ON SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

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The following exercise appears in [1, page 129, #16]: Can $y = \sin(x^2)$ be a solution on an interval containing $x = 0$ of an equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

with continuous coefficients?

The answer is negative, and an easy way to see it is to use the following theorem:

THEOREM ([1, p.122]): *Consider the initial value problem:*

$$y'' + py' + qy = g(x), \;\; y(x_0) = y_0, \;\; y'(x_0) = y'_0, \tag{2}$$

where p, q, and g are continuous on an open interval I containing x_0. Then there is exactly one solution $y = \phi(x)$ of this problem, and the solution exists throughout the interval I.

To apply this theorem to the exercise, we observe that $\sin(x^2)|_{x=0} = (\sin(x^2))'|_{x=0} = 0$. Therefore, if $\sin(x^2)$ is a solution of (1), then it is a solution of (2) with $g(x) = 0$ on I and $x_0 = y_0 = y'_0 = 0$. At the same time the constant zero function on I is a solution of (2) with the same initial values. Then by the uniqueness part of the theorem, $\sin(x^2)$ cannot be a solution of (2), and therefore of (1).

The question we asked at this point is: which function can be solutions of (1)? More precisely: for which functions f with continuous second derivative on I there exist continuous on I functions p, q, such that f is a solution of (1)?

Repeating the argument above we conclude that if $f(x_0) = f'(x_0) = 0$ for some $x_0 \in I$, and f is not the constant zero function on I, then f cannot be a solution of (1). Therefore a necessary condition on a solution f of (1) is that both f and f' have no common zero on I, of equivalently,

$$f^2 + (f')^2 \neq 0 \;\text{ on }\; I. \tag{3}$$

QUESTION: *Is condition (3) sufficient for f with continuous second derivative on I to be a solution of (1) for some p, q continuous on I?*

The answer is yes. In two particular cases, namely, $f \neq 0$ on I or $f' \neq 0$ on I, it can be obtained immediately: just take $p = 0, q = -f''/f$ or $p = -f''/f', q = 0$, respectively. A solution of the general case is given below. It is elementary, constructive, and short, but it was not easy to find.

This note is motivated by the question and the solution. We believe that it is probably discussed in the literature, but we failed to find a reference. Conversations with our colleagues confirmed our feeling that whatever the status is, this very basic question and its answer deserve attention of a wider audience and can be of interest to many instructors of differential equation courses.

THEOREM: *A function f with continuous second derivative on I is a solution of (1) for some p, q continuous on I if and only if f and f' have no common zero on I.*

Proof. We have already discussed the necessity of the condition. Let us show that it is also sufficient. Since $f^2 + (f')^2 \neq 0$ on I, the function

$$F = \frac{f^2 + f'^2}{f^4 + f'^2}$$

is continuous and positive on I. Since $F + 1/F \geq 2$, then

$$(F + 1/F)(1 + (f')^2) + f' > 0.$$

Therefore function $g = [(F + 1/F)(1 + (f')^2) + f']^{-1}$ is positive and continuously differentiable on I, and

$$g(F + 1/F)(1 + (f')^2) + gf' = 1. \tag{4}$$

It is easy to see (just rewrite F and g in terms of f and f') that

$$g(F + 1/F)(1 + (f')^2) = \frac{a_1 f^2 + a_2 (f')^2}{b_1 f^2 + b_2 (f')^2},$$

where a_1, a_2, b_1, b_2 are polynomials in f and f' and the denominator is positive on I. Therefore the derivative of the left hand side of (4) can be written in the form $uf + vf' + gf''$ where u and v are rational functions of f, f', f'' with positive (on I)

denominators. Thus we have $uf + vf' + gf'' = 0$. Dividing both sides by g ($g > 0$ on I), we obtain $f'' + pf' + qf = 0$, with both p and q continuous on I. ■

REMARK 1. A more natural approach to prove the sufficiency part of the theorem could be the following. Starting with f, show existence of a function h, having continuous second derivative on I, such that the Wronskian $W = fh' - f'h$ does not vanish on I. Then p and q could be easily found by solving the system:

$$f'' + pf' + qf = 0, \; h'' + ph' + qh = 0.$$

We did not see an elementary way to prove the existence of such h. Of course, it follows from the solution above and the fact that the solution space of (1) is two dimensional.

REMARK 2. The problem and the solution can be easily generalized. A function f with continuous n-th derivative on I, $n \geq 1$, is a solution of

$$f^{(n)} + p_1 f^{(n-1)} + \cdots + p_n f^{(0)} = 0$$

for some $p_1, \ldots, p_n$ continuous on I if and only if $f^{(n-1)}, \ldots, f^{(0)} = f$ have no common zero on I.

A simpler solution of the problem appeared in the corresponding solution section of the American Mathematical Monthly.

Acknowledgements. The authors are grateful to Dr. L. Berezansky and to several colleagues from their Department for discussions relative to this note.

REFERENCES

1. W.E. Boyce, R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, Fifth Edition, John Wiley & Sons, Inc., New York, 1996.