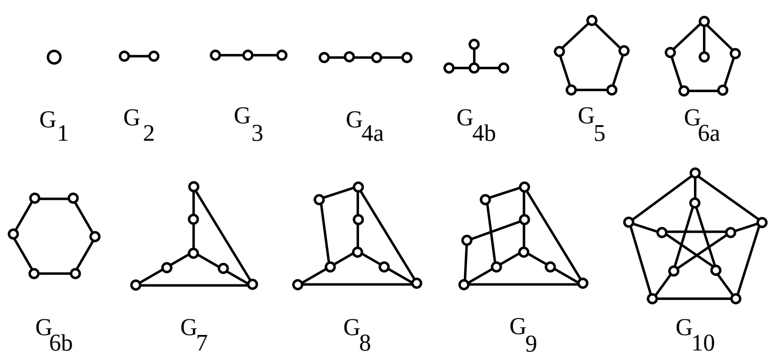

Figure 1: All extremal graphs for the orders up to 10

We now provide the exact value of $f(v), 11 \leq v \leq 24$, with a case-by-case proof based on the values of v. To show that $f(v) = m$, we use the following strategy. First, with the aid of computer programs (as described in [11]), we generate a graph $G_v \in \mathcal{F}_v$ with m edges to show that $f(v) \geq m$. Next, assume that there is a graph $G \in \mathcal{F}_v$ with more than m edges. Using Propositions 2.6–2.8, we compute the possible values of δ and Δ. Finally, assuming that G contains $S_{\Delta,\delta-1}$, we obtain a contradiction. Below we prove the theorem for several values of v. The cases chosen are representative of the techniques and the ideas we used, as well as the difficulties we encountered. Proofs for the omitted cases appear in [12].

$\boldsymbol{f(12) = 18}$. Since $G_{12} \in \mathcal{F}_{12}$ (see Figure 2), $f(12) \geq 18$. Since $f(12) \geq 19$ would contradict Proposition 2.8, we must have $f(12) = 18$. The proof is similar when $v \in \{11, 13, 15, 17, 18, 19, 20, 24\}$.

$\boldsymbol{f(14) = 23}$. G_{14} in Figure 2 gives $f(14) \geq 23$. Assume there exists $G \in \mathcal{F}_{14}$ with 24 edges. Propositions 2.6 and 2.7 imply that $\delta = 3$ and $\Delta = 4$. Thus G has 8 vertices of degree 3 and 6 vertices of degree 4. Further, G cannot contain a pair of adjacent vertices x and y each having degree 3, since $G - \{x, y\}$ would have 19 edges, contradicting $f(12) = 18$. Since the sum of the degrees of the degree 3 vertices equals the sum of the degrees of the degree 4 vertices, we conclude that every edge of G connects a degree 3 vertex to a degree 4 vertex. If for each degree 3 vertex we represent its neighborhood as a triple, then the existence of G implies the existence of a design of 8 triples on 6 elements (the six vertices of degree 4), where each element occurs in 4 triples and each distinct pair of elements occurs in at most 1 triple. But since 6 elements constitute 15 distinct pairs, and each triple specifies 3 pairs, such 8 triples will specify $8 \times 3 = 24$ pairs, and therefore cannot exist. Thus $f(14) < 24$.

$\boldsymbol{f(16) = 28}$. G_{16} is displayed in Figure 2. The upper bound $f(16) \leq 29$ follows from Proposition 2.8. Suppose there exists $G \in \mathcal{F}^*_{16}$ with 29 edges, then $(\delta, \Delta) = (3, 4), (3, 5)$. Note that any two vertices x and y of degree 3 cannot be adjacent, otherwise $G - \{x, y\}$ would contain 24 edges, which contradicts $f(14) = 23$.

If $\Delta = 5$, then G contains $S_{5,2}$ and $d(r_i) = 3$ for each $1 \leq i \leq 5$. Then $d(x) \geq 4$ for all $x \in R$, so that G has a minimum degree sum of 60, which contradicts $|E| = 29$. Therefore $\Delta = 4$ and G has 6 vertices of degree 3 and 10 vertices of degree 4. Consider any (4,2)-star S, at least one of the r_i's has degree 3. Suppose $N(r)$ contains 3 vertices of degree 3, then the removal of these vertices and r yields a graph with 19 edges, which contradicts $f(12) = 18$. Thus we may assume $d(r_1) = 3$ and $d(r_3) = d(r_4) = 4$; then $d(r_{1,1}) = d(r_{1,2}) = 4$.

Suppose $d(r_2) = 4$. For each $2 \leq i \leq 4$, since r_i can be regarded as the root of a (4,2)-star, it follows that there is at least one degree 4 vertex in R_i. We may assume $d(r_{i,3}) = d(r_{4,2}) = 4$ for $2 \leq i \leq 4$, and $d(r_{2,j}) = d(r_{3,j}) = d(r_{4,1}) = 3$ for $j = 1, 2$. There exists a matching between R_4 and each of R_i, $1 \leq i \leq 3$. So we may assume these edges are $r_{4,1}r_{2,3}$, $r_{4,1}r_{3,3}$, $r_{4,2}r_{i,2}$, $r_{4,3}r_{i,1}$ for $1 \leq i \leq 3$. The vertices $r_{2,1}$ and $r_{2,2}$ can only connect to $R_1 \cup \{r_{3,3}\}$. Suppose $r_{2,1}r_{1,2}, r_{2,2}r_{1,1} \in E$. One of $r_{3,1}$ and $r_{3,2}$ must also be incident to R_1. To avoid a triangle, $r_{3,1}r_{1,1}$ and $r_{3,2}r_{1,2}$ cannot be in E; thus $r_{3,1}r_{1,2} \in E$ or $r_{3,2}r_{1,1} \in E$, which will complete the 4-cycle $r_{3,1}r_{1,2}r_{2,1}r_{4,3}r_{3,1}$ or $r_{3,2}r_{1,1}r_{2,2}r_{4,2}r_{3,2}$, respectively. Therefore we may assume $r_{2,1}r_{3,3}, r_{2,2}r_{1,1} \in E$. Similarly, one of $r_{3,1}$ and $r_{3,2}$ must be adjacent to $r_{2,3}$, and the other adjacent to $r_{1,1}$ or $r_{1,2}$. Since $T = \{r_{2,3}, r_{3,3}\} \subset N(r_{4,1})$, the remaining two edges incident to T form a matching between T and R_1. Thus $r_{3,2}r_{1,1} \notin E$. But $r_{3,2}$ cannot be adjacent to $r_{1,2}$ either, because such an edge will complete a triangle. So we must have $r_{3,2}r_{2,3}, r_{3,1}r_{1,2} \in E$. Finally, $r_{3,3}$ is adjacent to $r_{1,1}$ or $r_{1,2}$, which will create the 4-cycle $r_{3,3}r_{1,1}r_{4,3}r_{2,1}r_{3,3}$ or $r_{3,3}r_{1,2}r_{3,1}r_3r_{2,3}$, respectively.

We have succeeded in showing that $d(r_2) = 3$. In other words, every degree 4 vertex must be adjacent to two degree 3 vertices and two degree 4 vertices. It follows that the ten degree 4 vertices must have 20 edges to degree 3 vertices, but these latter vertices have only 18 edges to them; thus, by contradiction, $f(16) \neq 29$.

$\boldsymbol{f(21) = 44}$. See G_{21} in Figure 2 for $f(21) \geq 44$. Assume there exists $G \in \mathcal{F}_{21}$ with 45 edges. Propositions 2.6 and 2.7 imply that $\delta = 4$ and $\Delta = 5$. Thus there are 6 vertices of degree 5 and 15 vertices of degree 4 in G. The (5,3)-star S it contains will have 5 leaves of degree 5 and 10 leaves of degree 4; therefore, there must be at least 3 branches in S with at least 2 leaves of degree 4. On any such branch, the two degree 4 leaves, together with their parent, form a P_3 of degree 4 vertices. The existence of such a P_3 of degree 4 vertices implies that the subgraph induced by the remaining 18 vertices will have 35 edges, contradicting $f(18) = 34$. Therefore, $f(21) < 45$. The proof for $f(22)$ is similar.

See [11] for a proof of $f(23) = 50$ and the derivation of $F(v)$ for $1 \leq v \leq 10$. Computation of $F(v)$ is more involved when $v > 10$. Values of $F(v)$ for $11 \leq v \leq 21$ can be found in [13]. However, it is easy to show that $F(19) = F(20) = 1$ based on the order of the $(4, 5)$-cage. An (r, g)-*cage* is defined as a graph of the smallest order that is r-regular and has girth g. See [22] for more about cages. The $(4, 5)$-cage was discovered by Robertson [17]; it has 19 vertices and 38 edges and is known to be unique. By Propositions 2.6 and 2.7, all elements of $\mathcal{F}^*_{19}$ are 4-regular. Thus, the Robertson graph (G_{19} in Figure 2) is the unique element of $\mathcal{F}^*_{19}$.

G_{19} has a unique set of three vertices that are mutually distance 3 apart (the shaded vertices outside the dodecagon in G_{19} in Figure 2). By Propositions 2.6 and 2.7, $\delta = 3$ for any extremal graph G of order 20. So G has a degree 3 vertex x such that $G - x$ is isomorphic to G_{19}. In addition, $N(x)$ consists of the three vertices in G_{19} that are pairwise distance 3 apart. Hence $F(20) = 1$.

$\boldsymbol{f(50) = 175}$ and $\boldsymbol{F(50) = 1}$. The Hoffman-Singleton graph is $\{C_3, C_4\}$-free and attains the upper bound on $f(50)$ from Theorem 2.2. The uniqueness of the graph was shown in [14], [15], and by Cong and Schwenk in [18]. ■

We note that some recent results of Professor McKay [16], who used an exhaustive computer search in several related problems, confirm our values of $f(v)$.

4 Constructive lower bounds on $f(v)$ for $v \leq 200$

We have developed and implemented algorithms, combining hill-climbing [21] and backtracking techniques, that attempt to find maximal graphs without triangles or quadrilaterals; the algorithms are described in [11]. We have succeeded in generating graphs, with sizes greater than the lower bound presented in Section 2, for all orders from 25 to 200. We present these results in Table 1. Adjacency lists for the graphs appear

$f(v)$	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**5**	**6**	**8**	**10**	**12**
10	**15**	**16**	**18**	**21**	**23**	**26**	**28**	**31**	**34**	**38**
20	**41**	**44**	**47**	**50**	**54**	**57**	**61**	**65**	**68**	**72**
30	**76**	80	85	87	90	94	99	104	109	114
40	120	124	129	134	139	144	150	156	162	168
50	**175**	176	178	181	185	188	192	195	199	203
60	207	212	216	221	226	231	235	240	245	250
70	255	260	265	270	275	280	285	291	296	301
80	306	311	317	323	329	334	340	346	352	357
90	363	368	374	379	385	391	398	404	410	416
100	422	428	434	440	446	452	458	464	470	476
110	483	489	495	501	508	514	520	526	532	538
120	544	551	558	565	571	578	584	590	596	603
130	610	617	623	630	637	644	651	658	665	672
140	679	686	693	700	707	714	721	728	735	742
150	749	756	763	770	777	784	791	798	805	812
160	819	826	834	841	849	856	863	871	878	886
170	893	901	909	917	925	933	941	948	956	963
180	971	979	986	994	1001	1009	1017	1025	1033	1041
190	1049	1057	1065	1073	1081	1089	1097	1105	1113	1121
200	1129									

Table 1: Constructive lower bounds on $f(v)$

in [10]. We include in Table 1 the exact values of $f(v)$ where they are known; those values are in bold face. The values of $f(v)$ for $25 \leq v \leq 30$ are determined in [13]. The fact that our algorithm repeatedly finds the unique extremal graph of order 50 in several seconds on a DECstation 5000 rated at 24 mips leads us to believe that these computed lower bounds are good.

Figure 3 compares the computational and theoretical bounds. The upper bound is attained for $v = 50$, but after that point our computational results fall away. It is not clear whether the upper or lower bound seems to be closer to the asymptotic value of $f(v)$; the upper bound might be attained on 3250 vertices.

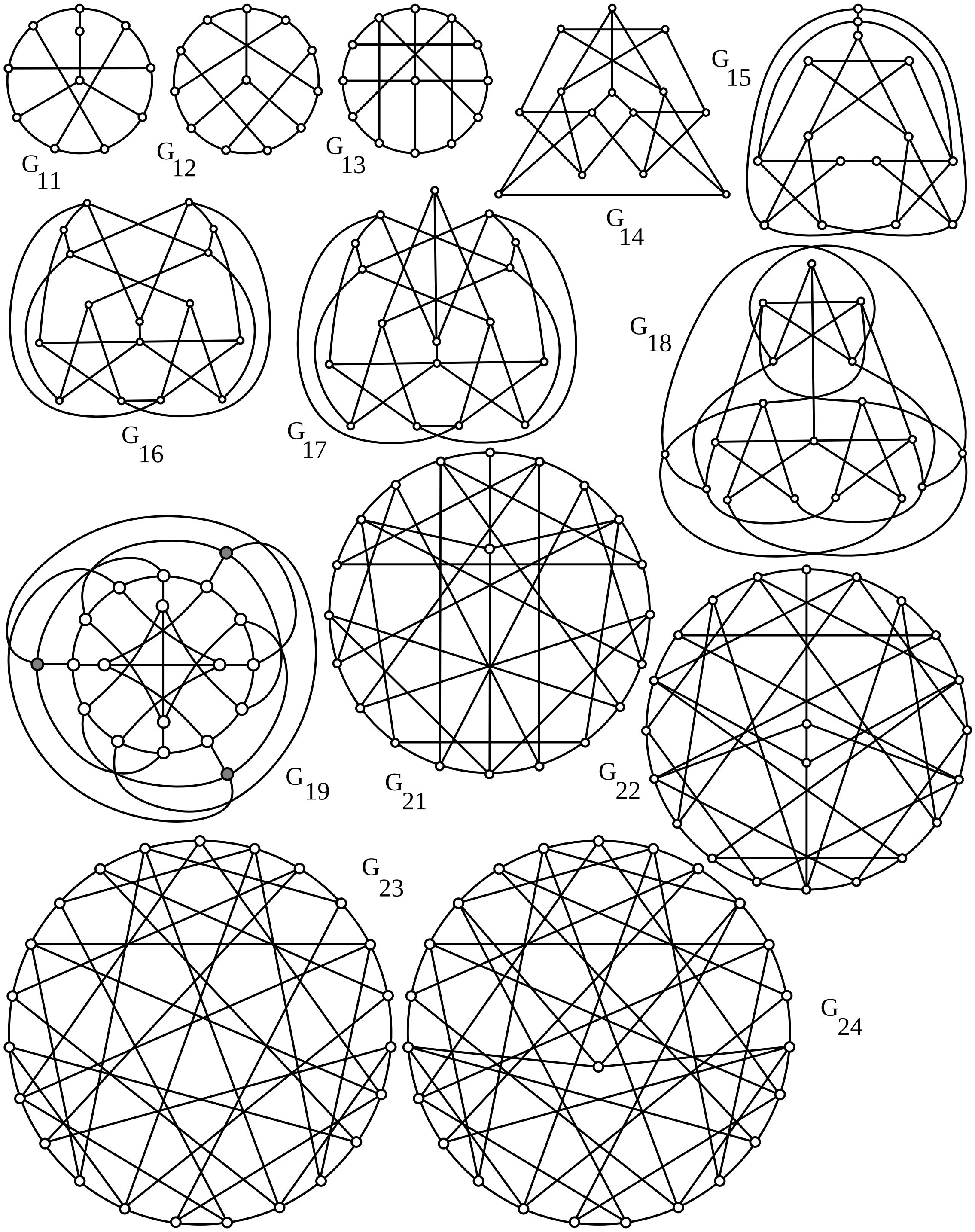

Figure 2: Extremal graphs of orders 11 to 24

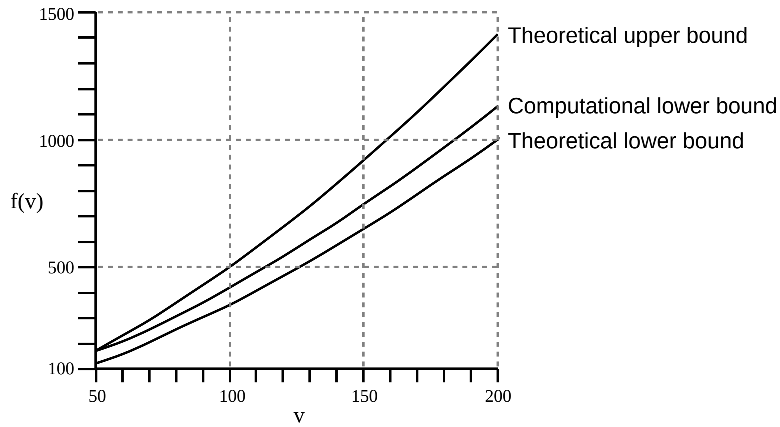

Figure 3: Computational and theoretical bounds on $f(v)$

Erdős' conjecture that $f(v) = (1/2 + o(1))^{3/2} v^{3/2}$ remains unsolved. We hope this paper sheds some light on the problem.

To improve the theoretical bounds of $f(v)$, more properties of the extremal graphs need to be recognized. The extremal graphs seem to have a highly symmetric structure; it may be worthwhile to study the groups associated with them. Another interesting topic concerns the values of $F(v)$. In particular, what are the necessary and/or sufficient conditions for $F(v) = 1$?

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