# A Tale of a Happy Marriage.

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Newton's *'Principia* does not use the language of Calculus explicitly. Very little trace of calculus techniques is to be found in the *Principia.*

Since the beginning of the 18th century to the end of the 19th, mathematicians are referred to as "geometers".

Calculus was often referred to as Geometry of Curves and Surfaces.

Often Largange was called the geometer par excellence, though in the preface to his *Mechanique analytique* (1788), he wrote that "One will not find any Figures in this Work. The methods that I expound here do not require [...] geometrical reasonings".

## My Methodological Opinion:

Every mathematics course should ENHANCE the overall mathematical strength of students. My observation that often it does not happen. Students forget almost as much as they learn in the course.

A solution is to make it an important goal of teaching, to revisit the past, review important techniques and facts, and add new ones.

I claim that often this can be done without taking too much time from a new course we teach. But it does require both the teacher and the students to work harder

# Plan for today

- Geometry is married to Algebra by the coordinate method. Maybe this is “the royal way to Geometry” nowdays. Understanding this connection deeper is useful.
- Algebra or Calculus can help Geometry, and vice versa.
- Supporting Examples.

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- An ideal program.
- The program we have, and some thoughts on how it can be improved.

How important is to be good in solving problems of Elementary Geometry at the end of the college? For math majors, for engineers?

What does Elementary Geometry cover? Polygons, circles, their elements, areas, extremal problems? conics?

A list of suggested facts for the beginning of Calculus that will allow students to do (plane) geometry. They can be established by using facts about parallel lines and similarity of triangles.

- On a line: Given $A : (x_A)$, $B : (x_B)$, $x_A \leq x_B$. Then
$$AB = x_B - x_A$$
- Given $A : (x_A)$, $B : (x_B)$, $C$ on line $AB$, $C \neq B$, $k \geq 0$ and and $CA/CB = k$. Find $x_C$.

$$x_C = \frac{a + kb}{1 + k}, \text{ for } C \in \overline{AB} \quad \text{or}$$
$$x_C = \frac{a - kb}{1 - k}, \text{ for } C \notin \overline{AB} \quad (k \neq 1)$$

- On a plane: Given $A:(x_A, y_A)$, $B:(x_B, y_B)$, and $k \geq 0$. $C$ on line $AB$, $C \neq B$, and $CA/CB = k$. Find $(x_C, y_C)$.

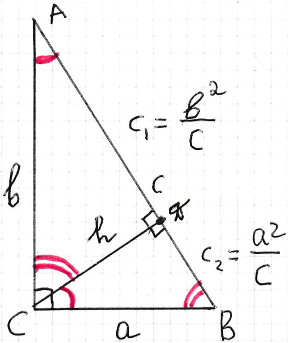
$$(x_C, y_C) = \left(\frac{x_A + kx_B}{1+k}, \frac{y_A + ky_B}{1+k}\right) \text{ for } C \in \overline{AB} \quad \text{or}$$

$$(x_C, y_C) = \left(\frac{x_A - kx_B}{1-k}, \frac{y_A - ky_B}{1-k}\right) \text{ for } C \notin \overline{AB} \quad (k \neq 1)$$

- Equations of lines (general axes): standard form, point-slope form (non-vertical lines), slope-intercept, $xy$-intersept $(x/a + y/b = 1,\ ab \neq 0)$.
- Lines are parallel $\Leftrightarrow$ $m_1 = m_2$ or both vertical.
- Cartesian coordinate system: distance formula, equation of a circle, $m = \tan\theta$ for non-vertical lines.

  Pythagorean Theorem is the must! Otherwise later students will think that it follows from the distance formula or from the identity $\cos^2 t + \sin^2 t = 1$.

# Generalized Pythagorean Theorem




$$c = c_1 + c_2 = \frac{a^2 + b^2}{c}$$

$$c^2 = a^2 + b^2$$

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$$K_1 = \frac{b}{c} \qquad K_2 = \frac{a}{c}$$

$$\boxed{K_1^2 + K_2^2 = 1}$$

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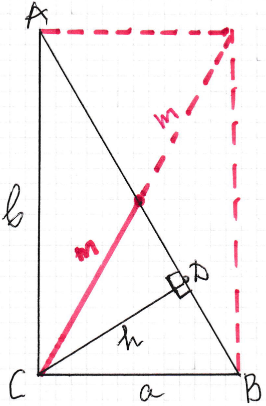
LET $X$ BE THE LENGTH OF A SEGMENT IN $\Delta ABC$

THEN $X_1 = K_1 X$ AND $X_2 = K_2 X$

BE THE LENGTHS OF THE CORRESP. SEGMENTS IN SMALLER TRIANGLES. THEN

$$\boxed{X^2 = X_1^2 + X_2^2}$$

# AGM




$AD = c_1$

$BD = c_2$ $\quad c = c_1 + c_2 = 2m$

So $m = \frac{1}{2}(c_1 + c_2)$

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$$\frac{h}{BD} = \frac{AD}{h} \rightarrow h = \sqrt{c_1 c_2}$$

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HENCE,

$$\sqrt{c_1 c_2} \leq \frac{c_1 + c_2}{2}$$

$$(h \leq m)$$

- For non-perpendicular lines with slopes $m_1, m_2$ that form angle of measure $\alpha$ $(0 \leq \alpha < \pi/2)$,

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

  Non-vertical lines are perpendicular $\Leftrightarrow$ $m_1 m_2 = -1$ or both vertical.

- The distance $d(P, l)$ from point $P : (x_P, y_P)$ to line $l : ax + by + c = 0$ is given by

$$d(P, l) = \frac{|\, ax_P + by_P + c \,|}{\sqrt{a^2 + b^2}}$$

75 minutes is more than sufficient to do all this. Good handouts with proofs and examples help.

Spending time for introducing students to Mathematica, or Maple, or any CAS is highly recommended. 30 minutes in class and a handout explaining how to access it on line, how to see what the commands are, and 20 examples of using the commands will do.

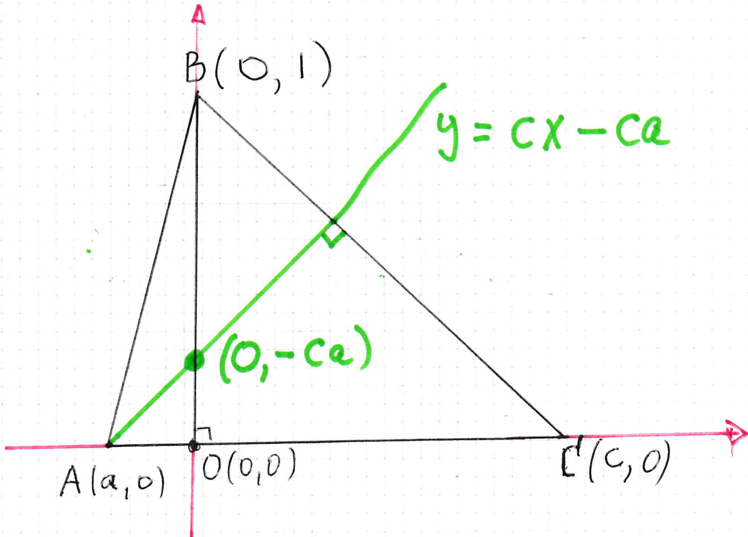
Here are some of the homework problems that I have in mind. Assume the students have seen numeric examples for the basic facts.

<u>**Problem 1:**</u> *Consider* $A : (4, 2)$, $B : (-1, 3)$, $C : (-1, 1)$. *The lines that contain altitudes of the triangle meet at one point.*

<u>Plan of a solution of Problem 1.</u>

1. Write an equation of the line $h_A$ through $A$ perpendicular to the line $BC$. Do the same for lines $h_B$ and $h_C$ passing through other two vertices $B$ and $C$ perpendicular to the corresponding opposite sides.
2. Find the coordinates of the point of intersection $H$ of lines $h_A$ and $h_B$.
3. Check that $H$ is on line $h_C$. This demonstrated that the lines that contain altitudes of a triangle meet at one point.
4. If you first find the intersection of two other lines, say $h_A$ and $h_C$, will you necessarily obtain the same point $H$? Justify your answer.

**Problem 2:** *Given an arbitrary triangle $\triangle ABC$. Prove that the lines that contain altitudes of the triangle meet at one point.*



Plan of a solution of Problem 2.

0. Can you solve it without using the coordinate method? Think about it, or find such a solution on the web, and understand it. If you cannot, ask me.

1. Suppose we wish to use the coordinate method. First we have to introduce a coordinate system. It is clear that it may better be Cartesian. Do you think that the validity of the statement you have to prove depends on the way you introduce the coordinate system?

Do you think the difficulty of the proof depends on the way we introduce our Cartesian coordinate system?

With vertices labelled as in Problem 1, make line $AC$ to be $x$-axis, line $h_B$ – $y$-axis, their intersection, call it $O$, to be the origin, and chose the $y$ coordinate of $B$ be 1. Then $A : (a, 0)$, $B : (0, 1)$, and $C : (c, 0)$, for some numbers $a$ and $b$. If $a = 0$ or $c = 0$, $\triangle ABC$ is a right triangle, and the statement is obvious. So $a \neq 0$ and $c \neq 0$.

2. Write an equation of the line $h_A$ through $A$ perpendicular to the line $BC$. Look at its $y$-intercept. The equation is $y = c(x - a) = cx - ca$ so $y$-intercept is $ca$.

Do the same for line $h_C$. Do we have to? No. No, it is clearly, $ac$.

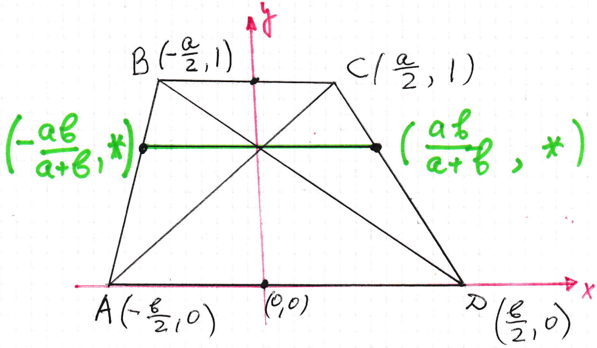
3. As $h_A$ and $h_C$ intersect $h_B : x = 0$ at the same point, the statement is proven.

| Our proof of the general statement is EASIER !!! |
|---|

For Math 243: Is the similar statement true for any tetrahedron?

For Math 349: Solutions with dot product of vectors (coordinate free). State an analog of this statement for a simplex in $\mathbb{R}^n$, $n \geq 2$. Is it correct?

**Problem 3.** *Given a trapezoid with bases of length a and b. A line parallel to the bases is drawn through the intersection of its diagonals. Find the length of the segment of this line cut out by the lateral sides of the trapezoid.*



## System of equations revisited.

1. *Solve the following system:* $2x + 3y = 11, \quad 4x - 2y = -2.$

Solution. Multiplying both sides of the first equation by 2, and of the second by 3, and adding we get $16x = 16$, or $x = 1$, and, subsequently $y = 3$. $\square$

Geometrically, we claim that the the pairs of lines

$$2x + 3y = 11, \quad 4x - 2y = -2$$

and

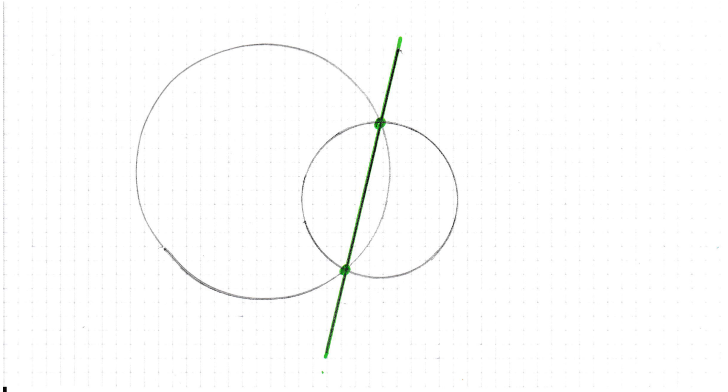
$$2x + 3y = 11, \quad x = 1$$

have the same intersection.

2. *Find an equation of the line passing through the points of intersection of two circles given respectively by the equations*

$$x^2 + (y-3)^2 = 9 \quad \textit{and} \quad (x-4)^2 + (y-2)^2 = 16.$$

Solution. Expanding squares and subtracting equations we obtain $4x - y = 2$. This is the answer!!! □



**Theorem.** *Let $F_1(x, y)$ and $F_2(x, y)$ be algebraic expressions. Let $\alpha, \beta$ be real numbers and $\beta \neq 0$. Then the following systems are equivalent.*

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0$$

$$F_1(x, y) = 0, \qquad \alpha F_1(x, y) + \beta F_2(x, y) = 0$$

3. *Take three circles in the plane such that every two of them intersect at two points. Prove that three common chords are concurrent.*

Solution. Let $F_i(x, y) = (x - a_i)^2 + (y - b_i)^2 = r_i^2$, i=1,2,3, be equations of the circles.
The lines

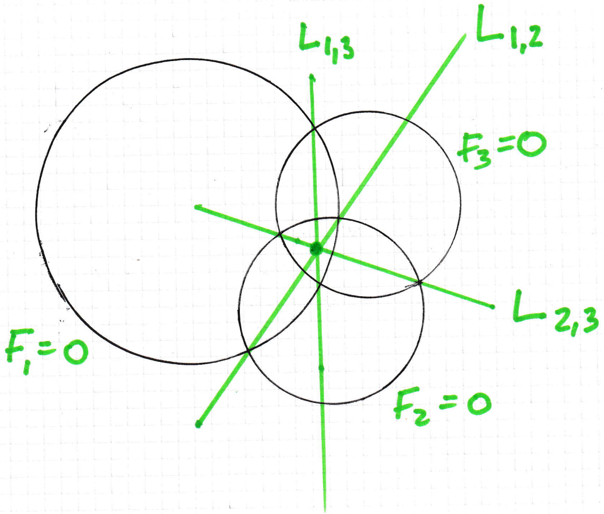
$$L_{1,2} : F_1 - F_2 = 0 \quad \text{and} \quad L_{2,3} : F_2 - F_3 = 0$$

intersect at the same point as lines

$$L_{1,2} : F_1 - F_2 = 0 \quad \text{and} \quad L_{1,3} : F_1 - F_3 = 0,$$

since

$$1 \cdot (F_1 - F_2) + 1 \cdot (F_2 - F_3) = F_1 - F_3. \quad \square$$


$L_{1,3}$
$L_{1,2}$
$F_3=0$
$L_{2,3}$
$F_1=0$
$F_2=0$

4. *Two parabolas with perpendicular axes intersect at four points. Prove that the four points lie on a circle.*

Solution. There is a coordinate system such that the equation nof the first parabola is

$$F_1(x,y): \qquad y-(a_1x^2+b_1x+c_1)=0$$

and

$$F_2(x,y): \qquad x-(a_2y^2+b_2y+c_2)=0,$$

where $a_1, a_2 \neq 0$. The system

$$F_1=0, \qquad F_2=0$$

is equivalent to

$$F_1=0, \qquad a_2F_1+a_1F_2=0.$$

As the second equation is of the form

$$Ax^2+Ay^2+Bx+Cy+D=0,$$

where $A=a_1a_2\neq 0$. Since it has solutions, it is of a circle. $\square$.

# Linear algebra and geometry

Suppose $A$ is a $n \times n$ nonsingular matrix, $n = 2, 3$, $\overrightarrow{b} \in \mathbb{R}^n$
Consider a transformation of $\mathbb{R}^n \to \mathbb{R}^n$ given by

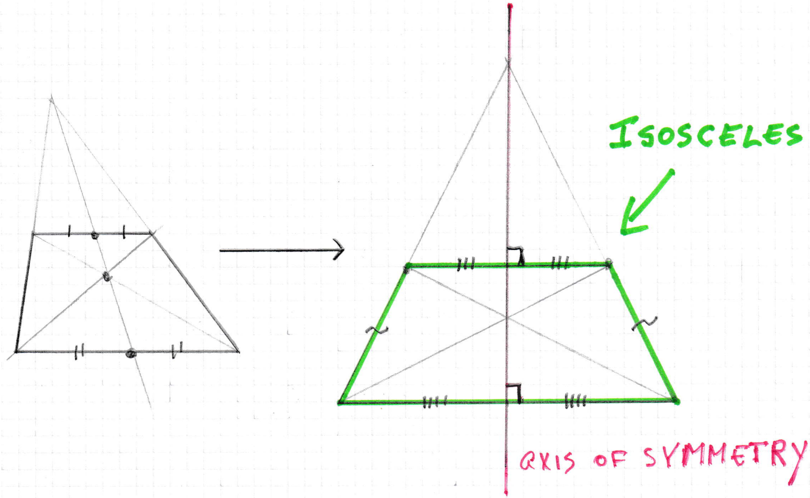
$$\overrightarrow{x} \mapsto A\overrightarrow{x} + \overrightarrow{b}$$

Properties (invariants): (can be done with proofs in 50 or 75 minutes)

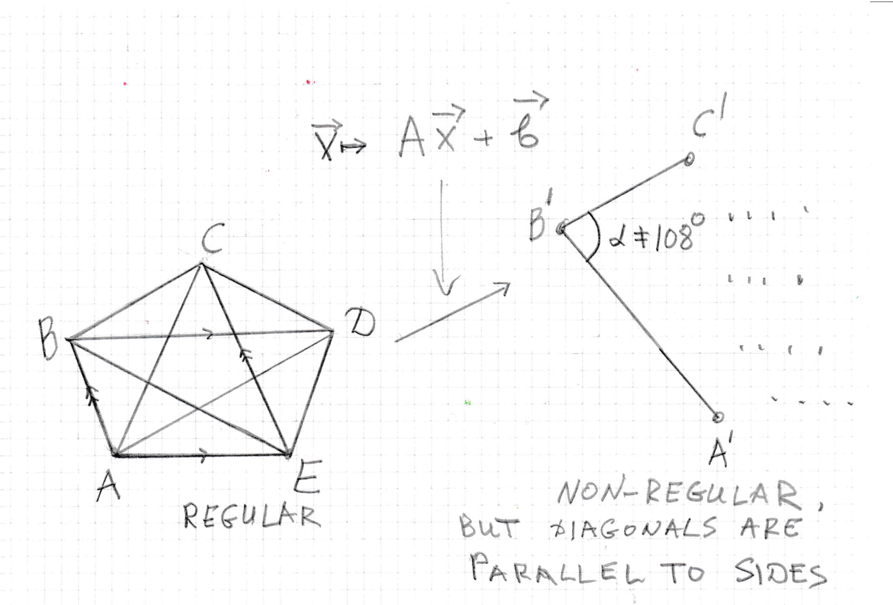
- Lines are mapped to lines, planes to planes. Property of lines (planes) being parallel is preserved.
- Ratio of lengths of parallel segments is preserved (so a midpoint of a segment is mapped to the midpoint of the image segment)
- Ratio of areas (volumes) of figures is preserved (all areas/volumes get multiplied by $|\det A|$)
- Any triangle (tetrahedron) can be mapped into a triangle (tetrahedron) similar to a given one.

Geometry was THE MAIN source for the development of the notion of a vector space and Linear algebra in general.

1. *In any trapezoid the points of intersection of lateral sides, the midpoints of the bases, and the point of intersection of the diagonals are collinear.*



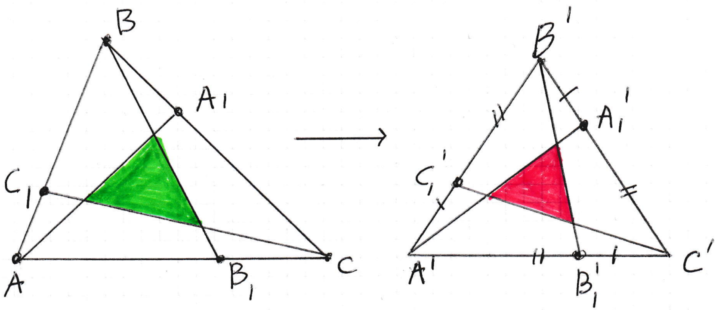
2. *In a regular pentagon each diagonal is parallel to a side of the pentagon. Can it still hold for a non-regular convex pentagon?*



3. *Given a $\triangle ABC$ of area 1. Points $A_1, B_1, C_1$ are taken on sides $\overline{BC}, \overline{CA}, \overline{AB}$, respectively, so that*

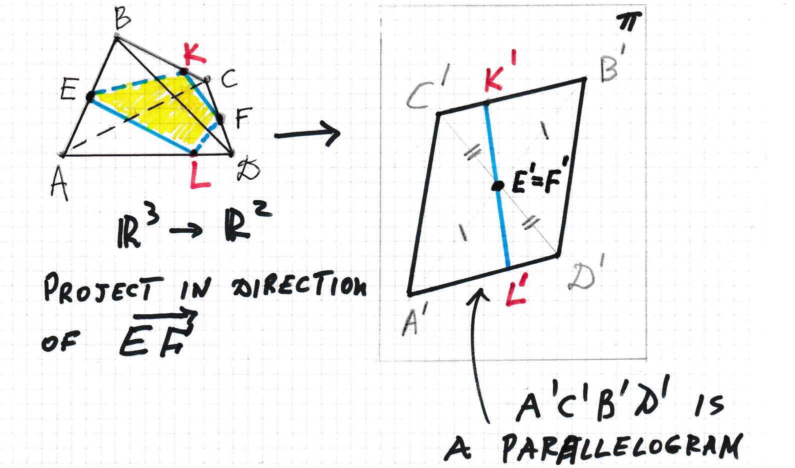
$$BA_1/A_1C = CB_1/B_1A = AC_1/C_1B = 2.$$

*Find the area of the triangle formed by intersections of lines $AA_1$, $BB_1$, and $CC_1$.*

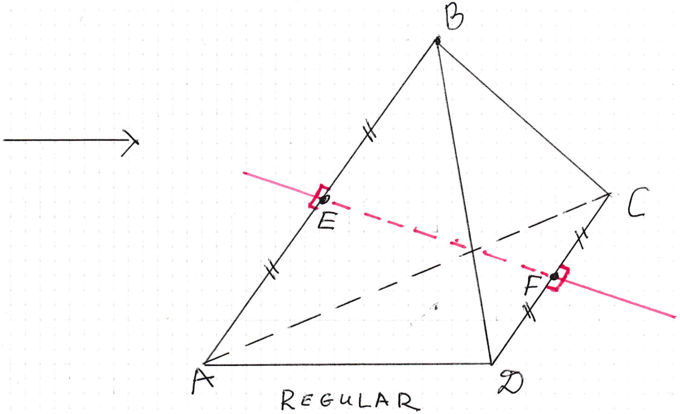


4. *Given a tetrahedron ABCD. Let points E and F be the midpoints of the sides AB and CD.*

*(i) Show that plane that passes through E and F divides the two sides it intersects in the same ratio.*



(ii) *Show that plane that passes through E and F divides the tetrahedron in two polyhedra of equal volume.*

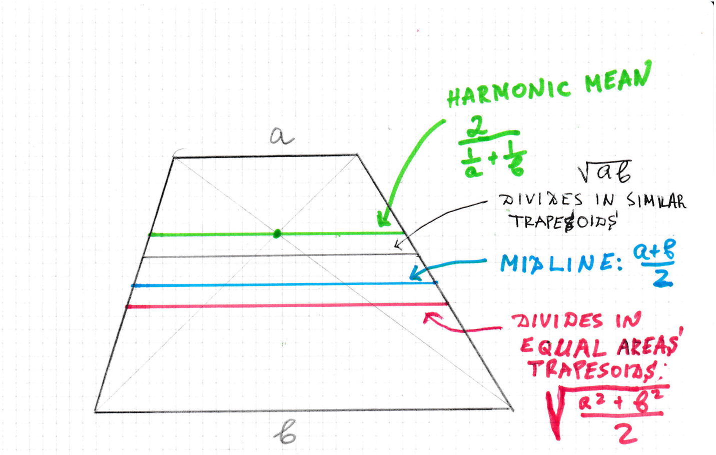


# Geometry and classical inequalities

**Theorem.** *Suppose* $0 \leq a \leq b$. *Then*

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}},$$

*with each equality holding if and only if* $a = b$.



Most of the optimization problems in our Calculus courses follow from them.

Generalization for 3 numbers, i.e., $\sqrt[3]{abc} \leq (a+b+c)/3$ comes from

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) =$$

$$(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2]/2 \geq 0.$$

Generalization from $n = 2$ to $n = 4$ is trivial. E.g.,

$$\sqrt[4]{abcd} = \sqrt{\sqrt{ab}\sqrt{cd}} \leq \sqrt{((a+b)/2)\cdot((c+d)/2)} \leq$$

$$((a+b)/2 + (c+d)/2)/2 = (a+b+c+d)/4.$$

In order to generalize these inequalities to $n$ non-negative numbers, it is possible to prove Jensen's Inequality for convex (up or down) functions by induction. Then use it for functions $f = 1/x, \; \ln x, \; x^2$.

**Problem.** *Prove that* $(1 + 1/n)^n < (1 + 1/(n+1))^{n+1}, n \geq 1$. Take $x_1 = x_2 = \cdots = x_n = 1/n$ and $x_{n+1} = 1/(n+1)$ and use

$$\sqrt[n+1]{x_1 x_2 \cdots x_n x_{n+1}} < (x_1 + x_2 + \cdots + x_n + x_{n+1})/(n+1)$$

This implies that compound interest gives more when the number of compoundings increases.

Try to do it with Calculus, by showing that $f(x) = (1 + 1/x)^x$ increases on $([1, \infty)$. Not too easy.

Most problems in Calculus (especially Optimization problems) can be solved by using these classical inequalities.

$$ax^2 + bx + c = a(x + b/2a)^2 - (b^2 - 4ac)/(4a)$$

$$x(5 - x) \leq [(x + (5 - x)]/2]^2 \leq 25/4$$

$$x(5 - 4x) \leq \frac{1}{4}[4x(5 - 4x)] \leq \frac{1}{4}(5/2)^2$$

$$6x + \frac{1}{x^2} = 3x + 3x + 1/x^2 \geq \cdots$$

$$6x^2 + 1/x = 6x^2 + 1/(2x) + 1/(2x) \geq \cdots$$

$$\sqrt{2 + x} + \sqrt{10 - x} \leq 2\sqrt{((2 + x) + (10 - x))/2} = 2\sqrt{6}$$

$$ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3(abc)^{2/3}$$

$$a + b + c \geq 3\sqrt[3]{abc}$$

The last two problems allow to solve optimization problems for surface areas, perimeters, and volume of a rectangulat parallelepiped without study of Calculus of several variables.

Some sources:

1. O. Byer, F. Lazebnik, D. Smeltzer: *Methods for Euclidean Geometry*, Second edition, Dover Publ., 2021.

2. I.M. Yaglom, *Geometric Transformations I, II, III, IV*, The MAA. 1960 – 1970.