# On the Structure of Extremal Graphs of High Girth*

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**Abstract**

Let $n \geq 3$ be a positive integer, and let $G$ be a simple graph of order $v$ containing no cycles of length smaller than $n+1$ and having the greatest possible number of edges (an extremal graph). Does $G$ contain an $n+1$–cycle? In this paper we establish some properties of extremal graphs and present several results where this question is answered affirmatively. For example, this is always the case for (i) $v \geq 8$ and $n = 5$, or (ii) when $v$ is large compared to $n$: $v \geq 2^{a^2+a+1}n^a$, where $a = n - 3 - \lfloor \frac{n-2}{4} \rfloor$, $n \geq 12$. On the other hand we prove that the answer to the question is negative for $v = 2n + 2 \geq 26$.

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# 1 Introduction

All graphs in this paper are assumed to be simple (undirected, no loops, no multiple edges) and finite. In our notations we follow Bollobás [1], and all missing definitions can be found there. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. $|V(G)| = v(G) = v$ is called the *order* of $G$, and $|E(G)| = e(G) = e$ is called the *size* of $G$. If $G$ contains a cycle, then the *girth* of $G$, denoted by $g = g(G)$, is the length of a shortest cycle in $G$. Any cycle of $G$ whose length is $g(G)$ is called a *girth cycle* of $G$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of $G$, respectively. By $C_n$ we will denote the cycle of length $n$, $n \geq 3$. Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called *$\mathcal{F}$–free* if it contains no subgraph isomorphic to a graph from $\mathcal{F}$. By $ex(v, \mathcal{F})$ we denote the the greatest number of edges in an $\mathcal{F}$–free graph of order $v$, and by $EX(v, \mathcal{F})$ we denote the set of all $\mathcal{F}$–free graphs of order $v$ with $ex(v, \mathcal{F})$ edges. We refer to graphs from $EX(v, \mathcal{F})$ as *extremal $\mathcal{F}$–free* graphs of order $v$, or just *extremal*.

For a given family $\mathcal{F}$, the question of the structure of extremal $\mathcal{F}$–free graphs is natural and has been addressed many times. It is a very hard question in general, but some results were obtained for particular families $\mathcal{F}$ and particular values of $v$ (see, e.g., Bollobás [1], Simonovits [ 11, 12] for many of such results.).

In [2] Erdős and Sachs showed that an $r$-regular graph of girth at least $n + 1$ of the smallest order must have girth equal to $n + 1$. (A proof of this result can be found in Lovász [10, pp. 66, 384, 385], see also the references therein.)
In this paper we consider a similar problem:

*What is the girth of an extremal $\{C_3, C_4, \cdots, C_n\}$–free graph of order $v$?*
*Is it always $n + 1$ or it can be greater?*

In [3] and [4] several properties of extremal $\{C_3, C_4\}$–free graphs were presented. In [4], Garnick and Nieuwejaar proved that for $v \geq 7$, the girth of an extremal $\{C_3, C_4\}$–free graph is 5. In the same paper they asked the following question:

*Is there a constant $c$ such that for all $n \geq 5$ and all $v \geq cn$, the girth* *of any extremal $\{C_3, C_4, \cdots, C_n\}$–free graphs is $n+1$?*

Our investigation was motivated by this problem. In section 2 we present our results on the structure of extremal $\{C_3, C_4, \cdots, C_n\}$–free graphs and prove them in section 3.

## 2 Results

Our first theorem concerns the structure of some extremal $\{C_3, C_4, \cdots, C_n\}$ – free graphs.

Theorem 1 *Let $\mathcal{F} = \{C_3, C_4, \cdots, C_n\}$ , $n \geq 3$ and $v \geq n+1$. Then*

1. *there exists an extremal graph with minimum degree $\delta \geq 2$;*
2. *there exists an extremal graph of girth $n+1$;*
3. *if $v \neq n+2$, there exists an extremal graph with minimum degree $\delta \geq 2$ and girth $n+1$.*

Our second theorem states that an extremal graph with maximum degree $\Delta \geq n$ is necessarily of girth $n+1$.

Theorem 2 *Let $n \geq 3$, $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$, and $\Delta(G) \geq n$. Then $g(G) = n+1$.*

The following theorem is analogous to the result of Garnick and Nieuwejaar [4] mentioned above, and, in a certain sense, "extends" it. The idea of our proof differs from the one in [4].

Theorem 3 *For $v \geq 8$, the girth of an extremal $\{C_3, C_4, C_5\}$–free graph is 6.*

Our next theorem states that the girth of an extremal $\{C_3, C_4, \cdots, C_n\}$–free graph is necessarily $n+1$ provided $v$ is large in comparison with $n$.

Theorem 4 *Let $n \geq 12$, $a = n - 3 - \lfloor \frac{n-2}{4} \rfloor$, $v \geq 2^{a^2+a+1} n^a$, and $G \in EX(v, \{C_3, C_4, \ldots, C_n\})$. Then $g(G) = n+1$.*

Theorem 5 below implies that for a positive answer to the question of Garnick and Nieuwejaar, the constant $c$ has to be greater than 2.

Theorem 5 *For $n \geq 12$, $ex(2n+2, \{C_3, C_4, \ldots, C_n\}) = 2n+4$, and there exist $G \in EX(2n+2, \{C_3, C_4, \ldots, C_n\})$ with $g(G) = n+2$.*

## 3 Proofs

**Proof of Theorem 1.**

**i.** Let $G$ be an extremal graph with the smallest number of vertices of degree 1. Every extremal graph must be connected, hence $\delta(G) \geq 1$. If $\delta(G) \geq 2$, the statement is proven. Let $\delta(G) = 1$ and $x \in V(G)$ be a vertex of degree 1. Since $G$ is extremal and $v \geq n+1$, then $e(G) \geq e(C_v) = v$, and $G$ contains a cycle. Since $x$ is not a vertex of the cycle, then by deleting $x$ from $G$ and subdividing any edge of the cycle, we obtain a graph $G'$ of the same order and size as $G$, with the girth at least as large as the girth of $G$. Then $G'$ is an extremal graph having one less vertex of degree 1, a contradiction.

**ii.** Since $G$ is extremal and $v \geq n+1$, then $e(G) \geq e(C_v) = v$, and every extremal graph must contain a cycle. Let $G$ be an extremal graph of the smallest girth. If $g(G) = n+1$, the statement is proven. If $g(G) > n+1$, let $C$ denote a girth cycle and let $uv$ denote an edge of $C$. Since the length of $C$ is at least 5, contracting $G$ by $uv$ leads to a graph $G'$ of girth $g(G) - 1 > n$ which has one less vertex and one less edge than $G$. Introducing a new vertex and joining it to an arbitrary vertex of $G'$, we obtain an extremal graph of girth less than $g(G)$, a contradiction.

**iii.** From part (ii), there is an extremal graph of girth $n+1$. Let $G$ be one of these extremal graphs, with the smallest number of vertices of degree 1, and let $C$ be a girth cycle of $G$. If $v = n+1$, then $G = C = C_{n+1}$ and $\delta(G) = 2$. Let $v \geq n+3$. Since $G$ is connected, $\delta(G) \geq 1$. If $\delta(G) \geq 2$, the statement is proven. If $G$ contains a vertex of degree 1, then delete it and subdivide an edge of $G$ which is not an edge of $C$ (such an edge exists, since $e(G) - e(C) \geq e(C_v) - e(C) = v - (n+1) \geq 2$ ) . We obtain an extremal graph of girth $n+1$ having one less vertex of degree 1, a contradiction. If $v = n+2$, $n \geq 4$, then there are exactly two extremal graphs: one is isomorphic to $C_{n+2}$, and another is isomorphic to $C_{n+1}$ with one of its vertices connected to a vertex of degree 1. Therefore the statement is false for $v = n+2$, which justifies the restriction in the statement of the theorem. ■

**Proof of Theorem 2.** Let $n \geq 3$, $G \in EX(v; \{C_3, C_4, \ldots, C_n\})$, and $\Delta(G) \geq n$. If $g(G) = n+1$, then the theorem is proven. Therefore we assume that $g(G) \geq n+2$. Let $x$ be a vertex of $G$ of maximum degree $\Delta$, and let $x_1, x_2, \ldots, x_\Delta$ be all neighbors of $x$. Let $G'$ be a graph obtained from $G$ by first deleting $\Delta - 2$ edges $xx_3, \ldots, xx_\Delta$, and then adding $\Delta - 1$ new edges $\{x_2x_3, x_3x_4, \ldots, x_{\Delta-1}x_\Delta, x_\Delta x_1\}$. Then $G'$ and $G$ are of the same order $(= v)$ but $G'$ has one more edge. Since $G$ is extremal, $G'$ must contain a cycle of length less than $n+1$. Let $C'$ be a shortest cycle in $G'$. Then its length is at most $n$, and it is not a subgraph of $G$. Therefore at least one of the newly introduced edges of $G'$ must be an edge of $C'$. On the other hand $E(C')$ must contain edges of $G$, since the newly introduced edges of $G'$ do not induce a cycle in $G'$. Hence $C'$ contains a subgraph $P$ which is a path in both $G$ and $G'$ with endpoints $x_i$ and $x_j$, for some $i \neq j$. Clearly the length of $P$ is at most $n-1$. To finish the proof we consider the following two cases.

Case 1: $x \notin V(P)$.

Then joining $x$ to both endpoints of $P$ we obtain a cycle $C$ in $G$ of length at most $n+1$, a contradiction.

Case 2: $x \in V(P)$.

Then $x_1xx_2$ is a sub-path of $P$. If all other edges of $C'$ are new, then it must contain all of them; and hence its length is at least $2 + (\Delta - 1) = \Delta + 1 \geq n+1$, a contradiction. Therefore $E(C')$ contains some new edges and some edges of $G$ other than $x_1x$ and $xx_2$. This implies the existence of a path in $C'$ which is also a path in $G$ of the form $x_1a \ldots x_i$, $a \neq x$, or of the form $x_2b \ldots x_j$, $b \neq x$. Then $C = xx_1a \ldots x_ix$ or $C = xx_2b \ldots x_jx$ is a cycle in G of length at most $n$, a contradiction. ■

**Proof of Theorem 3.** Let $xy \in E(G)$ and let $G/xy$ be the contraction of $G$ by an edge $xy$, i.e., $G/xy$ is obtained from $G$ by (i) deleting $x, y$ and all edges of $G$ incident with $x$ or $y$, (ii) introducing a new vertex $z$ , (iii) joining $z$ to every vertex $w \in V(G) \setminus \{x, y\}$, where $wx \in E(G)$ or $wy \in E(G)$.

Let $G$ be an extremal $\{C_3, C_4, C_5\}$–free graph of order $v \geq 8$. Then $|E(G)| \geq v$ and $G$ contains a cycle. If $g(G) = 6$, then the theorem is proved. Therefore we assume that $g(G) \geq 7$. If $g(G) \geq 8$, then contracting $G$ by two arbitrary edges of a girth cycle we obtain a graph $G'$ containing a cycle of length at least 6. This cycle contains two vertices $x$ and $y$ at distance at least 3 in $G'$ (and, hence, in $G$). Since $G$ contains no triangles, $e(G') = e(G) - 2$.

Let graph $G''$ be obtained from $G'$ by introducing two new vertices, $u$ and $v$, and two new edges $ux$ and $vy$. It is easy to see that $d_{G''}(u, v) \geq 5$. Hence we can add another edge $uv$ to $G''$ without creating a cycle of length less than six. The resulting graph $G''' = G'' + uv$ has the same order as $G$, $g(G''') \geq 6$, and $e(G''') = e(G'') + 1 = e(G') + 3 = (e(G) - 2) + 3 = e(G) + 1$. But this contradicts the extremality of $G$. It follows that $g(G) = 7$.

Let $C = x_1x_2 \cdots x_7x_1$ be a 7–cycle of $G$. Since $v \geq 8$ and $G$ is connected, there exists a vertex, say $u$, joined to a vertex of $C$, say $x_1$. If $ux_1$ and $x_4x_5$ are not edges of a 7–cycle of $G$, then applying the same argument as above (i.e., contracting $G$ by $ux_1$ and $x_4x_5$, etc.) we arrive at a contradiction. If $ux_1$ and $x_4x_5$ are edges of a 7–cycle of $G$, then the distance between them along the cycle is at most 2. But this implies that $G$ contains a cycle of length smaller than 7 ( see Figure 1), a contradiction. Therefore $g(G) = 6$. ■

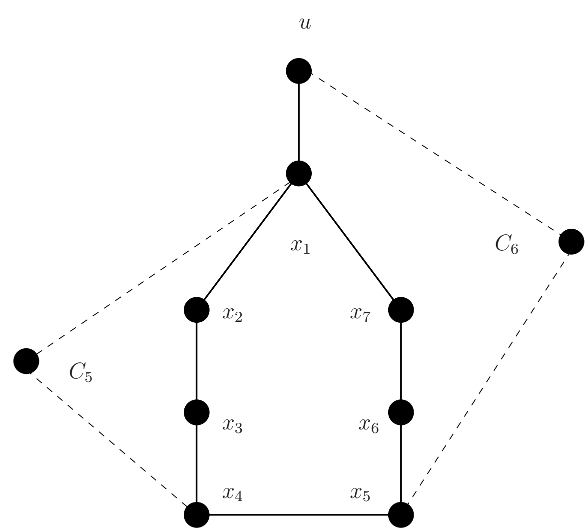


**FIGURE 1. Proof of Theorem 3.**

**Proof of Theorem 4.** The idea of the proof is to show that $\Delta(G) \geq n$, and then use Theorem 2. Since $\Delta(G)$ is greater or equal to the average degree $2e/v$ of $G$, it is sufficient to show that $e = ex(v, \{C_3, C_4, \ldots, C_n\}) \geq nv/2$. To prove this inequality we use graphs $CD(k, q)$ described below. Our choice of these graphs is due to their property that, with few exceptions on the values of $v$ and $n$, they provide the best known asymptotic lower bound on $ex(v, \{C_3, C_4, \cdots, C_n\})$.

Let $q$ be a prime power, and let $P$ and $L$ be two copies of the countably infinite dimensional vector space $V$ over $GF(q)$. By $D(q)$ we denote a bipartite graph with the bi-partition $(P, L)$ and edges defined as follows. We say that vertices $(p) = (p_1, p_2, p_3, \ldots) \in P$ and $(l) = (l_1, l_2, l_3, \ldots) \in L$ are adjacent, if and only if the following relations on their coordinates hold:

$$
\begin{aligned}
l_2 - p_2 &= l_1 p_1 && (3.1)\\
l_3 - p_3 &= l_2 p_1 && (3.2)\\
l_{4i} - p_{4i} &= l_1 p_{4i-2} && (3.3)\\
l_{4i+1} - p_{4i+1} &= l_1 p_{4i-1} && (3.4)\\
l_{4i+2} - p_{4i+2} &= l_{4i} p_1 && (3.5)\\
l_{4i+3} - p_{4i+3} &= l_{4i+1} p_1 && (3.6)\\
\text{for} \quad \text{all} \quad i &= 1, 2, \ldots && (3.7)
\end{aligned}
$$

For each positive integer $k \geq 2$, let $P_k$ and $L_k$ be canonical projections of $P$ and $L$ onto their $k$ initial coordinates. Imposing the first $k-1$ adjacency relations on vectors from $P_k$ and $L_k$, we obtain a bipartite graph of order $2q^k$ with bi-partition $(P_k, L_k)$ which is an induced subgraph of graph $D(q)$. We denote this graph by $D(k, q)$.

Graphs $D(q)$ and $D(k, q)$ where introduced by Lazebnik and Ustimenko in [7], where it was shown that they are $q$–regular; that their automorphism groups are transitive on each bi-partition and on edges; and, the most important, that for odd $k$, the girth $g(D(k, q)) \geq k + 5$.

In [8], Lazebnik, Ustimenko and Woldar proved that for $k \geq 6$, graphs $D(k, q)$ are disconnected. Since all connected components of $D(k, q)$ are isomorphic, we denote any one of them by $CD(k, q)$. Thus for each odd integer $k \geq 1$ and any prime power $q$, the graph $CD(k, q)$ is bipartite, $q$–regular, edge–transitive and its girth is at least $k+5$. In [9], the same authors proved that for all $k \geq 6$ and $q$ odd, the order of $CD(k, q)$ is $2q^{k-\lfloor \frac{k+2}{4} \rfloor + 1}$.

To continue our proof of Theorem 4, let $n \geq 12$ be given, and let $a = n - 3 - \lfloor \frac{n-2}{4} \rfloor$. By Chebyshev's Theorem, (see, e.g., [5, p. 343]), there exists an odd prime $q$ such that

$$\frac{1}{2}(v/2)^{1/a} \leq q \leq (v/2)^{1/a}.$$

Consider a graph $H = CD(n-4, q) + (v - 2q^a)K_1$, i.e., a graph obtained from $CD(k, q)$ by adding $v - 2q^a$ isolated vertices. Then $v(H) = 2q^a + (v - 2q^a) = v$,

$e(H) = e(CD(k,q)) = q^{a+1}$, and $g(H) = g(CD(n-4,q)) \geq (n-4)+5 = n+1$. Thus $H$ is a graph of order $v$ and the girth of $H$ is at least $n+1$. Since $G$ is extremal, $e(G) \geq e(H)$ and

$$\Delta(G) \geq 2e(G)/v \geq 2e(H)/v = 2q^{a+1}/v \geq q/2^a \geq \frac{1}{2}(v/2)^{1/a}/2^a \geq n.$$

Thus $\Delta(G) \geq n$, and, by Theorem 2, $g(G) = n+1$.

**Proof of Theorem 5.** First we show that an extremal graph $G$ must be planar. If $G$ is not planar, then by Kuratowski's Theorem [6], $G$ must contain a subgraph $H$ which is a subdivision of the complete bipartite graph $K_{3,3}$ or a subdivision of the complete graph $K_5$.

Case 1: $H$ is a subdivision of $K_{3,3}$.

The number of 4-cycles in $K_{3,3}$ is 9 and the number of 4-cycles passing through every edge of $K_{3,3}$ is 4. Let us refer to cycles of $H$ which are subdivisions of 4-cycles of $K_{3,3}$ as special cycles of $H$. Then $H$ contains 9 special cycles and the number of special cycles passing through every edge of $H$ is 4. For an arbitrary ordering of the 9 edges of $K_{3,3}$, let $l_i$, $i = 1, \ldots, 9$, denote the length of the path in $H$ obtained by the subdivision of the $i$-th edge of $K_{3,3}$. Then by counting in two different ways the cardinality of $\{(uv, C) : uv \in E(C), C \text{ is a special cycle in } H\}$, we obtain

$$4e(H) = 4\sum_{i=1}^{9} l_i = \sum_{C}(\text{length of } C) \geq \sum_{C}(n+1) = 9(n+1).$$

Subdividing an edge in a graph by one vertex we increase both the order and the size of the graph by one. Therefore $e(H) - v(H) = e(K_{3,3}) - v(K_{3,3}) = 9 - 6 = 3$, and $e(G) \geq e(H) = \sum_{i=1}^{9} l_i = v(H) + 3 \geq \frac{9}{4}(n+1)$. Therefore $2n+2 = v(G) \geq v(H) \geq \frac{9}{4}(n+1) - 3$. But this is impossible for $n \geq 12$. Hence $G$ contains no subgraph which is a subdivision of $K_{3,3}$.

Case 2: $H$ is a subdivision of $K_5$.

Repeating the argument of Case 1 for 10 triangles of $K_5$, we obtain $2n + 2 = v(G) \geq v(H) \geq \frac{10}{3}(n+1) - 5$, which again is impossible for $n \geq 12$. Hence $G$ contains no subgraph which is a subdivision of $K_{3,3}$.

Therefore $G$ is planar. It is well known (and follows immediately from Euler's formula) that for a planar graph of order $v$, size $e$ and girth $g$, $e \leq$

$(v-2)g/(g-2)$. Therefore

$$e(G) \leq 2ng/(g-2) = 2n(1+2/(g-2)) \leq 2n(1+2/(n-1)) = 2n+4+4/(n-1).$$

Since $n \geq 12$, $e(G) = ex(2n+2, \{C_3, C_4, \ldots, C_n\}) \leq 2n+4$. On the other hand it is easy to construct several non-isomorphic $\{C_3, C_4, \ldots, C_n\}$–free graphs of order $2n+2$ and size $2n+4$. See Figure 2a, 2b, 2c. The graph depicted on Figure 2c has girth $n+2$, and thereby the proof is completed. ∎

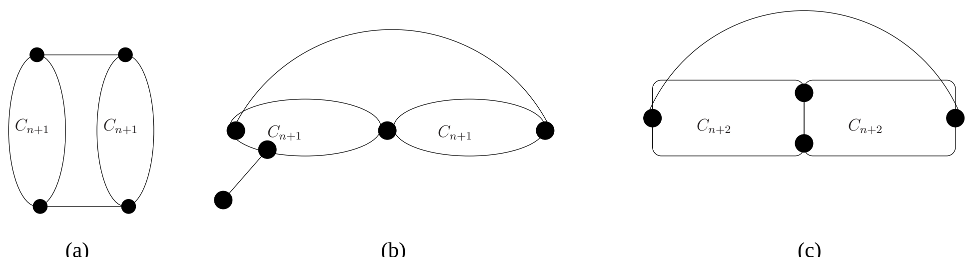


**FIGURE 2. Extremal graphs of order $2n+2$.**

# 4 Conclusion

We believe that we have a proof of a theorem similar to our Theorem 5, but for $v = 3n+3$. At the moment the proof is long and much harder than the one of Theorem 5. The result implies that for a positive answer to the question of Garnick and Nieuwejaar, the constant $c$ (if it exists) has to be greater than 3. Does such a constant exist? We do not have sufficient evidence to conjecture an answer.

Finally we would like to suggest the following open problems.

**Problem 1.** Is it possible to decrease the exponential (in $n$) lower bound on $v$ in Theorem 4 to a polynomial one?

**Problem 2.** "Extend" Theorem 3 to $\{C_3, C_4, C_5, C_6\}$-free graphs.

**Problem 3.** Let $g_{\max} = g_{\max}(n)$ be the largest girth for a graph from $EX(v, \{C_3, C_4, \ldots, C_n\})$. What is the largest value (over all $v$) of the difference $g_{\max} - (n+1)$?

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