In Elliptic World I.

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UD Discrete Mathematics/Algebra Seminar,
December 2, 2021.

I do not think that I am the only one who can enjoy a song in foreign language without understanding words or even without knowing what it is about. That happened when I listened for the first time to *Beatles*, or *ABBA*, or songs in French or Spanish.

Similarly, with reading mathematics that I understand very poorly. I can do it for hours and enjoy it. That was exactly what happened to me for years when I read mathematics I plan to discussed in this talk. Finally, I decided to understand more, and I wish to share what I understood with you.

One difficulty in understanding the origins of terms/notions like elliptic integrals, elliptic functions, elliptic curves (which surprisingly are not ellipses), theta functions, $\wp$-function, modular functions and others, is that the order of their treatments in different sources follow various permutations. This is a lesser problem if one wants to understand the related mathematics (just stick to your favorite source), but answering questions about their history and motivations behind their appearances was a hard task for me. But that was **what I wanted understand the most**.

Sometimes I strongly felt like inventing my own history of a notion, which, in my mind had more meaning, but I fought the temptation.

Needless to say that understanding related mathematics was hard enough, and often I stopped, and took some claims on faith. By no means I claim understanding this mathematics well.

On the other hand, I was surprised to learn that something that I thought was hard, actually was not. Especially what was done in the 18th and 19th centuries, often amounted to just algebraic manipulations. Sometimes ingenious and sometimes natural, sometimes long, but always within a possibility of checking them using a CAS in seconds.

I was also surprised with the ease that mathematicians in 18th and early 19th century used functions of complex variables, and their readiness to introduce new functions and to use them. Often it was done by treating them as functions of two real variables. But most modern treatments use the language and the notions from Complex Analysis that appeared later.

We know sin, cos and other trigonometric functions for so long that they seem "very natural" to us. It is, of course, just a habit.

First they appear as ratios of sides in a right triangle. Then we look at them as coordinates of points of unit circle $x^2 + y^2 = 1$ that correspond to some angle θ... For small angles we recovered the equivalence with the triangle definition, and extended the terms to all angles (thinking about them as measures of rotations). Having the **addition formula** for sin, we can compute all the values (ignore the signs):

$$\sin(\alpha + \beta) = \sin\alpha\sqrt{1 - \sin^2\beta} + \sin\beta\sqrt{1 - \sin^2\alpha}.$$

In the language of Euclidean geometry this is a theorem by **C. Ptolemy (100- 170 AD)** about sides and diagonals of an inscribed quadrangle.

Note that one does not need cos in the addition formula, and all values of cos, tan, cot, sec and cosec functions can be obtained via the addition formula with using special values of sin, like $\sin 0 = \sin\pi = 0$, or $\sin\pi/2 = 1$, and many developed formuli relating these functions.

Here we see in mathematics the phenomena that have been taken place in people's languages: the unnecessary expansion of the language.
Recent emigrants can get by with very few words in English. And *Elochka-The Cannibal* – one of the characters in a famous Russian satirical novel by Ilf and Petrov *"Twelve chairs"* – used only 30 words, and never felt that her vocabulary was limited.
Well, William Shakespeare used more than 20,000 words, and about 1,700 words were introduced to English language by him!!!
Later we got used to think about trigonometric functions as of "polynomials of infinite degree", i.e., as power series and infinite products:

$$\sin x = x - x^3/3! + x^5/5! - \dots \quad \textit{or} \quad \sin x = x \prod_{n \geq 1} \left(1 - \frac{x^2}{\pi^2 n^2}\right)$$

And even a little before that, we learned that $y = \sin x$ can be thought as the **inverse function** for $y \in [0, \pi/2]$ of

$$x = \int_0^y \frac{dt}{\sqrt{1 - t^2}}.$$

Elliptic Integrals.

In Calculus, using the the general formula for the **arc length** C of the curve $y = f(x)$ on $[x_1, x_2]$, or its parametric version, we have

$$C = \int_{x_1}^{x_2} \sqrt{1 + [dy/dx]^2}\, dx = \int_{t_1}^{t_2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

Consider some classical curves, like an **ellipse** or a **hyperbola**, or a **lemniscate**. All of these curves can be defined as loci of points in the plane. The computation of their arc length lead to indefinite integrals that cannot be expressed in terms of values of "elementary functions". Like for the circle, the inverses of these intergrals can be used to define new functions:

$$x = \int_0^y \frac{dt}{\sqrt{1-t^2}}, \qquad x = \int_0^y \frac{dt}{\sqrt{1+t^2}}, \qquad x = \int_0^y \frac{dt}{\sqrt{1-t^4}}.$$

$$y = \sin x, \qquad y = \sinh x, \qquad y = \operatorname{sinl} x$$

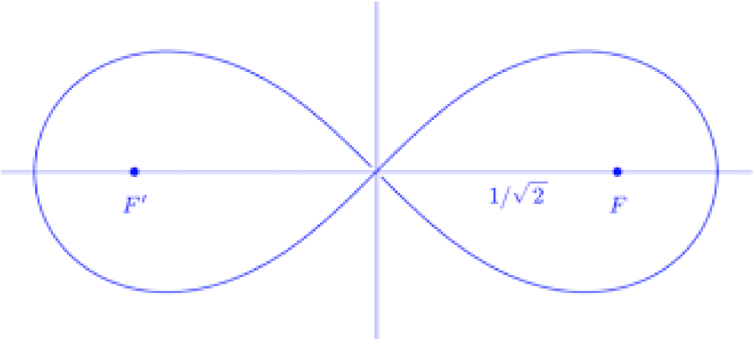
Lemniscate of J. Bernoulli (1654 - 1705) is a plane curve defined from two given points F and F', known as foci, at distance $2c$ from each other as the locus of points P so that

$$PF \times PF' = c^2.$$

In Latin *lemniscatus* = *"decorated with ribbons"*.

If $F' = (-1/\sqrt{2}, 0)$ and $F = (1/\sqrt{2}, 0)$, and $c = 1/\sqrt{2}$, we have

$$(x^2 + y^2)^2 = x^2 - y^2, \quad r^2 = \cos 2\theta$$



$(x^2+y^2)^2 = x^2 - y^2$ can be parameterized as

$$(x(r), y(r)) = \left(\sqrt{\frac{1}{2}r^2(1+r^2)}, \sqrt{\frac{1}{2}r^2(1-r^2)}\right)$$

with $r \in [0,1]$ for a quarter of the lemniscate. So the arc length from the origin to a point $(x(r), y(r))$ is

$$\int_0^r \sqrt{x'(t)^2 + y'(t)^2}\, dt =$$

$$\int_0^r \sqrt{\frac{(1+2t^2)^2}{2(1+t^2)} + \frac{(1-2t^2)^2}{2(1-t^2)}}\, dt =$$

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} \quad (=: \quad \operatorname{arcsinl} r.)$$

G. Fagnano (1682 - 1766): Addition Law of Lemniscate:

$$\int_0^\alpha \frac{dx}{\sqrt{1-x^4}} + \int_0^\beta \frac{dx}{\sqrt{1-x^4}} = \int_0^\gamma \frac{dx}{\sqrt{1-x^4}},$$

where

$$\gamma = \frac{\alpha\sqrt{1-\beta^4} + \beta\sqrt{1-\alpha^4}}{1+\alpha^2\beta^2}.$$

Note the resemblance with

$$\int_0^\alpha \frac{dx}{\sqrt{1-x^2}} + \int_0^\beta \frac{dx}{\sqrt{1-x^2}} = \int_0^\gamma \frac{dx}{\sqrt{1-x^2}},$$

which gives

$$\gamma = \alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2},$$

if we know

$$\arcsin\alpha + \arcsin\beta = \arcsin\gamma.$$

For **L. Euler (1707 – 1783)**, the interest in Fagnano's result was two-fold: (i) addition formuli and (ii) expressing the antiderivative in terms of elementary function.

(i) Addition for Generalized Sine: If $P(x) = 1 + mx^2 + nx^4$, then

$$\int_0^{\alpha} \frac{dx}{\sqrt{P(x)}} + \int_0^{\beta} \frac{dx}{\sqrt{P(x)}} = \int_0^{\gamma} \frac{dx}{\sqrt{P(x)}},$$

where

$$\gamma = \frac{\alpha\sqrt{P(\beta)} + \beta\sqrt{P(\alpha)}}{1 - n\alpha^2\beta^2}.$$

For $m = -1, n = 0$, we get the addition for sin.
For $m = 1, n = 0$, we get the addition for sinh.
For $m = 0$ and $n = -1$, we the addition for sinl, i.e. the Fagnano's result.

Denote the integrals by u, v, w, and s be the inverse function for $t = \int_0^\omega dz/\sqrt{P(z)}$, so $s(t) = \omega$. Then $s(u) = \alpha$, $s(v) = \beta$, and $s(w) = \alpha + \beta$. Assume $w = u + v$ and w fixed. Then $d(w) = 0$, or

$$\frac{d\alpha}{\sqrt{P(\alpha)}} + \frac{d\beta}{\sqrt{P(\beta)}} = 0.$$

We show that there exists a polynomial $F(\alpha, \beta)$ such that $F(\alpha, 0) = \alpha$, $F(0, \beta) = \beta$, and

$$d(F) = \Phi(\alpha, \beta)\Big(\frac{d\alpha}{\sqrt{P(\alpha)}} + \frac{d\beta}{\sqrt{P(\beta)}}\Big) = 0.$$

Euler used the method of undetermined coefficients and found F in the form

$$F(\alpha, \beta) = \alpha^2 + \beta^2 + A\alpha^2\beta^2 + 2B\alpha\beta - C^2 = 0,$$

and showed that A, B, C can be found in terms of m, n.

(ii) An observation by **J. Bernulli (1704)**:

$$\sqrt{1 - x^4}$$

cannot be rationalized by substitution $x = f(t)$ for any rational function f with rational coefficients. Bernulli knew about the result of Fermat:

$$X^4 - Y^4 = Z^2$$

has no integer solutions with $XYZ \neq 0$. In his life, Fermat only left one proof in relation to number theory, and exactly for this equation. And here he used his method of infinite descent. Fermat actually tried to solve this problem:

No right triangle with integral sides has area that is an integral square

Rationalizing the integral could lead to rational solutions of $z^2 = 1 - x^4$.

Euler's results influenced **A.-M. Legendre (1752 – 1833)**.

Louis Legendre (1752 - 1797): Not Adrien.

Euler's results influenced **A.-M. Legendre (1752 – 1833)**.

!!!Adrien-Marie Legendre!!!

An elliptic integral is an integral of the form

$$\int R(x, \sqrt{G(x)})\, dx,$$

where $R(x, y)$ is a rational function of two variables, and $G(x)$ is a polynomial of degree 3 or 4 without multiple roots.

If the degree of G is 1 or 2, or G has a multiple root, then $\int R(x, y)\, dx$, $y = \sqrt{G(x)}$, can be expressed in terms of elementary functions.
His main result is similar to the main theorem about integrals of rational functions:

Theorem (Legendre): *The elliptic integral I can be represented as a linear combination of: a rational function in x and y, an integral of a rational function of x, and of the integrals*

$$\int \frac{dx}{y}, \quad \int \frac{xdx}{y}, \quad \int \frac{x^2 dx}{y}, \quad \text{and} \quad \int \frac{dx}{(x-c)y}.$$

In the case where

$$y^2 = G(x) = (1 - x^2)(1 - k^2 x^2),$$

the first two integrals in Legendre's theorem can be transformed using the change of variables to $x = \sin\phi$ to

$$F(\phi) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \quad \text{and} \quad E(\phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 t}\, dt,$$

the first and the second form of Legendre's integrals.

Application to pendulum of length L, $k = \sin\phi_0/2$:

$$\frac{d^2\phi}{d\,t^2} - \frac{g}{L}\sin\phi = 0$$

$$T = 4\sqrt{\frac{L}{g}}\, F(\pi/2).$$

Elliptic functions.

Let G is a polynomial of degree 3 or 4 with no multiple roots.
In 1820s, **N. Abel (1802 - 1829)** and **C. Jacobi (1804 - 1852)** realized that inverting elliptic integrals over $\mathbb{C}$,

$$u = g^{-1}(z) = \int_0^z \frac{dt}{\sqrt{G(t)}},$$

i.e., using $z = g(u)$ can lead to a new type of very interesting functions, in particular **doubly periodic meromorphic functions**.

Let us begin with a simpler case: $u = \int_0^z dt/\sqrt{1-t^2}$ should lead to a periodic function $z = \sin u$.

How can one see that the inverse function for $u = \int_0^z dt/\sqrt{1-t^2}$ is actually a periodic function $z = \sin u$?

But <u>first:</u> What is $\sin u$ for $u \in \mathbb{C}$, and why is it periodic?

(Even if we know that $\sin u = \sum_{k \geq 0} (-1)^k u^{2k+1}/(2k+1)!$, that it is not obvious that the sum $\sum_{k \geq 0} (-1)^k (u+T)^{2k+1}/(2k+1)!$ for some positive T represents the same.)

In some classical treatments of complex analysis, after covering power series and analytic functions, trigonometric functions are introduced via the exponential function defined as

$$e^u := \sum_{k \geq 0} u^k/k!.$$

Then getting the addition formula $e^a \cdot e^b = e^{a+b}$
Then proving that e^{iz} has real periods, with the smallest positive being **denoted !!!** by 2π.
Then introducing

$$\sin u := (e^{iu} - e^{-iu})/2 = \sum_{k \geq 0} u^{2k+1}/(2k+1)!.$$

Hence, 2π is a period of sin.

Second: still why $\sin u$ is the inverse of $u = \int_0^z dt/\sqrt{1-t^2}$?

One way to see it to write

$$(1-t^2)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k} (-t^2)^k$$

using the binomial formula, and then integrate it term by term from 0 to z, getting $\sum_{m\geq 1} b_m z^m$.
Then trying to find the unknown coefficients a_i of the power series that represents the inverse function by a carefully carried substitution and solving

$$\sum_{n\geq 1} a_n \Big(\sum_{m\geq 1} b_m t^m \Big)^n = t.$$

This is a well understood process. It version can use about $(n+1)^2/2$ operations to compute the first n coefficients a_i.
What is crucial here is that we need $b_0 = 0$ for the method to work, and we have it here. The result will be $a_{2n} = 0$ and $a_{2n+1} = (-1)^n/(2n+1)!$.

Now, why the inverse function to $\int_0^z dt/\sqrt{1-t^4}$ is doubly periodic?

Abel showed that extended to complex plane, the inverse function to this integral, which we denoted as sinl, has two periods: $w_1 = (1-i)\overline{\omega}$, $w_2 = (1+i)\overline{\omega}$, and so all these:

$$\{mw_1 + nw_2 : \quad m, n \in \mathbb{Z}\},$$

where $\overline{\omega}$ is called the **lemniscate constant** and

$$\overline{\omega} = 2\int_0^1 dt/\sqrt{1-t^4} = 2.6221\ldots$$

In particular, sinl also has these two other periods: a real one $2\overline{\omega}$, and pure imaginary $2\overline{\omega}i$.
Geometrically $\overline{\omega}$ is the ratio of the perimeter of Bernoulli's lemniscate to its diameter, like π for the circle!!!

A function f on $\overline{\mathbb{C}}$ is **doubly periodic** if there exist some $w_1, w_2 \in \mathbb{C}$ such that $w_1/w_2 \notin \mathbb{R}$, and for any $m, n \in \mathbb{Z}$, and any $z \in \mathbb{C}$,

$$f(z + mw_1 + nw_2) = f(z).$$

A meromorphic doubly periodic function is called **elliptic functions**. After they where discovered, they became the main interest of Abel and Jacobi.
All periods

$$\Lambda = \{mw_1 + mw_2 : \quad m, n \in \mathbb{Z}\}$$

form an abelian group, or a **period lattice**. The **fundamental parallelogram** generated by w_1 and w_2 is

$$\{\mu w_1 + \nu w_2 : \quad \mu, \nu \in [0, 1]\}$$

Its copies tile the plane. Everything that happens in the fundamental domain repeats in all the others. Therefore we view elliptic functions as functions with $\mathbb{C}/\Lambda$ as their domain.
Identifying opposite sides of the parallelogram, we get a torus.

Parts II and III of this talk and the bibliography will follow.