Combinatorics and Graph Theory I (Math 688). Problems and Solutions.

May 17, 2006

PREFACE

Most of the problems in this document are the problems suggested as homework in a graduate course Combinatorics and Graph Theory I (Math 688) taught by me at the University of Delaware in Fall, 2000. Later I added several more problems and solutions. Most of the solutions were prepared by me, but some are based on the ones given by students from the class, and from subsequent classes. I tried to acknowledge all those, and I apologize in advance if I missed someone.

The course focused on Enumeration and Matching Theory.

Many problems related to enumeration are taken from a 1984-85 course given by Herbert S. Wilf at the University of Pennsylvania.

I would like to thank all students who took the course. Their friendly criticism and questions motivated some of the problems.

I am especially thankful to Frank Fiedler and David Kravitz. To Frank – for sharing with me LaTeX files of his solutions and allowing me to use them. I did it several times. To David – for the technical help with the preparation of this version.
Hopefully all solutions included in this document are correct, but the whole set is by no means polished. I will appreciate all comments. Please send them to `lazebnik@math.udel.edu`.

– Felix Lazebnik

Problem 1. *In how many 4–digit numbers abcd (a, b, c, d are the digits, $a \neq 0$)*

(i) $a < b < c < d$*?*

(ii) $a > b > c > d$*?*

Solution. (i) Notice that there exists a bijection between the set of our numbers and the set of all 4–subsets of the set $\{1, 2, \ldots, 9\}$. If $\overline{abcd}$ denotes a 4-digit number, a, b, c, d being the digits, then the bijection can be given by the map $f : \overline{abcd} \mapsto \{a, b, c, d\}$. f is obviously a function, $1-1$, and it is onto, since any four element subset of the set $\{1, 2, \ldots, 9\}$ can be uniquely ordered in increasing order. Therefore there are $\binom{9}{4}$ numbers with the property.

(ii) Similarly to (i), but consider all 4–subsets of the set $\{0, 1, \ldots, 9\} = 126$. There are $\binom{10}{4} = 210$ such numbers.

Problem 2. *How many positive factors does the number* $N = 2^3 5^4 7^3 11^5$ *have ?*

Solution. From the Unique Factorization Theorem for integers, A divides $N = p_1^{k_1} \ldots p_s^{k_s}$ (p_i's are distinct primes) iff $A = p_1^{l_1} \ldots p_s^{l_s}$, with $0 \leq l_i \leq k_i$, $i = 1, \ldots, s$. By the product rule, there are $(k_1 + 1)(k_2 + 1) \ldots (k_s + 1)$ choices for A, since l_i's can be chosen independently in $k_i + 1$ ways each. In our case, the number of positive factors is $4 \cdot 5 \cdot 4 \cdot 6 = 480$.

Problem 3. *(i) What is the greatest number of parts (bounded or unbounded) in which n lines can divide a plane?*

(ii) What is the greatest number of parts (bounded or unbounded) in which n planes can divide the space?

(iii) What is the greatest number of parts (bounded or unbounded) in which n hyperplanes can divide $\mathbf{R}^n$*?*

Solution. (i) Let a_n be the greatest number of regions in which $n \geq 0$ lines can divide a plane. Then $a_0 = 1$ and $a_1 = 2$. Suppose $n \geq 1$ lines are given. Let us call the regions they divide the plane the "new regions". Fix one of these lines, say line l. See Fig. 1. The remaining $n - 1$ lines divide the plane in at most a_{n-1} regions. Refer to them as "old regions". Each of these remaining $n - 1$ lines intersects l in at most one point, so all of them intersect l at no more than at $n - 1$ points. These points of intersections divide l in at most n pieces, each piece being an interval, a ray or a line (in case of zero pieces). Each piece lies inside of some "old region", and it divides it into two "new regions". Hence each such piece increases the total number of regions by one. Therefore

$$a_n \le a_{n-1} + n, \; n \ge 1, \; a_0 = 1.$$

Solving this recurrence we get:

$$a_n \le a_{n-1} + n \le (a_{n-1} + (n-1)) + n \le \ldots \le a_0 + \sum_{i=1}^{n} i = 1 + \frac{n(n+1)}{2}.$$

It is clear that he greatest value of a_n is $\boxed{1 + n(n+1)/2}$ and it attained if and only if all inequalities become equalities, i.e., the lines are in general position. The latter means that every two lines intersect at a point, and no three lines are concurrent. The existence of such n lines for every n can be easily proven by induction on n or by an explicit construction. $\square$

(ii) Let us change the notation used in (i) from a_n to $a_2(n)$. Let $a_3(n)$ be the greatest number of regions in which $n \ge 0$ planes divide the space. Repeat the argument given in (i). The only difference is that the pieces now are regions of a fixed plane. This gives $a_3(n) \le a_3(n-1) + a_2(n-1)$. We have shown that $a_2(n) = 1 + \frac{n(n+1)}{2} = 1 + \binom{n+1}{2}$. To get a closed formula for $a_3(n)$, solve the recurrence and use either the facts $\sum\limits_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$, $\sum\limits_{i=1}^{k} i = \frac{k(k+1)}{2}$, or the fact $\sum\limits_{i=2}^{k} \binom{i}{2} = \binom{k+1}{3}$. To see the last formula, rewrite $\binom{2}{2}$ as $\binom{3}{3}$. Then the sum will "collapse". The maximum is achieved if and only if the n planes are in general position, i.e., every three of them intersect at a point and no four are concurrent. The existence of such n planes for every n can be easily proven by induction on n or by an explicit construction. A closed formula is:

$$\boxed{a_n = 1 + n + \binom{n+1}{3}, \; n \ge 0}$$

(iii) An (affine) hyperplane in $\mathbb{R}^m$ is an (affine)subspace of dimension $m-1$.

Intersection of two distinct hyperplanes is a subspace of dimension $m-2$ (this follows from the general fact: for any two subspaces U and W of a vector space V, dim $(U+W)+$ dim $(U \cap W) =$ dim $U+$ dim W. Therefore, if $a_m(n)$ denotes the greatest number of regions in which n hyperplanes in

$\mathbb{R}^m$ divide $\mathbb{R}^m$, then, similarly to the cases $m = 2, 3$, we can show (but will not go in the details here) that

$$a_m(n) \leq a_m(n-1) + a_{m-1}(n-1),\ n \geq 1, m \geq 2, a_m(0) = 1.$$

This implies that

$$\boxed{a_m(n) = 1 + n + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{m}}$$

Notice, that for $m = 2, 3$, we could write the answers as $1 + n + \binom{n}{2}$ and $1+n+\binom{n}{2}+\binom{n}{3}$, respectively. The upper bound is achieved if the hyperplanes are in general position, i.e., every m of them intersect at a point and no $m+1$ of them are concurrent. The existence of such n hyperplanes for every n can be proven by induction on n or by an explicit construction.
The problem and the answer also make sense in case when $m = 1$. The hyperplanes, in this case are points. □

Problem 4. *Let $n \geq 1$ circles be drawn on the plane. What is the greatest number of parts (bounded or unbounded) can they divide the plane?*

Hint: if a_n is the number, then it is easy to show (similarly to 3(i))) that

$$a_n \leq a_{n-1} + 2(n-1)\ , n \geq 2,\ \ a_1 = 2.$$

This leads to $\boxed{a_n = n^2 - n + 2\ ,\ \ n \geq 1.}$

<u>Remark.</u> Using argument similar to the one in Problem 3(ii), one can show that in $\mathbb{R}^3$, n spheres can divide the space in at most $\boxed{n(n^2 - 3n + 8)/3}$ parts. Again, the existence of such n circles or spheres for every n can be easily proven by induction on n or by an explicit construction. It seems that the generalization for n spheres in $\mathbb{R}^m$ is similar to the one in Problem 3(iii). Can we get a nice expression in this case?

<u>Remark.</u> In both Problem 3 and Problem 4, one can try to answer a more interesting question: for a given n, what values the total number of parts can take? It is clear that some values between the smallest and the greatest ones will be missing.

Problem 5. *Let $n \geq 1$ points be taken on the circumference of a circle. Through any two of them a chord is drawn. No three chords intersect at one point inside the circle.*

(i) How many points of intersections of these chords are inside the circle?

(ii) Into how many parts do these chords divide the circle?

Solution. (i) Solution #1. Out of 6 lines determined by 4 points on the circle, exactly 2 always intersect inside the circle. Since no 3 chords pass through the same point, there are as many points of intersection of chords inside the circle as the number of 4–subsets of the set of our n points. Thus the answer is $\binom{n}{4}$. □

Solution #2. Let a_n denote the number. The study of the difference table for $a_n,\ n = 3, 4, 5, 6, 7$, suggests that a_n is a polynomial of n of degree 4. Write it with unknown coefficients and determine them by solving the system of 5 linear equations with 5 unknowns. Then prove the formula by induction (which is still not too easy). □

Solution #3. Let a_n denote the number. Prove that

$$a_{n+1} = a_n + [(n-2) + 2(n-3) + 3(n-5) + \ldots + (n-2)\cdot 1] =$$

$$a_n + \sum_{k=2}^{n-1}(k-1)(n-k), n \geq 4,\ a_4 = 1.$$

This comes from the fact that the chord $A_{n+1}A_k$ intersects exactly $(k-1)(n-k)$ chords coming from the set $\{A_1, \ldots, A_{k-1}\}$ to $\{A_{k+1}, \ldots, A_n\}$, and doesn't intersect any other chords determined by points $A_1, \ldots A_n$. The summation can be worked this way:

$$\sum_{k=2}^{n-1}(k-1)(n-k) = n\sum_{k=2}^{n-1}(k-1) - \sum_{k=2}^{n-1}k^2 + \sum_{k=2}^{n-1}k =$$

$$n\frac{(n-2)(n-1)}{2} - \left[\frac{n(n+1)(2n+1)}{6} - n^2 - 1^2\right] + \left[\frac{(n-1)n}{2} - 1\right] =$$

$$\frac{n(n-1)(n-2)}{6} = \binom{n}{3}.$$

So $a_{n+1} = a_n + \binom{n}{3}$, $a_4 = 1, n \geq 4$. From here it is easy to get a closed formula for a_n: work it "backwards" and use formula for $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$, $\sum_{i=1}^{n} i^3$ or the fact $\sum_{m=3}^{n}\binom{m}{3} = \binom{n+1}{4}$, (the trick is $\binom{3}{3} = \binom{4}{4}$). The answer is: $\boxed{\binom{n}{4}.}$ $, n \geq 0.$

(ii) Solution #1. How does the number of regions change if we remove a chord which intersects with other chords at k points (inside the circle)? Removing the chord we remove $k+1$ segment, and each segment divided the region it lied in into two. So the total number of regions decreases by $k+1$. Thus, removing one chord with k inner points on it results in the decrease of the number of regions by $k+1$. So, removing all chords we decrease the total number of regions by [(the total number of points of intersection of chords inside the circle) + (the number of chords)] $\underset{\text{(part (i))}}{=}$ $\binom{n}{4}+\binom{n}{2}$. After removing all chords we are left with 1 part. So the total number of parts all chords divide the circle is $\boxed{1+\binom{n}{2}+\binom{n}{4}}$, $n \geq 0$. □

Solution #2. (by G. Fiorini and O. Byer.) Use Euler formula for planar graphs: $v+f=e+2$, where

$$\begin{aligned} v &= \# \text{ of vertices} \\ f &= \text{the unknown } \# \text{ of faces (including the outer face)} \\ e &= \# \text{ of edges} \end{aligned}$$

We have: $v=n+\binom{n}{4}$, $f=\#$ of regions, e is determined from $2e=$ sum of degrees of all vertices $=(n+1)n+4\cdot\binom{n}{4}$. Solving $v+f=e+2$ for f, and subtracting one for the outer region, we get $\boxed{1+\binom{n}{2}+\binom{n}{4}}$, $n \geq 0$. □

Problem 6. *How many monic square-free polynomials of degree $n \geq 1$ over the finite field $GF(q)$ are there? (A polynomial is called square-free if it is not divisible by a square of a non-constant polynomial).*

Solution. Let a_n be the number of all monic square-free polynomials of degree $n \geq 1$ over $GF(q)$. Obviously, $a_1=q$. It will be convenient to set $a_0=1$. Every monic polynomial $f \in GF(q)[x]$ can be written uniquely in the form $f=g^2h$, where h is monic and square-free (if f is square-free, then $g=1$ and $h=f$). The uniqueness of such a representation follows immediately from the uniqueness of the representation of f as a product of monic irreducible polynomials in $GF(q)[x]$. If the degree of g is $k \geq 0$, then the degree of h is $n-2k$. Therefore there exists a bijection between the set of all monic polynomials of degree n over $GF(q)$ and the set of all ordered pairs (g,h), where g is an arbitrary monic polynomial in $GF(q)[x]$ of degree k, $0 \leq k \leq n/2$, and h is an arbitrary square-free polynomial in $GF(q)[x]$ of degree $n-2k$. This gives the following recurrence:

$$q^n=\sum_{k=0}^{n/2} q^k a_{n-2k}\,,\ n \geq 1,\quad a_0=1$$

Solving for $a_i,\ i \in [4]$, we get: $q = q^0 a_1$, so $a_1 = q$ (as it should be);
$q^2 = a_2 + qa_0$, so $a_2 = q^2 - q$;
$q^3 = a_3 + qa_1$, so $a_3 = q^3 - q^2$;
$q^4 = a_4 + qa_2 + q^2 a_0$, so $a_4 = q^4 - q(q^2 - q) - q^2 = q^4 - q^3$.
This pattern suggests a conjecture:

$$a_n = q^n - q^{n-1},\ n \geq 2.$$

It can be proven immediately via induction on n. For $n = 2$ it is correct. Suppose $n \geq 2$ and that the statement is proven for all i, $2 \leq i < n$. Then, if $n = 2m \geq 2$, we have

$$a_n = q^n - \sum_{k=1}^{m} q^k a_{2m-2k} = q^n \sum_{k=1}^{m-1} q^k (q^{2m-2k} - q^{2m-2k-1}) - a_0 q^m =$$

$$q^n - \sum_{k=1}^{m-1} (q^{2m-k} - q^{2m-k-1}) - q^m = q^n - q^{n-1}.$$

If $n = 2m + 1 \geq 3$, we have

$$a_n = q^n - \sum_{k=1}^{m} q^k a_{2m+1-2k} = q^n - \sum_{k=1}^{m-1} q^k (q^{2m+1-2k} - q^{2m-2k}) - a_1 q^m =$$

$$q^n - \sum_{k=1}^{m-1} (q^{2m+1-k} - q^{2m-k}) - q^{m+1} = q^n - q^{n-1}.$$

Thus $a_0 = 1,\ a_1 = q$, and $a_n = q^n - q^{n-1}$ for all $n \geq 2$. □

Problem 7. *Consider all $m \times n$ matrices whose entries are equal to 1 or -1 only. In how many of these matrices the product of elements in each row and each column $= 1$?*
*(***Hint:** *Let's call a matrix A "good" if its entries are equal to 1 or -1 only and the products of elements in each row and each column = 1. First prove that the number of "good" $m \times n$ matrices is equal to the number of all $(m-1) \times (n-1)$ matrices whose entries are equal to 1 or -1 only).*

Solution. Let us show that starting with an arbitrary $(m-1)\times(n-1)$ matrix of -1 and 1 as entries we can construct a unique $m \times n$ "good" matrix by adding one row and one column. Let (a_{ij}) be an arbitrary $(m-1) \times (n-1)$

matrix. For each $i \in [m-1]$, define $a_{in} = \prod_{k=1}^{n-1} a_{ik}$. This leads us to an $(m-1) \times n$ matrix. Then, for each $j \in [n]$, define $a_{mj} = \prod_{k=1}^{m-1} a_{kj}$. This leads us to an $m \times n$ matrix. The only thing to check is that product of all elements in the m-th row of the obtained $m \times n$ matrix is 1, i.e., $\prod_{j=1}^{n} a_{mj} = 1$. We have

$$\prod_{j=1}^{n} a_{mj} = \prod_{j=1}^{n} \left(\prod_{k=1}^{m-1} a_{kj} \right) = \prod_{k=1}^{m-1} \left(\prod_{j=1}^{n} a_{kj} \right) = \prod_{k=1}^{m-1} (a_{kn})^2 = \prod_{k=1}^{m-1} (1) = 1$$

Thus each ± 1–$(m-1) \times (n-1)$ matrix "defines" a unique "good" $m \times n$ matrix, and obviously every "good" $m \times n$ matrix is obtained via this process from its submatrix defined by the first $m-1$ rows and $n-1$ columns. So there are $2^{(m-1)(n-1)}$ "good" $m \times n$ matrices. □

Problem 8. *Consider a tournament which starts with $n \geq 1$ teams. In the first round, all teams are divided into pairs if n is even, and the winner in each pair passes to the next round (no ties). If n is odd, then one random (lucky) team passes to the next round without playing. The second round proceeds similarly. At the end, only one team is left – the winner. Find a <u>simple</u> formula for the total number of games played in the tournament.*

<u>Solution #1.</u> The number of games played is equal to the number of teams which lost their game: every team but the winner loses exactly one game and the every game eliminates exactly one team. Therefore the number of games in the tournament is $n-1$. □

<u>Solution #2.</u> Let $f(n)$, $n \geq 1$, denotes the number of all games played. Assume $f(1) = 0$. Working out examples for small n (or for $n = 2^k$), we come to a conjecture $f(n) = n-1$. We can prove it by induction. Suppose it is established for all m, $1 \leq m < n$. Then, if $n = 2k$, we have

$$f(n) = f(k) + k = (k-1) + k = 2k - 1 = n - 1.$$

If $n = 2k+1$, we have

$$f(n) = f(k+1) + k = k + k = 2k = n - 1.$$

Thus the statement is proven. □

Problem 9.

(i) Find the value of a_{50} in the following expansion:

$$\frac{x-3}{x^2-3x+2} = a_0 + a_1 x + \cdots + a_{50}x^{50} + \cdots$$

(ii) Solve the recurrence: $a_0 = 1$; $a_n = a_{n-1} + 2a_{n-2} + +3a_{n-3} \ldots + na_0$, for $n > 0$.

Solution. (i) $f(x) = \frac{x-3}{x^2-3x+2} = \frac{x-3}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$.
In order to find A, B, we can do, for example, this:

$$\begin{array}{rcl} A & = & \lim_{x\to 2}[f(x)(x-2)] = \lim_{x\to 2} \frac{x-3}{x-1} = \frac{-1}{1} = -1 \\ B & = & \lim_{x\to 1}[f(x)(x-1)] = \lim_{x\to 1} \frac{x-3}{x-2} = \frac{-2}{-1} = 2 \ . \end{array}$$

Thus

$$f(x) = \frac{x-3}{x^2-3x+2} = \frac{-1}{x-2} + \frac{2}{x-1} = \frac{1}{2-x} - \frac{2}{1-x} = \frac{1}{2}\sum_{i\geq 0}\left(\frac{x}{2}\right)^i - 2\sum_{i\geq 0} x^i =$$

$$\sum_{i\geq 0}\left(\frac{1}{2^{i+1}} - 2\right) x^i = \sum_{n\geq 0} a_n x^n.$$

Thus $\boxed{a_{50} = \frac{1}{2^{51}} - 2.}$

(ii) Solution #1. Obviously, $\{a_{n+1}\}_{n\geq 0}$ is an "ordinary" convolution of $b_n = \{n\}_{n\geq 0}$, and $\{a_n\}_{n\geq 0}$. Let $A(x) = \sum_{k\geq 0} a_k x^k$ and

$$B(x) = \sum_{k\geq 0} b_k x^k = \sum_{k\geq 0} kx^k = xD(\frac{1}{1-x}) = \frac{x}{(1-x)^2}.$$

Then we have:

$$A(x) - a_0 = A(x)B(x) = A(x) \cdot \frac{x}{(1-x)^2}$$

Using $a_0 = 1$ and solving for $A(x)$, we get

$$A(x) = \frac{x^2-2x+1}{x^2-3x+1}.$$

Using partial fractions we get

$$a_n = c_1\alpha^n + c_2\beta^n,$$

where $\alpha, \beta = (3 \pm \sqrt{5})/2$, respectively, and $c_{1,2} = (5 \mp \sqrt{5})/10$, respectively. $\square$

<u>Solution #2.</u> Using the recurrence for a_n and a_{n-1}, we get:

$$a_n - a_{n-1} = a_{n-1} + a_{n-2} + \ldots + a_0$$

Therefore

$$a_{n-1} - a_{n-2} = a_{n-2} + a_{n-3} + \ldots + a_0,$$

and we get

$$(a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) = a_{n-2}.$$

This implies the recurrence $a_n = 3a_{n-1} - a_{n-2}$, $n \geq 2$, $a_0 = a_1 = 1$. Therefore $a_n = C_1\alpha^n + C_2\beta^n$, where α, β are the roots of the characteristic equation $\lambda^2 - 3\lambda + 1 = 0$, and we obtain the explicit formula from (i). $\square$

Problem 10. *Evaluate* $\sum_{0 \leq k \leq n/2} \binom{n-2k}{k} (\frac{-4}{27})^k$ *in closed form. Hint:* $x^3 - x^2 + 4/27 = (x + 1/3)(x - 2/3)^2$.

Solution #1. Let $f_n(x) = \sum_{0 \leq k \leq n/2} \binom{n-2k}{k} x^k$. Then $f_n(x) = f_{n-1}(x) + x f_{n-3}$, $n \geq 3$, $f_0 = f(1) = f(2) = 1$. Indeed, the verification of the recurrence amounts to showing that

$$\binom{n-2k}{k} = \binom{n-1-2k}{k} + \binom{n-3-2(k-1)}{k-1}, \ 1 \leq k \leq n/2,$$

which is clearly correct. Therefore, the characteristic equation for $f_n(x)$ is $\lambda^3 - \lambda^2 - x = 0$. For $x = -4/27$, it is equivalent to $\lambda^3 - \lambda^2 + 4/27 = (\lambda + 1/3)(\lambda - 2/3)$. Therefore

$$f_n(-4/27) = C_1(-1/3)^n + (C_2 n + C_3)(2/3)^n.$$

Using the initial conditions, we get:

$$\boxed{f_n(-4/27) = (1/9)(-1/3)^n + ((2/3)n + (8/9))(2/3)^n, \ n \geq 0.}$$

Solution #2. (by Frank Fiedler) Hint to look for a similar method in [5] provided by Sukhendu Mehrotra. Solution is based on [5], p.122. Interchanging orders of summation is done formally without consideration of convergence. Let $y = -\frac{4}{27}$. Consider the generating function with coefficients $\sum_{k \le \frac{n}{2}} \binom{n-2k}{k} y^k$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n \sum_{k \le \frac{n}{2}} \binom{n-2k}{k} y^k &= \sum_{k=0}^{\infty} y^k \sum_{n \ge \frac{k}{2}}^{\infty} \binom{n-2k}{k} x^n \\
&= \sum_{k=0}^{\infty} y^k x^{2k} \sum_{n \ge \frac{k}{2}}^{\infty} \binom{n-2k}{k} x^{n-2k}
\end{aligned}$$

which by [5] (4.3.1) becomes

$$\begin{aligned}
&= \sum_{k=0}^{\infty} y^k x^{2k} \frac{x^k}{(1-x)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{1}{1-x} \left(\frac{yx^3}{1-x} \right)^k \\
&= \frac{1}{1-x} \cdot \frac{1}{1 - \frac{yx^3}{1-x}} \\
&= \frac{1}{-yx^3 + 1 - x} \\
&= \frac{1}{\frac{4}{27}x^3 - x + 1} \\
&= \frac{\frac{4}{27}}{(x+3)(x-\frac{3}{2})^2}
\end{aligned}$$

which (using Maple) factors into

$$\begin{aligned}
&= \frac{1}{3}\frac{1}{x+3} + 6\frac{1}{(2x-3)^2} - \frac{2}{3}\frac{1}{2x-3} \\
&= \frac{1}{3}\frac{1}{x+3} + \frac{2}{3}\frac{1}{(1-\frac{2}{3}x)^2} - \frac{2}{3}\frac{1}{2x-3}
\end{aligned}$$

The $\dfrac{1}{(1-\frac{2}{3}x)^2}$ can be understood as a derivative:

$$\begin{aligned}
&= \frac{1}{9}\cdot\frac{1}{1-(-\frac{1}{3}x)} + D\left(\frac{1}{1-\frac{2}{3}x}\right) + \frac{2}{9}\cdot\frac{1}{1-\frac{2}{3}x} \\
&= \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{1}{3}x\right)^n + D\left(\sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n\right) \\
&\quad + \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{2}{3}x\right)^n
\end{aligned}$$

Hence the n-th coefficient is

$$\begin{aligned}
&\frac{1}{9}\left(-\frac{1}{3}\right)^n + (n+1)\left(\frac{2}{3}\right)^{n+1} + \frac{2}{9}\left(\frac{2}{3}\right)^n \\
&= \left(-\frac{1}{3}\right)^{n+2} + (n+1)\left(\frac{2}{3}\right)^{n+1} + \left(\frac{2}{3}\right)^{n+2} \\
&= \frac{(-1)^n + 2^{n+1}(n+1) + 2^{n+2}}{3^{n+2}}
\end{aligned}$$

□

Problem 11. *(i) Find* $\sum_{n\geq 0} H_n/10^n$*, where* $H_n = 1 + 1/2 + \ldots + 1/n$ *is the n-th harmonic number,* $n \geq 1$*.*

(ii) Let F_i *denote the i-th Fibonacci number:* $F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$*,* $n \geq 2$*. Find a closed formula for*

$$\sum_{m>0}\sum_{\lambda} F_{k_1}F_{k_2}\cdots F_{k_m},$$

where λ *runs over all compositions* $k_1+k_2+\ldots k_m = n$ *of a nonnegative integer n with all* $k_i \geq 1$*. Hint: some of these problems have short solutions based on generating functions.*

Solution. (i) Let $H(x) = \sum_{n\geq 0} H_n x^n$ be the o.g.f. for $\{H_n\}_{n\geq 0},\ H_0 = 0$. Then

$$H(x) = (x+1/2x^2+1/3x^3+\ldots)(1+x+x^2+x^3+\ldots) = \ln(1/(1-x))(1/(1-x)).$$

The functional equality is obviously correct for $|x| < 1$. Hence

$$\boxed{H(1/10) = (10/9) \cdot \ln(10/9).}$$

(ii) Let a_n, $n \geq 1$, denote the double sum we have to simplify. Let $f(x)$ be the o.g.f. for the Fibonacci sequence, i.e., $f(x) = x/(1 - x - x^2)$. Then

$$\sum_{m \geq 1} f^m(x) = f(x)/(1 - f(x)) = x/(1 - 2x - x^2)$$

is the o.g.f. for $\{a_n\}$. Using partial fractions, we get:

$$\boxed{a_n = \frac{1}{2\sqrt{2}}(\alpha^n - \beta^n),}$$

where α, β are $1 \pm \sqrt{2}$, respectively.

Problem 12. *Let $\sigma(n)$ denote the sum of all positive divisors of a positive integer n, and $p(n)$ denote the number of partitions of integer n, i.e., the multisets of positive integers which add to n. Prove the Euler's identity:*

$$np(n) = \sum_{0 \leq m < n} \sigma(n - m)p(m), \quad n \geq 1, p(0) = 1$$

(Hint: One can use the ordinary generating function $P(x)$ of $\{p(n)\}_{n \geq 0}$ in the following way: first compute $xP'(x)$ directly, and then as $[x(\log P(x))']P(x)$; equate the results.)
Check that the formula is correct for $n = 1, 2, 3, 4, 5, 6$

Solution. Let

$$(*) \qquad p(x) = \sum_{n \geq 0} p(n)x^n \underset{\substack{\text{proven}\\ \text{before}}}{=} \prod_{i \geq 1} (1 - x^i)^{-1}$$

Therefore

$$(**) \qquad xP'(x) = \sum_{n \geq 1} nP(n)x^n \quad .$$

On the other hand, taking log of both sides of $(*)$ and then applying $\left(x\frac{d}{dx}\right)$ to both sides we get

$$\frac{xP'(x)}{P(x)} = \frac{xd}{dx}\left[\sum_{i\geq 1} -\log(1-x^i)\right] = \sum_{i\geq 1}\frac{ix^i}{1-x^i}$$

Therefore

$$\begin{aligned} xP'(x) &= \left(\sum_{i\geq 1}\frac{ix^i}{1-x^i}\right)\cdot P(x) = \sum_{i\geq 1}\frac{ix^i}{1-x^i}\sum_{j\geq 0}P(j)x^j \\ &= \left[\sum_{i\geq 1} ix^i \sum_{k\geq 0}(x^i)^k\right]\left[\sum_{j\geq 0}P(j)x^j\right] = \\ &= \left[\sum_{i\geq 1}\sum_{k\geq 0} ix^{(k+1)i}\right]\left[\sum_{j\geq 0}P(j)x^j\right] \end{aligned}$$

The first double summation can be rewritten as $\sum_{m\geq 1}\left(\sum_{i|m} i\right)x^m$, since the coefficient i appears at those powers of x whose exponents are divisible by i. Thus

$$xP'(x) = \left[\sum_{m\geq 1}\left(\sum_{i|m} i\right)x^m\right]\left[\sum_{j\geq 0}P(j)x^j\right] =$$

$$\left[\sum_{m\geq 1}\sigma(m)x^m\right]\cdot\left[\sum_{j\geq 0}P(j)x^j\right] =$$

$$(***) \qquad \sum_{n\geq 1}\left(\sum_{\substack{m+j=n\\ m\geq 1, j\geq 0}}\sigma(m)P(j)\right)x^n$$

Comparing $(**)$ and $(***)$ and replacing m by $n-j$ we finish the proof.

Remark. A combinatorial proof of this Euler identity can be found, e.g., in A. Nijenhuis & H.S. Wilf "*Combinatorial Algorithms*", 2nd edition, pp. 73–74.

Problem 13. *Let $f(n)$ be the number of paths with n steps starting from $(0,0)$, with steps of the type $(1,0)$,$(-1,0)$,or $(0,1)$, and never intersecting themselves. Assume $f(0)=1$.*

(i) Find a simple recurrence for $f(n)$;

(ii) Then find the ordinary generating function for the sequence $\{f(n)\}_{n\geq 0}$;

(iii) Then find an explicit formula for $f(n)$.

(iv) Suppose that all paths of length n constructed with given steps are equally likely to appear. Show that for large n, the probability of getting a path of length n which does not intersect itself is about $1.2071(.804738)^n$.

Solution. (i) Each path which doesn't intersect itself can be coded by a string $a_1a_2\ldots a_n$, when each $a_i \in \{L, R, U\}$, L corresponds to a step to the left, R–to the right, U–up, and the condition of non–intersecting itself is translated as $a_ia_{i+1} \neq LR$ or RL, $i = 1, \ldots, n-1$. Let $f(n)$ be the number of such strings, $n \geq 0$. We have $f(0) = 1$ (by definition), $f(1) = 3$. If $n \geq 2$, we partition all $f(n)$ strings of length n into 5 groups.

Group 1:	Strings ending on	U
Group 2:	Strings ending on	LL
Group 3:	Strings ending on	RR
Group 4:	Strings ending on	UL
Group 5:	Strings ending on	UR

There are $f(n-1)$ strings in Group 1. There are $f(n-2)$ strings in Group 4. All strings in the union of Groups 2,3,5 are in a bijective correspondence with all valid strings of length $n-1$, since each of such string ends by L, R, or U and these 3 letters are the first letters in words LL, RR, UR. Thus the union of Groups 2,3,5 contains $f(n-1)$ elements.
Therefore, we get

$$\boxed{f(n) = 2 \cdot f(n-1) + f(n-2), n \geq 2.}$$

(ii) By using the recurrence above, we get the ordinary generating function for $\{f(n)\}_{n\geq 0}$ in our usual way. We get:

$$\boxed{F(x) = \sum_{n\geq 0} f(n)x^n = \tfrac{1+x}{1-2x-x^2}.}$$

(iii) Let $x_{1,2} = 1 \pm \sqrt{2}$ be the roots of the polynomial $1-2x-x^2$. Proceeding as in Problem 1 of this Homework, or by using Linear Algebra, we get

$f(n) = C_1 x_1^n + C_2 x_2^n$, where C_1, C_2 are unknown constants. To find C_1, C_2 we use: $f(0) = 1, f(1) = 3$. We get $C_1, C_2 = \frac{1}{2}(1 \pm \sqrt{2})$, Thus

$$\boxed{f(n) = \tfrac{1}{2}x_1^{n+1} + \tfrac{1}{2}x_2^{n+1}, n \geq 0.}$$

(iv) First we notice that the second addend in the formula for $f(n)$ above approaches zero exponentially fast when $n \to \infty$. Moreover, $f(n)$ is the closest integer to $\frac{1}{2}x_1^{n+1}$. There are 3^n strings over the alphabet $\{R, L, U\}$ which correspond to all paths of length n. We assume that all of them equally likely to appear. Let A denote the event that a path doesn't intersect itself. Then $\text{Prob}(A) = \frac{f(n)}{3^n} = \frac{x_1}{2}\left(\frac{x_1}{3}\right)^n + \frac{x_2}{2}\left(\frac{x_2}{3}\right)^n$. When $n \to \infty$, the second addend goes to zero much faster than the first. Thus $\text{Prob}(A) \approx \frac{x_1}{2}\left(\frac{x_1}{3}\right)^n = \frac{1+\sqrt{2}}{2}\left(\frac{1+\sqrt{2}}{3}\right)^n$.

Problem 14. *Let n, and k be fixed positive integers. How many sequences (ordered k–tuples) $(X_1, X_2, \ldots, X_k)$ are there of subsets of $[n] = \{1, 2, \ldots, n\}$ such that $X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset$?*
Though this problem has a remarkable solution which does not use generating functions (can you see it?), we can solve it in the following steps. Let $f(n, k)$ denote the number.

(i) Prove that $f(n, k) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} f(n - i, k - 1)$

(ii) Prove that the exponential generating function $F_k(x)$ for $\{f(n, k)\}_{n \geq 0}$ satisfies the property $F_k(x) = e^x F_{k-1}(2x)$.

(iii) Find an explicit expression for $F_k(x)$, and using it, conclude that $f(n, k) = (2^k - 1)^n$.

Solution. (By Frank Fiedler)

(i) Let $\{X_1, \ldots, X_{k-1}\}$ be a collection of subsets of $[n]$. Let $X = \bigcap_{j=1}^{k-1} X_j$ and say $|X| = i$ for some $0 \leq i \leq n$. Then any set X_k such that $\{X_1, \ldots, X_k\}$ is a sequence with $\emptyset = \bigcap_{j=1}^{k}$ must not contain the elements in X. Hence X_k is a subset of the remaining $n - i$ elements. There are 2^{n-i} such sets. On the other side, $\{X_1 \cap X, \ldots, X_{k-1} \cap X\}$ is a sequence with the required properties. There are $f(n - i, k - 1)$ such sequences. For any such sequence we can select a set X of cardinality $n-i$ to obtain a sequence $\{X_1, \ldots, X_{k-1}\}$ in $\binom{n}{i}$ ways. Hence

$$f(n,k) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} f(n-i, k-1)$$

(ii)

$$\begin{aligned} e^x F_{k-1}(2x) &= \left(\sum_{i=0}^{\infty} \frac{x^i}{i!}\right) \left(\sum_{n=0}^{\infty} f(n,k-1) \frac{(2x)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{1}{i!} f(n-i,k-1) x^i 2^{n-i} \frac{x^{n-i}}{(n-i)!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} f(n-i,k-1) 2^{n-i} \binom{n}{i}\right) \frac{x^n}{n!} \end{aligned}$$

since $\frac{1}{i!(n-i)!} = \frac{\binom{n}{i}}{n!}$

$$\begin{aligned} &= \sum_{n=0}^{\infty} f(n,k) \frac{x^n}{n!} \\ &= F_k(x) \end{aligned}$$

(iii) With $f(n,1) = 1$ (only $X_1 = \emptyset$ is admissible) we get

$$\begin{aligned} F_1(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= e^x \end{aligned}$$

and

$$\begin{aligned} F_k(x) &= e^x F_{k-1}(2x) \\ &= e^x \left(e^{2x} F_{k-2}(4x)\right) \\ &\vdots \\ &= e^{\sum_{i=0}^{k-2}(2^i x)} F_1(2^{k-1}x) \\ &= e^{(2^{k-1}-1)x} e^{2^{k-1}x} \\ \Longrightarrow \qquad F_k(x) &= e^{(2^k-1)x} \\ &= \sum_{n=0}^{\infty} (2k-1)^n \frac{x^n}{n!} \end{aligned}$$

and thus

$$f(n,k) = (2^k-1)^n$$

$\square$

Problem 15. *What is the average size of the largest element of a k-subset of* $[n]$ *(*n, k *are fixed).*

Solution. If the largest element of a k–subset is i, then other $k-1$ elements must be chosen from the set $\{1, 2, \ldots, i-1\}$. There are $\binom{i-1}{k-1}$ such choices, so there are $\binom{i-1}{k-1}$ k–subsets of $[n]$ with the largest element i. Thus the average size of the largest element in a k–subset of $[n]$ is

$$\frac{\sum\limits_{i=k}^{n} i\binom{i-1}{k-1}}{\binom{n}{k}} \underset{i\binom{i-1}{k-1}=k\binom{i}{k}}{=} \frac{\sum\limits_{i=k}^{n} k\binom{i}{k}}{\binom{n}{k}} = \frac{k\sum\limits_{i=k}^{n}\binom{i}{k}}{\binom{n}{k}} = \frac{k\binom{n+1}{k+1}}{\binom{n}{k}}.$$

The identity $\sum\limits_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$ was proven before. Simplifying the obtained ratio, we get the following simple expression for the average:

$$\boxed{\frac{k(n+1)}{k+1}.}$$

$\square$

Problem 16. *This problem can be thought as counting compositions of an integer "by modulo p". Let $p > 2$ be prime. Consider an equation $\sum_{i=1}^{n} a_i x_i = b$, where all $a_i \in GF^*(p) = GF(p) \setminus \{0\}$, $b \in GF(p)$.*

(i) Prove that the number of all solutions of the equation in $(GF(p))^n$ is p^{n-1}.

(ii) Prove that the number of all solutions of the equation in $(GF^*(p))^n$, i.e., solutions with all $x_i \neq 0$, is

$$\frac{1}{p}((p-1)^n + (-1)^n(p-1)), \quad \text{if} \quad b = 0, \quad \text{and}$$

$$\frac{1}{p}((p-1)^n + (-1)^{n+1})), \quad \text{if} \quad b \neq 0.$$

Solution. (by Frank Fiedler)

(i) $\sum_{i=1}^{n} a_i x_i = b$ implies $x_n = a_n^{-1}(b - \sum_{i=1}^{n-1} a_i x_i)$. Hence for any choice of $\{x_i \in \mathsf{GF}(p)\}_{i=1}^{n-1}$ there exists exactly one solution for x_n. Therefore the number of solutions is p^{n-1}.

(ii) Let $f_{(=)}(n)$ denote the number of solutions if $b = 0$, $f_{(\neq)}(n)$ the number for $b \neq 0$. First consider $b = 0$ and

$$\sum_{i=1}^{n} a_i x_i = 0$$
$$\implies \qquad x_1 = -\sum_{i=2}^{n} a_1^{-1} a_i x_i$$

Hence for any fixed $x_1 \in \mathsf{GF}(p)^*$ there are $f_{\neq}(n-1)$ solutions and therefore

$$f_{=}(n) = (p-1) f_{\neq}(n-1)$$

Now start proving the formulas. Proof for $b \neq 0$ by induction. Note that

$$f_{\neq}(1) = 1 \qquad = \frac{1}{p}\left((p-1) + (-1)^2\right)$$
$$f_{\neq}(2) = p - 2 \qquad = \frac{1}{p}\left((p-1)^2 + (-1)^3\right)$$

Suppose the formula holds for all $k < n$. Consider

$$\sum_{i=1}^{n} a_i x_i = b$$
$$\implies \qquad x_1 = a_1^{-1} b - \sum_{i=2}^{n} a_1^{-1} a_i x_i$$

Since $b, a_1^{-1} \in \mathsf{GF}(p)^*$ their product is not zero. There are $p-1$ choices for $x_1 \in \mathsf{GF}(p)^*$. Fix x_1. If $x_1 = a_1^{-1}b$ then there are $f_=(n-1)$ solutions $\{x_i\}_{i=2}^{n}$. If $x_1 \neq a_1^{-1}b$ (this is the case for $(p-2)$ of the choices) then $x_1 - a_1^{-1}b \neq 0$ and there are $f_{\neq}(n-1)$ solutions. Hence

$$\begin{aligned}
f_{\neq}(n) &= f_=(n-1) + (p-2) f(n-1) \\
&= (p-1) f_{\neq}(n-2) + (p-2) f_{\neq}(n-1) \\
&= \frac{p-1}{p}\left((p-1)^{n-2} + (-1)^{n-1}\right) \\
&\quad + \frac{p-2}{p}\left((p-1)^{n-1} + (-1)^{n}\right) \\
&= \frac{1}{p}\left((p-1)^{n-1} + (-1)^{n-1}(p-1)\right. \\
&\quad \left. + (p-1)^{n} - (p-1)^{n-1} + (-1)^{n}(p-2)\right) \\
&= \frac{1}{p}\left((p-1)^{n} + (-1)^{n+1}\left((p-2)-(p-1)\right)\right) \\
&= \frac{1}{p}\left((p-1)^{n} + (-1)^{n+1}\right)
\end{aligned}$$

With the recurrence relation for $f_=(n)$ it then follows

$$\begin{aligned}
f_=(n) &= (p-1) \cdot \frac{1}{p}\left((p-1)^{n-1} + (-1)^{n}\right) \\
&= \frac{1}{p}\left((p-1)^{n} + (-1)^{n}(p-1)\right)
\end{aligned}$$

□

Problem 17. *(i) Suppose $f(n)$ and $g(n)$ are multiplicative functions. Prove that the function*

$$h(n) = \sum_{d|n} f(d) g(\frac{n}{d})$$

is multiplicative.

(ii) Prove that the function $\sigma(n)$ = the sum of all positive divisors of a positive integer n is multiplicative.

(iii) A positive integer n is called perfect if the sum of all its positive divisors is $2n$ (e.g. 6, 28). Prove that if $n = 2^a(2^{a+1} - 1)$, a is a positive integer and $2^{a+1} - 1$ is a prime number, then n is perfect.

Remark. Euler proved that every even perfect number must be of this form (the proof is not hard). The existence of an odd perfect number is an open problem.

Solution. (i) We have to show that $h(1) = 1$ and $h(mm) = h(m)h(n)$, for $(m, n) = 1$.

$$h(m)h(n) = \left[\sum_{a|m} f(a)g\left(\frac{m}{a}\right)\right]\left[\sum_{b|n} f(b)g\left(\frac{n}{b}\right)\right] = \sum_{\substack{a|m\\b|n}} f(a)g\left(\frac{m}{a}\right) f(b)g\left(\frac{n}{b}\right).$$

If $(m, n) = 1$, and $a|m, b|n$, then $(a, b) = 1$, and $\left(\frac{m}{a}, \frac{n}{b}\right) = 1$. Using multiplicativity of f and g, we can rewrite the last sum as

$$\sum_{a|m,b|n} f(ab)g\left(\frac{m \cdot n}{a \cdot b}\right) = \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) \quad .$$

The last equality is true, since all divisors d of mn have a unique decomposition $d = a \cdot b$, where $a|m$ and $b|n (a, b > 0)$. Thus $h(m)h(n) = \sum\limits_{d|mn} f(d)g\left(\frac{mn}{d}\right) = h(mn)$. Also $h(1) = \sum\limits_{d|1} f(d)g\left(\frac{1}{d}\right) = f(1)g(1) = 1 \cdot 1 = 1$.

Therefore $h(n)$ is multiplicative. □

Remark. It would be sufficient to prove that $h(p^a)h(q^b) = h(p^a q^b)$ for all prime powers $p^a, q^b, p \neq q$.

(ii) By definition, $\sigma(n) = \sum\limits_{a|n} a$, which can be rewritten as $\sum\limits_{a|n} f(a)g\left(\frac{n}{a}\right)$, where $f(x) = x, g(x) = 1$ for all $x \in \mathbf{N}$. Both f and g are obviously multiplicative, hence by (i), and $\sigma(1) = 1$, $\sigma(n)$ is multiplicative. □

(iii) We have:

$$\sigma(n) = \sigma(2^a(2^{a+1} - 1)) = \sigma(2^a)\sigma(2^{a+1} - 1) = \left(\sum_{d|2^a} d\right)\left(\sum_{d|2^{a+1}-1} d\right) =$$

$$(1+2+\ldots+2^a)(1+(2^{a+1}-1)) = (2^{a+1}-1)\cdot 2^{a+1} = 2(2^a(2^{a+1}-1)) = 2n\ .$$

The fourth equality sign is due to the fact that $2^{a+1}-$ is prime. So n is perfect.

Problem 18. *(i) A string (or a sequence, or a "word") $a_1a_2\ldots a_n$, where each $a_i = 0$ or 1, is called primitive if it is not a concatenation of several identical shorter strings. Prove that every string is uniquely expressible as a concatenation of some number, n/d, of a primitive string of length d, $td = n$.*

(ii) Does the statement from (i) hold when the alphabet $\{0,1\}$ is replaced with an arbitrary finite alphabet?

(i) Suppose a string $a = a_1a_2\ldots a_n$ is a concatenation of n/k copies of a primitive string x and of n/m copies of a primitive string y. We want to show that $k = m$ and $x = y$. The latter follows from the former, since $x = a_1a_2\ldots a_k = a_1a_2\ldots a_m = y$. Each string can be thought as a binary representation of an integer, and we use the same notation for both the number and the string. Then $2^{k-1} \le x < 2^k$ and $2^{m-1} \le y < 2^m$, and

$$a = x + 2^k x + \cdots + 2^{n-k}x = y + 2^m x + \cdots + 2^{n-m}y.$$

This implies that $\frac{2^n-1}{2^k-1}x = \frac{2^n-1}{2^m-1}y$, or

$$(2^m-1)x = (2^k-1)y. \tag{1}$$

It is a well-know (and easy to prove, make sure you know how) that $d = (k,m) \iff 2^d - 1 = (2^k-1, 2^m-1)$. Reducing both sides of (1) by $2^d - 1$, we obtain

$$(1 + 2^d + \cdots + 2^{m-d})x = (1 + 2^d + \cdots + 2^{k-d})y,$$

where $(1 + 2^d + \cdots + 2^{m-d}, 1 + 2^d + \cdots + 2^{k-d}) = 1$. Therefore $x = (1 + 2^d + \cdots + 2^{k-d})x_1$. Since $2^{k-1} \le x < 2^k$, then $2^{d-1} \le x_1 < 2^d$. But this means that the binary string x is a concatenation of k/d copies of $x_1 = 1$. Since x is primitive, $x = x_1$, and hence $d = k$. A similar argument shows that $d = m$. Hence $k = m$. □

(ii) Yes. If the cardinality of the alphabet is $t \ge 2$, we can use the same argument, and the fact $d = (k,m) \iff t^d - 1 = (t^k - 1, t^m - 1)$. For $t = 1$ the uniqueness is obvious. □

Problem 19. *Read in H. Wilf's "generatingfunctionology" Section 2.6, Example 2, where cyclotomic polynomials are discussed and a 'fairly explicit' formula for a cyclotomic polynomial is obtained. This is another great application of the Möbius Inversion Formula.*
Prove or disprove the following statement: coefficients of cyclotomic polynomials can only be 0, or 1, or -1.

Solution. The statement is false. The smallest n which provides a counterexample is $n = 105$. This is easy to check both by hand computations and by using any computer algebra package. The question was posed by N.G. Chebotaröv as a note in a journal *Uspekhi Matematicheskih Nauk* in 1938, and the counterexample was given by V. Ivanov, in *Usp. Mat. Nauk*, 4, 313-317 (1941).

The following presentation is due to Frank Fiedler.

The Theorem of Migotti (cf. [3]) says that a necessary condition for $\Phi_n(x)$ not having only coefficients 0, 1, and -1 is that n is divisible by three odd primes. The smallest such n is 105. Rewriting $x^n - 1$ as product of roots (which splits into the product of primitive roots for all divisors of n) the n-th cyclotomic polynomial has a recurrence relation as follows (see [3])

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d<n}} \Phi_d(x)}$$

Thus

$$\Phi_{105}(x) = \frac{x^{105} - 1}{\Phi_1(x)\Phi_3(x)\Phi_5(x)\Phi_7(x)\Phi_{15}(x)\Phi_{21}(x)\Phi_{35}(x)}$$

and using the same arguments for $\Phi_n(x)$ where n is composite

$$
\begin{aligned}
\Phi_{15}(x) &= \frac{x^{15}-1}{\Phi_1(x)\Phi_3(x)\Phi_5(x)} \\
\Phi_{21}(x) &= \frac{x^{21}-1}{\Phi_1(x)\Phi_3(x)\Phi_7(x)} \\
\Phi_{35}(x) &= \frac{x^{35}-1}{\Phi_1(x)\Phi_5(x)\Phi_7(x)} \\
\implies \qquad \Phi_{105}(x) &= \frac{(x^{105}-1)\left(\Phi_1(x)\right)^2 \Phi_3(x)\Phi_5(x)\Phi_7(x)}{(x^{15}-1)(x^{21}-1)(x^{35}-1)} \\
&= \frac{(x^{105}-1)(x-1)^2(x^2+x+1)(x^4+x^3+x^2+x+1)}{(x^{15}-1)(x^{21}-1)(x^{35}-1)} \\
&\quad \cdot (x^6+x^5+x^4+x^3+x^2+x+1) \\
&= 1+x-x^6-x^5+x^2-2x^7-x^{24}+x^{12}-x^8 \\
&\quad +x^{13}+x^{14}+x^{16}+x^{17}-x^9+x^{15}+x^{48} \\
&\quad -x^{20}-x^{22}-x^{26}-x^{28}+x^{31}+x^{32}+x^{33} \\
&\quad +x^{34}+x^{35}+x^{36}-x^{39}-x^{40}-2x^{41}-x^{42} \\
&\quad -x^{43}+x^{46}+x^{47}
\end{aligned}
$$

Because of the term $-2x^7$, this is a counter-example to the conjecture that all coefficients in cyclotomic polynomials are 0, 1, or -1. (For the last step I used Maple.) □

Problem 20. *(i) Prove that the Euler function $\phi(n)$ is multiplicative.*

(ii) Find the Dirichlet generating function for $\phi(n)$.

(iii) Prove in two ways that $\sum_{d|n} \phi(n) = n$.

What is the significance of this number-theoretic fact in algebra?

Hint: One proof can be based on Problem 1 (i).

(iv) Prove that $\phi(n) = \sum_{d|n} \mu(n/d)d$.

First Proof.(i) The result follows immediately from the explicit formula

$$\phi(n) = n \prod (1 - 1/q),$$

where the product is taken over all distinct prime divisors q of n. This formula was proven in this course by the PIE. If $(m, n) = 1$, than the sets of distinct prime divisors p of m and q of n are disjoint and their union gives the set of distinct prime divisors r of mn. Therefore

$$\phi(mn) = mn \prod (1 - 1/r) = \left(m \prod (1 - 1/p)\right) \left(n \prod (1 - 1/q)\right) = \phi(m)\phi(n).$$

□

(ii) Since ϕ is multiplicative by (i), and $\phi(p^k) = p^k - p^{k-1}$, we have

$$\Phi(s) = \sum_{n \geq 1} \frac{\phi(n)}{n^s} = \prod_p \sum_{k \geq 0} \frac{\phi(p^k)}{p^{ks}} = \prod_p \left(1 + \sum_{k \geq 1} \frac{p^k - p^{k-1}}{p^{ks}}\right)$$

$$\prod_p \left(\sum_{k \geq 0} \frac{1}{p^{k(s-1)}} - \sum_{k \geq 1} \frac{1}{p^{ks-k+1}}\right) = \prod_p \left(\frac{1}{1 - p^{-(s-1)}} - \frac{p^{-s}}{1 - p^{-(s-1)}}\right) =$$

$$\prod_p \frac{1 - p^{-s}}{1 - p^{-(s-1)}} = \prod_p (1 - p^{-s}) \cdot \prod_p (1 - p^{-(s-1)})^{-1} =$$

$$(\zeta(s))^{-1} \cdot \zeta(s-1) = \frac{\zeta(s-1)}{\zeta(s)}.$$

□

(iii) **First Proof.** If we decide to use Problem 1(i), we set $f = \phi$ and $g = 1$. Since both these functions are multiplicative, so is $h(n) = \sum_{d|n} \phi(d)$. Therefore it is sufficient to show that $h(n) = n$, for n being a prime power, say p^k. Then

$$h(p^k) = \sum_{d|p^k} \phi(d) = \sum_{t=0}^{k} \phi(p^t) = 1 + \sum_{t=1}^{k} ((p^t - p^{t-1}) = p^k.$$

□

Second Proof. Use the facts that the Dirichlet's generating functions for the sequences $\{1\}$, $\{n\}$, and $\{\phi(n)\}$, are $\zeta(s)$, $\zeta(s-1)$ and $\Phi(s) = \frac{\zeta(s-1)}{\zeta(s)}$, respectively. The last fact was established in (ii). Since the product of the first and the third functions is the second one, the Dirichlet's convolution of $\{1\}$ and $\{\phi(n)\}$ is $\{n\}$. □

Third Proof. Each complex n-th root of unity is a d-th complex primitive root of unity for exactly one divisor d of n, and vice versa. But there are

exactly n complex roots of unity, and there are exactly $\phi(d)$ primitive d-th complex roots of unity.
The algebraic significance of this fact is that $x^n - 1$ is a product of all cyclotomic polynomials of degree d, $d|n$.

(iv) The fastest way is to use the Möbius Inversion Formula for $n = \sum_{d|n} \phi(d)$. Another way is to use the facts that the Dirichlet's generating functions for the sequences $\{n\}$, $\{\mu(n)\}$, and $\{\phi(n)\}$, are $\zeta(s-1)$, $(\zeta(s))^{-1}$ (proven in class), and $\Phi(s) = \frac{\zeta(s-1)}{\zeta(s)}$ (proven in part (ii)), respectively. □

Problem 21. *Prove (in any way you wish) the following recurrences for the number of derangements D_n:*

$$(i) \quad D_n = (n-1)(D_{n-1} + D_{n-2}), \quad D_1 = 0, D_2 = 1 \;\; n \geq 3.$$

$$(ii) \quad D_n = nD_{n-1} + (-1)^n, \quad D_1 = 0, \; n \geq 2.$$

(iii) Prove that D_n is even if and only if n is odd.

Solution. The fastest way (for both (i) and (ii)) is to use induction on n and the explicit formula for the number of derangements obtained via the PIE in class. (iii) follows immediately from (ii).
It is also clear that having one of (i) or (ii) proven, another follows by a simple induction.
For a proof of (ii) which makes use of the exponential generating function for D_n, namely $e^{-x}/(x-1)$, see H.S. Wilf, *generatingfunctionology*, solution to the Exercise 27 of Chapter 2.
A "combinatorial proof" of (i) can be obtained by partitioning all derangements on $[n]$ into two classes. The first class contains all derangements in which the element n belongs to a 2-cycle. The second class contains all derangements in which n belongs to a cycle of length at least three. Obviously the first class contains $(n-1)D_{n-2}$ permutations: D_{n-2} for each element $i \neq n$ such that (i,n) is the 2-cycle. Deletion of n from all permutations of the second class defines a $(n-1)$-to-1 function from the second class to the set of all derangements of $[n-1]$. Indeed, the obtained permutations will not contain a cycle of length 1 (a fixed point), so they are derangements of $[n-1]$. Inserting n in any cycle of length k, $k \geq 2$, of a derangement π of $[n-1]$ will result in k distinct permutations of the second class. Doing this for all cycles of π we get $n-1$ permutations of the second class. This proves (i). □

Problem 22. *Find a simple formula for the chromatic polynomial of C_n (a cycle with n vertices.)*

First Solution. Let e be an edge of C_n. Then $P_n = C_n - e$ is a path with n vertices, and C_n/e is C_{n-1} if $n > 3$, and P_2 if $n = 3$. Therefore, for $n > 3$, $P_{C_n}(\lambda) = P_{C_n - e}(\lambda) - P_{C_n/e}(\lambda) = P_{P_n}(\lambda) - P_{C_{n-1}}(\lambda) = \lambda(\lambda-1)^{n-1} - P_{C_{n-1}}(\lambda)$. This gives

$$P_{C_n}(\lambda) = \lambda(\lambda-1)^{n-1} - \lambda(\lambda-1)^{n-2} + \lambda(\lambda-1)^{n-3} - \cdots + (-1)^n\lambda(\lambda-1) =$$

$$\lambda(\lambda-1)\frac{(\lambda-1)^{n-1} + (-1)^n}{(\lambda-1)+1} =$$

$$(\lambda-1)^n + (-1)^n(\lambda-1).$$

□

Second Solution. According the Whitney Broken Circuit Theorem, for a connected graph G,

$$P_G(\lambda) = \sum_{i=0}^{n-1}(-1)^i a_i \lambda^{n-i},$$

where a_i is the number of i-subsets of $E(G)$ containing no broken circuits. There exists only one broken circuit of C_n having $n-1$ edges. Therefore, for $G = C_n$, $a_i = \binom{n}{i}$ for all $i = 0, 1, \ldots, n-2$, $a_{n-1} = \binom{n}{n-1} - 1 = n - 1$. Therefore

$$P_{C_n}(\lambda) = \sum_{i=0}^{n-2}(-1)^i\binom{n}{i}\lambda^{n-i} + (-1)^{n-1}(n-1)\lambda =$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\lambda^{n-i} - (-1)^{n-1}\lambda - (-1)^n = (\lambda-1)^n + (-1)^n(\lambda-1).$$ □

Problem 23. *(i) Describe all positive integers n such that $\phi(n)$ is a power of 2. Do you know why this question is of importance in algebra?*

(ii) Is it true that $\lim_{n\to\infty}\phi(n) = \infty$? Justify.

bf Solution. (i) Let $n = \prod_{i=1}^{s} {p_i}^{k_i}$ is the prime decomposition of n, where $p_1 < p_2 < \ldots < p_s$. Then

$$\phi(n) = \prod_{i=1}^{s}({p_i}^{k_i} - {p_i}^{k_i-1}) = \prod_{i=1}^{s}{p_i}^{k_i-1}(p_i - 1).$$

For $\phi(n)$ to be a power of 2, either $s = 1$ and $p_1 = 2$, or all $p_i - 1$ are powers of 2 and all $k_i = 1$ for $i > 1$. Therefore, for $i > 1$, each $p_i = 2^{t_i} + 1$ for some t_i.
For which t_i , $2^{t_i} + 1$ can be prime? If $t_i = ab$, where $a > 1, b > 1$, and a is an odd, then

$$(2^b + 1)|(2^{t_i} + 1) = 2^{ab} + 1 = (2^b + 1)((2^b)^{(a-1)} - (2^b)^{(a-2)} + \cdots - 2^b + 1).$$

Therefore t_i has no odd proper divisors, and hence must itself be a power of 2. Thus we have: $\phi(n)$ is a power of 2 if and only if n is a power of 2, or a product of a power of 2 and k distinct primes, each of the form $2^{2^m} + 1$:

$$n = 2^h p_1 p_2 \cdots p_k = 2^h (2^{2^m} + 1)(2^{2^{m_2}} + 1) \cdots (2^{m_k} + 1).$$

The algebraic significance of this result is that only for these values of n the Galois group of the polynomial $x^n - 1$ over $\mathbf{Q}$ is solvable, i.e., only for these n the unit circle can be divided into n congruent parts by means of a compass and a straightedge.(Gauss). $\square$

(ii) **First Proof.** For $n = p^k$, where p is an odd prime and an integer $k \geq 1$, one can check easily that $\phi(p^k) = p^{k-1}(p-1) > p^{k/2}$. Therefore, due to the multiplicative property,

$$\phi(n) \geq \sqrt{n} \quad \text{for all odd values of } n.$$

If n is even, then $n = 2^k m$, where an integer $k \geq 1$ and m is odd. Then $\phi(2^k m) = 2^{k-1}\phi(m) \geq \frac{1}{2} 2^{k/2}\sqrt{m} = \frac{1}{2}\sqrt{n}$, thus

$$\phi(n) \geq \frac{1}{2}\sqrt{n} \quad \text{for all even values of } n.$$

Therefore $\phi(n) \geq \frac{1}{2}\sqrt{n}$ for all $n \geq 1$, and $\lim_{n\to\infty} \phi(n) = \infty$. $\square$

Second Proof. Since $\phi(n) = n \prod_{p|n}(1 - 1/p)$, then

$$\log_2(\phi(n)) = \log_2(n) + \sum_{p|n} \log_2(1 - 1/p).$$

Since 2 may or may not be a factor of n, and n cannot have more than $\log_3 n$ factors,

$$\log_2(\phi(n)) = \log_2(n) + \sum_{p|n} \log_2(1 - 1/p) \geq$$

$$\log_2 n - 1 + \sum_{p|n, p\geq 3} \log_2(1 - 1/p) \geq \log_2 n - 1 + \log_3 n \cdot \log_2(2/3) =$$

$$\log_2 3 \log_3 n + \log_2 2/3 \log_3 n - 1 = (\log_2 3 + \log_2(2/3)) \log_3 n - 1 =$$

$$\log_3 n - 1 \to \infty, \quad \text{when} \;\; n \to \infty.$$

Since $\log_2 \phi(n)) \to \infty$, then $\phi(n)) \to \infty$. □

Remark. It is obvious that $\limsup_{n\to\infty} \phi(n)/n = 1$ (just take prime values of n). It is easy to show that $\lim_{n\to\infty} \phi(n)/n^\epsilon = \infty$ for all $\epsilon < 1$: just use the fact that the function $\phi(n)/n^\epsilon$ is multiplicative and check it for a prime power. Proofs of the following results can be found in G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford Science Publ., 1979:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi(i) = 6/\pi^2 \quad \text{– the average value of } \phi;$$

$$\liminf_{n\to\infty} \frac{\phi(n)}{\frac{n}{\ln \ln n}} = e^{-\gamma}, \quad \text{where } \gamma \text{ is the Euler's constant.}$$

Problem 24. *Prove that the number of different labeled trees T with vertices $v_1, v_2, \ldots, v_n$ and with $d_T(v_1) = k$ is $\binom{n-2}{k-1}(n-1)^{n-k-1}$, $n \geq 2$, $k \geq 1$.*

Solution #1. We know that the number of times a label of a vertex of a tree appears in its Prüfer code is one less than the degree of the vertex. Therefore the set of out trees is in bijective correspondence with the set of all codewords containing exactly $k-1$ v_1's. The remaining $(n-2)-(k-1) = n-k-1$ labels can be arbitrary, but not v_1. Therefore there are

$$\binom{n-2}{k-1}(n-1)^{n-k-1}, \; n \geq 2, \; k \geq 1.$$

□

Solution #2. We know that the number of trees T with vertices $v_1, \ldots, v_n$ and degrees $d(v_i) = d_i$, $i = 1, \ldots, n$, is $\frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!}$. Therefore there are

$$\sum_{\substack{k+d_2+\ldots+d_n=2(n-1)\\ d_i \geq 1}} \frac{(n-2)!}{(k-1)!(d_2-1)!\ldots(d_n-1)!}$$

trees with $d_1 = k$. This expression can be rewritten as

$$\sum_{\substack{s_2+\cdots+s_n=2n-2-k-(n-1)\\ =n-k-1}} \frac{(n-2)!}{(k-1)!s_2!\ldots s_n!},$$

where $s_i = d_i - 1 \geq 0$, or as

$$\binom{n-2}{k-1} \sum_{\substack{s_2+\cdots+s_n=n-k-1 \\ s_i \geq 0}} \frac{(n-k-1)!}{s_2! \dots s_n!} = \binom{n-2}{k-1} (\underbrace{1+1+\cdots+1}_{n-1})^{n-k-1}$$

$$= \binom{n-2}{k-1} (n-1)^{n-k-1}$$

Solution #3. (based on D. Chandler's solution.) Let $A(n,k)$ denote the set of all such trees.

Let $k \geq 2$, $T \in A(n,k)$, and $v_1 u \in E(T)$. Deleting this edge from T we create a forest with exactly two components. Let n_u denote the order of the component containing u. Connecting u to each vertex $w \neq v_1$ of the other component, we obtain $n-1-n_u$ trees T', such that $V(T) = V(T')$ and $deg_{T'}(v_1) = k-1$. Doing this for each neighbor u of v_1 in T, we obtain

$$\sum_{v_1 v_i \in E(T)} (n-1-n_i) = (k)(n-1) - \sum_{v_1 v_i \in E(T)} n_i = k(n-1)-(n-1) = (k-1)(n-1)$$

trees T'. Varying T over all $A(n,k)$ we can produce $(k-1)(n-1)|A(n,k)|$ such trees T', not all distinct.

How many times an arbitrary tree $T' \in A(n, k-1$ can be obtained this way? Choose a vertex u of T' distinct from v_1 and any neighbors of v_1 in T'. Let wu be the edge of T' on the (unique) $v_1 u$-path in T'. Deleting wu and connecting v_1 to u we obtain a tree in $T \in A(n,k)$, and T' can be obtained from T by the procedure described in the previous paragraph. Since there are $n-k$ ways to choose u, we get $(n-k)$ such T's. Therefore $(k-1)(n-1)|A(n,k)| = (n-k)|A(n,k-1)|$, or

$$|A(n,k)| = \frac{n-k}{(n-1)(k-1)} |A(n,k-1| =$$

$$\frac{(n-k)(n-k+1)}{(n-1)(k-1)(n-1)(k-2)} |A(n,k-2| = \cdots =$$

$$\frac{(n-k)(n-k+1)\cdots(n-2)}{(n-1)^{k-1}(k-1)!} |A(n,1| = \binom{n-2}{k-1} \frac{1}{(n-1)^{k-1}} |A(n,1)|$$

for all $n \geq 2$, and $k \geq 2$.

For $k=1$, there are $(n-1)^{n-3}$ labeled trees with vertex set $\{v_2, \dots, v_n\}$, and every tree $T \in A(n,1)$ can be obtained from each of them by joining v_1 to one of its vertices. Clearly such T is obtained this way only once.

Therefore, $|A(n,1)| = (n-1)\cdot(n-1)^{n-3} = (n-1)^{n-2}$. Hence $|A(n,k)| = \binom{n-2}{k-1}(n-1)^{n-k-1}, n \geq 2, k \geq 1$. $\square$

Problem 25. *Show that the number of spanning trees of a complete graph K_n which do not contain a fixed edge e of K_n is $(n-2)n^{n-3}$.*

Solution #1. From symmetry, each edge of K_n belongs to the same number of spanning trees of K_n. Let us call this number N. Then $\binom{n}{2}N$ is the total number of edges in all spanning trees of K_n. On the other hand this number is $(n-1)n^{n-2}$, since there are n^{n-2} spanning trees of K_n and each of them has $n-1$ edges. So $N = \frac{(n-1)n^{n-2}}{\binom{n}{2}} = 2n^{n-3}$, and for any edge $e \in E(K_n)$, the number of spanning trees of $K_n - e$ is $n^{n-2} - N = (n-2)n^{n-3}$. $\square$

Solution #2. (by J. Williford) Let $V(K_n) = \{v_1, \dots, v_n\}$. We may assume that $e = v_1v_2$, and we count the number N of spanning trees which contain the edge e. Let T be such a tree. Then $deg_T(v_1) + deg_T(v_2) = k+2$ for some integer $k \geq 0$. If $k = 0$, then $n = 2$, and the statements proven. Let $k > 1$. Given such a k, how many trees T have $deg_T(v_1) + deg_T(v_2) = k+2$? Call this number a_k.
Let T' be obtained from T by replacing the pair of vertices $\{v_1, v_2\}$ by a new vertex u and joining u to all neighbors of v_1 and v_2 in T (i.e., $T' = T/e$). Then T' is a tree of order $n-1 \geq 2$ and $deg_{T'}(u) = k \geq 1$. It is clear that $T \mapsto T'$ defines a 2^k-to-1 map from the set of trees on $\{v_1, \dots, v_n\}$ with $deg_T(v_1) + deg_T(v_2) = k+2$, to the set of trees on $\{u, v_3, \dots, v_n\}$ with $deg_{T'}(u) = k$, since every vertex v_i, $3 \leq i \leq n$, of T can be connected to either v_1 or v_2, but not both.
Using Problem 1, we conclude that there are $\binom{n-3}{k-1}(n-2)^{n-k-2}$ trees T'. Hence, there are $a_k = 2^k\binom{n-3}{k-1}(n-2)^{n-k-2}$, and

$$N = \sum_{k=1}^{n-2} a_k = \sum_{k=1}^{n-2} 2^k \binom{n-3}{k-1}(n-2)^{n-k-2} = 2\sum_{i=0}^{n-3}\binom{n-3}{i}(n-2)^{n-2-i}2^i =$$

$$2((n-2)+2)^{n-3} = 2n^{n-3}.$$

Therefore there are $n^{n-2} - N = n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$ spanning trees of K_n not containing e. $\square$

Solution #3. (F. Fiedler's with help from N.H. Abel.) Let $V(K_n) = \{v_1, \dots, v_n\}$. We may assume that $e = v_1v_2$, and we count the number N of spanning trees which contain the edge e. Each such tree T can be built via

the following construction in only one way:
Step 1: For every m, $0 \le m \le n-2$, choose an m-subset of vertices C of $V(K_n) \setminus \{v_1, v_2\}$;
Step 2: consider a pair of arbitrary trees T_1 and T_2 with $V(T_1) = C \cup \{v_1\}$, and $V(T_2) = V(K_n) \setminus V(T_1)$;
Step 3: join T_1 and T_2 by e. Therefore

$$N = \sum_{m=0}^{n-2} \binom{n-2}{m} (m+1)^{m-1} (n-m-1)^{n-m-3}. \qquad (*)$$

To simplify this sum we use the following identity of N.H. Abel, which is a deeper generalization of the binomial theorem:
For all x, y, z,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k}.$$

This great identity is not easy to prove. See, e.g., a proof by Lucas (was distributed in class). Another proof, from Lovász's *"Combinatorial Problems and Exercises"*, 1979, Problem 44(a) (was distributed in class), can be obtained from Lucas' by specialization $z := -1, y := y+n$, or independently as in Lovász's book The identity from the Problem 44(c) follows from 44(a) via 44 (b) fast. For $n \ge 2$, the sum (*) that we arrived to is obtained from 44(c) by trivial transformations, and it is $2n^{n-3}$. Therefore there are $n^{n-2} - N = n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$ spanning trees of K_n not containing e. □

Comment. Using any solution of our Problem 2 independent of Abel's identities, we can **prove** that (*) is equal to $2n^{n-3}$, and therefore the Abel's formula given in Problem 44(c) in Lovász's book.

Solution #4. We use the Kirchoff's Matrix-Tree Theorem. Let $V(K_n) = \{v_1, \ldots, v_n\}$ and $e = v_1v_2$. Then the corresponding $n \times n$ matrix $D - A$ is

$$\begin{pmatrix} n-2 & 0 & -1 & -1 & \cdots & -1 \\ 0 & n-2 & -1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \ldots & \ldots & -1 & n-1 \end{pmatrix}$$

The $(1,1)$–cofactor of this matrix is

$$(-1)^{1+1}\begin{vmatrix} n-2 & -1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & n-1 \end{vmatrix}$$

Subtract the first row from each other then add to the first column the product or $\frac{n-1}{n}$ and the ith column, for $i = 2, 3, \ldots, n-1$. This leads us to the determinant

$$\begin{vmatrix} \frac{n-2}{n} & -1 & -1 & \cdots & -1 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \cdots & n \end{vmatrix} = \frac{n-2}{n} n^{n-2} = (n-2)n^{n-3}$$

■

Solution #5. We obtain this result from much stronger statements. References can be found in Berge "Graph & Hypergraphs", pp. 45, 46 and Ref. List.

Proposition 1. (Moon [1967]). Let $V(K_n) = X_1 \dot\cup \ldots \dot\cup X_p$ be a partition of vertices of K_n, and let T_i be a tree with the vertex set X_i and some edge set E_i, $i = 1, \ldots, p$. Then the number of spanning trees of K_n that contain all T_i's as subgraphs is $n_1 n_2 \ldots n_p n^{p-2}$, where $n_i = |X_i|$.

Proof. If each set X_i is contracted to a unique vertex a_i, then, the number of trees $\overline{T}$ with $d_{\overline{T}}(a_i) = d_i$, $i = 1, 2, \ldots, p$, is

$$\binom{p-2}{d_1 - 1, d_2 - 1, \ldots, d_p - 1}$$

To each $\overline{T}$ correspond exactly $(n_1)^{d_1}(n_2)^{d_2} \ldots (n_p)^{d_p}$ different spanning trees of K_n (explain it!!!)

Hence, the number of spanning trees of K_n that contain T_i's as subgraphs is equal to

$$\sum_{\substack{d_i \geq 1 \\ d_1 + \ldots + d_p = 2(p-1)}} \binom{p-2}{d_1 - 1, d_2 - 1, \ldots, d_p - 1} n_1^{d_1} n_2^{d_2} \ldots n_p^{d_p} \underset{s_i = d_i - 1}{=}$$

$$\left[\sum_{\substack{s_i \geq 0 \\ s_1+\ldots+s_p=p-2}} \binom{p-2}{s_1, s_2, \ldots, s_p} n_1^{s_1} \ldots n_p^{s_p}\right] n_1 n_2 \ldots n_p =$$

$$n_1 n_2 \ldots n_p (n_1 + \ldots + n_p)^{p-2} = n_1 n_2 \ldots n_p n^{p-2}.$$

□

We can use Proposition 1 to solve our problem. Take $n_1 = \ldots = n_{n-2} = 1, n_{n-1} = 2$, i.e., $T_1 \cong \ldots \cong T_{n-2} \cong K_1$ (one vertex, no edges), $T_{n-1} \cong K_2$ (o—o). Then the number of spanning trees of K_n containing T_i's as subgraphs $= 1 \cdot 1 \cdot \ldots \cdot 2 \cdot n^{(n-1)-2} = 2n^{n-3}$. Therefore the number of spanning trees of K_n which do not contain $T_{n-1} = n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$. (Each spanning tree must contain $T_1 = \{V_1\}, \ldots, T_n = \{V_n\}$). □

The following statement generalizes the result of Problem 2.

Proposition 2. (Temperley [1964]). Let $E \subseteq E(K_n), V = V(K_n), T(V, E)$ be the number of different trees on V that do not contain any edge of E. For any subset of edges F of K_n by (V, F) we denote the graph with vertex set V and edge set F. Suppose (V, F) has p components of cardinalities $n_1, n_2, \ldots, n_p$ and

$$\nu(F) = \begin{cases} 0, & if graph (\text{V,F}) contains a cycle \\ n_1 n_2 \ldots n_p, & otherwise. \end{cases}$$

Then the number of different trees on set V that do not contain any edge in E is

$$T(V, E) = n^{n-2} \sum_{F \subseteq E} \nu(F) \left(\frac{-1}{n}\right)^{|F|} .$$

Proof. If $e \in E$, let $A_e =$ the set of all trees on V which contain edge e. For any $F \subseteq E$, if (V, F) has no cycles and p components, then by Proposition 1, $\left|\bigcap_{e \in F} A_e\right| = \nu(F) n^{p-2}$. Since each component of (V, F) is a tree (i.e., (V, F) is a forest), $p = n - |F|$. Hence

$$\left|\bigcap_{e \in F} A_e\right| = \nu(F) n^{n-|F|-2}$$

If (V,F) contains a cycle, the above formula is still valid, since both sides of the equality are zero. By Incl.–Excl. formula $T(V,E) = \left|\bigcap_{e\in E} \overline{A}_e\right| = \sum_{F\subseteq E} (-1)^{|F|} \left|\bigcap_{e\in F} A_e\right| = \sum_{F\subseteq E} (-1)^{|F|}\nu(F) n^{n-2-|F|} = n^{n-2} \sum_{F\subseteq E} \nu(F)\left(\frac{-1}{n}\right)^{|F|}$.

□

Of course, Proposition 2 can be used to solve Problem 2 too:
$T(V,\{e\}) = n^{n-2}\left(\nu(\emptyset)\left(\frac{-1}{n}\right)^0 - (-1)\nu(\{e\})\left(\frac{-1}{n}\right)^1\right) =$
$= n^{n-2}\left(1 + \underbrace{1\cdot 1\cdot \ldots \cdot 1}_{n-2}\cdot 2\left(\frac{-1}{n}\right)\right) = n^{n-2}\left(1-\frac{2}{n}\right) = (n-2)n^{n-3}$. □

Problem 26. *What is the average number of spanning trees that simple labeled graphs on n vertices have? The result can be written in a simple form with no summations in it.*

Solution. Let $\tau(G)$ be the number of spanning trees of a graph G, and let $g(T)$ be the number of simple graphs having tree T as a spanning tree. Then the average number of spanning trees in simple graphs on n vertices is

$$\frac{\sum_G \tau(G)}{2^{\binom{n}{2}}} = \frac{\sum_T g(T)}{2^{\binom{n}{2}}}.$$

The function $g(T)$ is easy to compute and it doesn't depend on T! Indeed: any graph G containing T as a spanning tree is completely defined by a choice of edges which are not in T. $|E(T)| = n-1$. There are $\binom{n}{2} - (n-1)$ possible edges on $V(T)$ which are not in T. Therefore $g(T) = 2^{\binom{n}{2}-(n-1)}$. Since there are n^{n-2} distinct trees on n vertices, then the average is

$$\frac{n^{n-2}\cdot 2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \frac{n^{n-2}}{2^{n-1}}.$$

□

Problem 27. *Suggest a reasonable way of counting the number of connected labeled graphs with n vertices and m edges. Use it to count the number for $n = 10$ and $m = 17$.*

Solution. Let us denote the number we are interested in by $c(n, m)$. We assume $c(x, y) = 0$ for all $y < x - 1$ or $y > \binom{x}{2}$. We know that $c(n, n-1) = n^{n-2}$ by Cayley's Theorem, since all connected graphs of order n and size $n-1$ are trees. We will first count the number N of all disconnected graphs G of order n and size m. Let v_1 be a vertex of G.
Each such graph G can be built via the following procedure in only one way:
Step 1: For every m, $0 \leq i \leq n-2$, choose an i-subset of vertices C all distinct from v_1;
Step 2: consider a pair of graphs (G_1, G_2), where G_1 is an arbitrary connected graph with $V(G_1) = C \cup \{v_1\}$ and $|E(G_1)| = j \leq m$, and G_2 is an arbitrary graph with $V(G_2) = V(G) \backslash V(G_1)$ and $|E(G_2)| = m - j$. Therefore

$$N = \sum_{i=0}^{n-2} \binom{n-1}{i} \sum_{j=0}^{m} c(i+1, j) \binom{\binom{n-i-1}{2}}{m-j}.$$

Therefore, omitting all zero addends, we have $c(n, m) = \binom{\binom{n}{2}}{m} - N$, or

$$c(n, m) = \binom{\binom{n}{2}}{m} - \sum_{i=0}^{n-2} \binom{n-1}{i} \sum_{j=i}^{\min\{\binom{i+1}{2}, m\}} c(i+1, j) \binom{\binom{n-i-1}{2}}{m-j}.$$

Using a computer program, we get $c(10, 17) = 1,016,662,746,825$. □
Several other formulae for $c(n, m)$ can be derived. For example:
(D. Chandler)

$$c(n, m) = \frac{1}{2m} \sum_{i=1}^{n-1} i(n-i) \binom{n}{i} \sum_{j=0}^{m-1} c(n-i, j) c(i, m-1-j) + \frac{1}{m} \left(\binom{n}{2} - m + 1 \right) c(n, m-1),$$

with obvious initial conditions.

Problem 28. *(i) <u>The Exchange Lemma:</u> Prove that: Given two spanning trees S and T of a connected graph G and $s \in E(S) \setminus E(T)$, there exists an edge $t \in E(T) \setminus E(S)$ such that both $S + t - s$ and $T + s - t$ are spanning trees of G.*

(ii) Let S and T be two distinct MST's of a connected graph G and s, t be a pair of edges described in the Exchange Lemma. Prove that s and t have equal weight and both $S + t - s$ and $T + s - t$ are MST's (i.e., spanning trees of minimum weight).

(iii) The weight sequence of a weighted graph G consists of weights of edges in G, arranged in non-decreasing order. Let S and T be two distinct MST's of a connected weighted graph G. Prove that S and T have the same weight sequence. (This fact was first pointed out to me by David Kravitz, who noticed this property).

Proof. (i) Let $s = xy \in E(S) \setminus E(T)$, and let X, Y be the components of $S - s$, where $x \in X$ and $y \in Y$. Since T is a spanning tree of G, there exists a unique x, y-path P_T in T. This path must contain an edge $t = ab$, where $a \in V(X)$ and $b \in V(Y)$. At the same time there exists a unique a, b-path P_S in S and the only edge from this path connecting a vertex from X with a vertex from Y is $s = xy$. Hence $t \notin E(S)$, $P_T \cup \{s\}$ is the unique cycle in $T + s$, and $P_S \cup \{t\}$ is the unique cycle in $S + t$. Therefore both $S + t - s$ and $T + s - t$ are spanning trees of G. □
(ii) Let w be the weight (or cost) function. If $w(s) \neq w(t)$, then one of the spanning trees $S + t - s$ or $T + s - t$ has weight smaller than $w(S) = w(T)$ – a contradiction with S and T being minimum. So $w(s) = w(t)$ and both trees $S + t - s$ and $T + s - t$ are MST's. □
(iii) The Exchange Lemma guarantees the existence of a sequence of exchanges transforming S into T. By part (ii), each tree along the way is an MST and the weight sequence never changes. □

Problem 29. *Please do not use Cameron's solutions while doing this problem.*

(i) Show that the smallest number of transpositions of $[n]$ whose product is an n-cycle is $n - 1$.

(ii) Prove that any n-cycle can be expressed as a product of $n - 1$ transpositions in n^{n-2} different ways.

Proof. It is clearly sufficient to prove both statements for the cycle $\sigma = (12 \ldots n)$, since the group of inner automorphisms of S_n acts transitively on the sets of all cycles of equal lengths.
(i) Let $\sigma = \tau_1 \cdots \tau_m$, and let G be a simple graph with $V(G) = [n]$ and $E(G) = \{xy \ : (xy) = \tau_j,\ j \in [m]\}$. If H is a component of G, then $\sigma(V(H)) = V(H)$. But the cyclic group $< \sigma >$ acts transitively on $[n]$. Therefore $H = G$. Connectedness of G implies $m = |E(G)| \geq n - 1$. □
(ii) See the solution by P. Cameron.

Problem 30. *(i) Let H be a finite bipartite graph with maximum degree $\Delta = \Delta(H)$. Show that there exists a finite bipartite Δ-regular graph G such that H is a subgraph of G.*

(ii) Prove that the edge-chromatic number of a bipartite graph H is $\Delta = \Delta(H)$.

Solution 1. (i) If $\Delta = 0$, there is nothing to prove: $G = H$. Suppose $\Delta \geq 1$. Let A and B denote the color classes of H, $|A| \leq |B|$. By adding $|B| - |A|$ isolated vertices to A, we get a bipartite graph G' with the properties:

(i) color classes A' and B of G' have the same cardinality $|B|$;

(ii) G' contains H as a subgraph;

(iii) $\Delta(G') = \Delta(H) = \Delta$.

Let G be a bipartite graph satisfying (1)-(3) with the greatest number of edges. If G is Δ-regular, the proof is finished. If not, then

$$|E(G)| = \sum_{a \in A'} deg_G a = \sum_{b \in B} deg_G b < \Delta n.$$

Then there exists $a \in A'$ and $b \in B$ both having degrees less than Δ. Joining them[1] by an edge, we obtain a bipartite graph G' satisfying (1)–(3) and having 1 more edge than G, a contradiction. □

Comment. It is an interesting question what is the smallest order of a simple regular graph containing a given graph H as a subgraph? For an answer, see P. Erdős and P. Kelly: The Minimal Regular Graph Containing a Given Graph. Amer. Math. Monthly 70 (1963), no. 10, 1074–1075.

(ii) We know that $\chi'(G) = \chi'(K_{\Delta,\Delta}) = \Delta$, since its edge set is a union of Δ edge-disjoint perfect matchings (a corollary of Hall's marriage theorem). Hence, using part (i), $\chi'(H) \leq \Delta$. Since $\Delta(H) = \Delta$, then $\chi'(H) \geq \Delta$. Therefore $\chi'(H) = \Delta$. □

Solution 2 (sketch). (i) Several other very clever constructions were found. Their main idea was to embed H into a graph of the same maximum degree

[1] Here we assume that our definition of a graph allows multiple edges. The whole theory can be extended to multigraphs almost without any change in the proofs of main theorems. I thank to Jathan Austin and Bobby DeMarco for pointing to this.

but greater minimum degree. There are several ways to do it. For example, it can be started by embedding H into disjoint union of two isomorphic copies of H, and introducing a matching between the vertices of minimum degrees which belong to corresponding color classes of the two copies. The obtained graph contains H as a subgraph, has the same maximum degree as H but its minimum degree is 1 greater. Then this construction is repeated, starting with the last constructed graph, until minimum degree becomes equal Δ. It leads to a graph having, in general, many more vertices than $2|B|$ (as in solution 1). □

Problem 31. *For each $k > 1$ find an example of a k–regular simple graph that has no perfect matching.*

Solution. For k even, the complete graph K_{k+1} provides such an example, since it has an odd number of vertices. For $k = 2n + 1$ (odd), an example can be constructed in the following steps.

(i) Consider a complete bipartite graph $K_{2n,2n}$ with color classes A and B.

(ii) Partition all $2n$ vertices of A into n pairs and join vertices of each pair.

(iii) Introduce a new vertex v and connect it to all $2n$ vertices of B. Denote the graph obtained in (1) – (3) by H.

(iv) Let H_i, $i \in [k]$, be a set of graphs each isomorphic to H. Denote by v_i, $i \in [k]$, their vertices which correspond to vertex v of H. Let H' be the disjoint union of all H_i.

(v) Introduce a new vertex u and connect it to all v_i's in H'. Call the obtained graph G.

It is obvious that G is k-regular. G has no perfect matching, since deletion of its vertex u disconnects the graph into more than 1 odd components (an easy part of Tutte's Theorem). □

Problem 32. *A 2-factor of a graph G is a 2-regular spanning subgraph of G, and G is 2-factorable if it is a union of its edge-disjoint 2-factors. Prove that every $2k$-regular graph G , $k \geq 1$, is 2-factorable.*
(Hint: each component of G is an Eulerian graph.)

Solution. Clearly it is sufficient to prove the statement for a component H of G. Since H is connected and every degree of H is even, then H is Eulerian. Let S be a sequence of vertices of H which corresponds to a closed Eulerian trail of H when it is traversed starting at some vertex. Every vertex of H appears in S exactly $2k$ times, since H is $2k$-regular. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. Consider a bipartite graph T with color classes $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ and edges defined in the following way: $x_i y_j \in E(T)$ if and only if v_i immediately precedes v_j somewhere in S. Then T is k-regular, since the closed Eulerian trail 'leaves' every vertex of H as many times as it 'enters' it. By a corollary from Hall's marriage theorem, (or by Problem 1 (ii) from this homework), T is 1-factorable, i.e. $E(T)$ is a disjoint union of k perfect matchings of T: $M_1, M_2, \ldots, M_n$. Let M be one of these matchings. If $x_i y_j \in M$, then $i \neq j$. Since M saturates all vertices of T, there exists $x_j y_k \in M$. Can $i = k$? No, since it will contradict the definition of a trail in a (simple) graph: it would imply that the edge $v_i v_j$ of G is traversed in the closed Eulerian trail twice. Let E_M be the set of edges of G which 'define' edges in M. Then the graph induced in G by E_M is 2-regular and spanning. So it is a union of vertex disjoint cycles of G, or it is a 2-factor of G. Since all E_{M_i}'s partition $E(G)$, uniting 2-factors defined by all E_{M_i}'s gives the required 2-factorization of G. □

Problem 33. *Two people play a game on a simple graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots$ such that, for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.*

Solution. Call the first player A and the second B.
Let G have a perfect matching M. Whatever vertex u of G A chooses, B chooses a vertex v, such that $uv \in M$. Since vertices cannot be repeated during the game, B always can make a move, and therefore always wins playing this way. Thus A has no winning strategy.
Let G have no perfect matching and let M be a maximum matching. By Berge's theorem, G contains no M-augmenting path. Let v_1 be M-unsaturated vertex (it exists since M is not perfect). A starts with v_1. If v_1 is an isolated vertex, A wins. Otherwise, B chooses a vertex v_2. If v_2 is M-unsaturated, we got an M-augmenting path v_1, v_2, which cannot happen. Therefore A can chose a vertex v_3 such that $v_2 v_3 \in M$. If B cannot continue, A wins the game. If B can continue by choosing v_4, then v_4 is M-saturated: otherwise $v_1 v_2 v_3 v_4$ would be an M-augmenting path. Then A can chose v_5 such that $v_4 v_5 \in M$. Continuing playing this way, A obviously wins, since if B can

make a move the chosen vertex must be M-saturated and A chooses the second vertex of the corresponding (new!) edge from M. □

Problem 34. *(König – Birkhoff – von Neumann Theorem) Let A be an $n \times n$ matrix of nonnegative real entries. Suppose that the row and column sums of A are all equal. We want to show that A can be written in the form*

$$A = \lambda_1 P_1 + \ldots\ldots + \lambda_m P_m$$

where the P's are permutation matrices and $\lambda_1, \ldots\ldots, \lambda_m$ are nonnegative real numbers. (A permutation matrix is an $n \times n$ matrix with entries $= 0$ or 1 only and which has exactly one 1 in each row and in each column.)

The proof is by induction on $w =$ the number of nonzero entries of A. Show the following:

(a) If $A \neq 0$, then $w \geq n$.

(b) If $w = n$, then $A = \lambda P$ where P is a permutation matrix and $\lambda > 0$.

(c) Show that the following is impossible: "all of the positive entries of A in a certain set of k rows lie in a certain set of $k-1$ columns".

(d) For each $i = 1, 2, \ldots, n$, let S_i be the set of columns of A such that $a_{ij} > 0$. Show that $\{S_1, \ldots, S_n\}$ satisfies Hall's condition, and therefore has a SDR.

(e) Show that we can subtract a positive scalar multiple of a certain permutation matrix from A, and the result will still have nonnegative entries and constant row and column sums.

(f) Complete the proof of the theorem stated above.

Solution. In what follows, the term 'line' means either a row or a column of A, and s denotes the common value of line sums of A. We also note that if $w = 0$, the statement is correct: $A =$ zero matrix $= 0 \cdot P$.

(a) If $w < n$, then there should be a line of A containing all zero entries. This implies that $s = 0$, hence A is zero matrix.

(b) If $w = n$, then either there exists a line of all zeros (and we continue as in part (a)), or each line contains only one nonzero entry. Since the line sums are all equal, these nonzero entries are all equal s. Then $A = s \cdot P$ and the theorem is proven.

(c) If this happens, then the sum of all entries of these k rows is ks. The sum of all entries of the $k-1$ columns is $(k-1)s$, but this sum must be *at least* as large as the first since it contains all its terms and the extra terms are nonnegative. A contradiction.

(d) Let K be a subset of $[n]$, $|K| = k$. If $|\bigcup_{i\in K} S_i| < k$, then we got k rows with the property that the set of all their positive entries lie in less than k columns. This is impossible by part (c). Therefore $|\bigcup_{i\in K} S_i| \geq k$, Hall's condition for $\{S_1, \dots, S_n\}$ is satisfied, and this family has a SDR.

(e) Let $\{i_t : t \in [n], i_t \in S_t$ be a SDR for $\{S_1, \dots, S_n\}$ which existence was proven in (d). Then all n entries a_{t,i_t} are positive and no two of this entries appear in one line of A. Let μ be the smallest of these numbers, and let P be the permutation matrix having 1's in positions (t, i_t). Then $A - \mu P$ has all its entries nonnegative and the line sums of $A - \mu P$ are all equal $s - \mu$.

(f) The matrix $A - \mu P$ has at least one less positive entry as A. By inductive hypothesis, $A - \mu P = \lambda_1 P_1 + \dots\dots + \lambda_m P_m$, all $\lambda_i \geq 0$, and therefore $A = \lambda_1 P_1 + \dots\dots + \lambda_m P_m + \mu P$. □

Problem 35. *A sequence of polynomials, called Chebyshev polynomials, satisfies the recurrence*

$$T_{n+1}(x) = (2x)T_n(x) - T_{n-1}(x), \quad n \geq 1; \quad T_0(x) = 1; \quad T_1(x) = x.$$

(i) Write out $T_n(x)$ for $n = 2, 3, 4, 5$.

(ii) Find the explicit formula for the generating function $\sum_{n\geq 0} T_n(x)t^n$.

(iii) Expand the generating function in partial fractions.

(iv) Use the result in part (iii) to obtain an explicit formula for $T_n(x)$.

Just answers. (i) $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$.

(ii) $f(t) = \frac{1-xt}{t^2-2xt+1}$.

(iii) $f(t) = \frac{A(x)}{t-t_1} + \frac{B(x)}{t-t_2}$, where $t_1 = x + \sqrt{x^2-1}$, $t_2 = x - \sqrt{x^2-1}$, $A(x) = (-x - \sqrt{x^2-1})/2$, and $B(x) = (-x + \sqrt{x^2-1})/2$.

(iv) $T_n(x) = \frac{1}{2}(t_1^n + t_2^n)$. □

Problem 36. *One deals out a deck of 52 cards, faced up, into a* 4×13 *array. Then one tries to select 13 cards, one from each column, in such a way as to get one card of each denomination (but not necessarily of the same suit.) What is the probability that it is possible?*

Solution. The probability is 1, i.e., it is always possible. One can apply Ph. Hall's theorem on SDR's in order to prove this. Consider thirteen sets, each set is the set of distinct denominations appearing in the ith column. The cardinality of each such set is at most four. If the union of some k of these sets contained less than k elements, than the total number of cards in the corresponding columns would be less than $4k$, which is impossible. Therefore the condition of Hall's theorem for this set system is satisfied and a system of distinct representatives exists. □

Problem 37. *Use an algorithm described in the proof of the Max-flow Min-Cut Theorem to find an* (u, v)*- integral flow of maximum value in the network below. The numbers on directed edges represent their capacities. Please, do not use 'ad hoc' approach. Find also a cut of minimum capacity.*

Solution. The solution of this problem is a straightforward application of the algorithm. The diagram will be provided later.

Problem 38. *(i) In how many permutations of* $[n]$ *the element 1 precedes the element 2 which precedes the element 3 (not necessarily immediately). For* $n = 6$*, examples of such permutations are (142365), (516243), (456123), etc.*

(ii) 10 distinct books are placed on 4 distinct shelves. Two placements are considered different if they differ by a content of at least one shelf or by an order of books in a shelf. Some shelves can be empty. How many different placements are there?

(iii) How many simple labeled graphs on n *vertices have no isolated vertices (i.e. vertices of degree zero)? If your answer involves a summation you do not have to simplify it. Check the correctness of your answer for* $n = 3$*. Compute numerical value of your answer for* $n = 10$*.*

Solution. (i) It is clear that for each ordering τ of $\{1, 2, 3\}$, the number of permutations of $[n]$ where 1,2,3 form τ does not depend on τ. Therefore each ordering τ appears in $n!/3!$ permutations. □

(ii) The first book can be placed in 4 ways. The second in 5 ways, since if it is on the same shelf as the first book, then placing it on the right or on the left of it leads to distinct placements of all books. Similarly the third book can be placed in 6 ways and so on. This leads to $4 \times 5 \times \ldots \times 12 \times 13$ placements.

Another way to count is the following. Consider 10 distinct books and 3 identical vertical bars. Permutation of this multiset is in bijective correspondence with our book placements: the segment of a permutation on the left of the leftmost bar is the placement of books on the first shelf, the segment of this permutation between the leftmost bar the second bar (from the left) is the placement of books on the second shelf, and so on. The number of such permutations is $\frac{(10+3)!}{3!1!1!\cdots 1!} = 13!/3! = 4 \times 5 \times \ldots \times 12 \times 13$. □

(iii) Let $[n]$ represent the set of vertices, and let A_i, $i \in [n]$, be the set of graphs in which vertex i is isolated. For any $I \subset [n]$, $|\bigcap_{i \in I} A_i| = 2^{\binom{n-|I|}{2}}$, since each such graph is uniquely determined by its induced subgraph on $[n] \backslash I$. Hence by the Inclusion-Exclusion Principle, the number of our graphs is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{\binom{n-k}{2}}.$$

For $n = 3$, this number is 4, and one can check it buy listing all the graphs: three trees on [3] and C_3. For $n = 10$, the number is 34,509,011,894,545. It was obtained by executing this line of Maple:

$$add((-1)^k * binomial(10,k) * 2^{binomial(10-k,2)}, k = 0..10);$$

□

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