Irregularity Strength of Dense Graphs

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Abstract

Let G be a simple graph of order n with no isolated vertices and no isolated edges. For a positive integer w, an assignment f on G is a function $f : E(G) \to \{1, 2, \ldots, w\}$. For a vertex v, $f(v)$ is defined as the sum $f(e)$ over all edges e of G incident with v. f is called irregular, if all $f(v)$ are distinct. The smallest w for which there exists an irregular assignment on G is called the irregularity strength of G, and it is denoted by $s(G)$. We show that if the minimum degree $\delta(G) \geq 10n^{3/4} \log^{1/4} n$, then $s(G) \leq 48(n/\delta)+6$. For these δ, this improves the magnitude of the previous best upper bound of A. Frieze, R.J. Gould, M. Karoński, and F. Pfender by a $\log n$ factor. It also provides an affirmative answer to a question of J. Lehel, whether for every $\alpha \in (0, 1)$, there exists a constant $c = c(\alpha)$ such that $s(G) \leq c$ for every graph G of order n with minimum degree $\delta(G) \geq (1 - \alpha)n$. Specializing the argument for d-regular graphs with $d \geq 10^{4/3}n^{2/3} \log^{1/3} n$, we prove that $s(G) \leq 48(n/d) + 6$.

Keywords: *irregular assignment; irregularity strength, spanning forest*

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1 Definitions and Notations

In this paper, all graphs are assumed to be simple, i.e. undirected with no loops or multiple edges. The terms related to graphs that we do not define can be found in Bollobás [8] or [6], and those related to probability – in Feller [14] or Ross [24].

By $V = V(G)$ and $E = E(G)$ we denote the set of vertices and the set of edges of a graph G, respectively. The *order* of G is the number of its vertices, and the *size* of G is the number of its edges. To simplify notations, an edge $\{u, v\}$ will be denoted by just uv or vu. The set of all vertices adjacent to a vertex v of G, or *neighbors* of v in G, is called the *neighborhood* of v in G and is denoted by $N(v) = N_G(v)$. The number $|N_G(v)|$ is called the *degree* of the vertex v of G, and is denoted by $d(v) = d_G(v)$. The minimum and the maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. $K_{s,t}$ denotes the bipartite graph with partitions having s and t vertices. For $a \geq 3$, each vertex from the 2-vertex partition of $K_{2,a}$ will be called *special*.

For any finite set S of positive integers, called *edge weights* or *labels*, an *assignment* with labels from S is a function $f : E(G) \to S$. The largest number in S is called the *strength* of f. For an edge e, $f(e)$ is called the *weight* of e. For a vertex v, the *weight* or the *label* $f(v)$ of v is defined as the sum of weights of all edges of G incident with v, i.e., $f(v) = \sum_{u \in N(v)} f(uv)$. We call an assignment f *irregular*, if the weights $f(v)$ of all vertices are distinct. The smallest strength of an irregular assignment on G is called the *irregularity strength* of G, and it is denoted by $s(G)$. If the minimum does not exist, we set $s(G) = \infty$. It is easy to argue that $s(G)$ is finite if and only if G has no component of order two and at most one isolated vertex.

2 Introduction

The notion of the irregularity strength was introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba in [9]. It is motivated by a well known fact that a simple graph of order at least 2 cannot be "completely irregular": it always has at least two vertices of equal degree. On the other hand, there are multigraphs in which all vertices have distinct degrees. Suppose we are allowed to create multiple edges in a graph G and our goal is to end up with a multigraph with all distinct degrees. Then $s(G)$ is equal to the smallest maximum multiplicity of an edge in a resulting multigraph.

For a survey of known results and many open questions about the irregularity strength, see Lehel [22]. For many results which were not mentioned in the survey and appeared since it was published, see [1], [11], [16], [12], [10], [2], [17], [21], [20], [26], [3], [18], [23], [25], [15], [5], [4].

The question which energized the study of the irregularity strength of regular graphs is due to M.S. Jacobson (as mentioned in [22]):

Q : *Does there exist an absolute constant c such that $s(G) \leq n/d + c$ holds for every d-regular graph G of order n?*

The question was motivated by the facts that

(i) for every d-regular graph of order n, $s(G) \geq \lceil (n-1)/d \rceil + 1$ (for a simple counting proof see [9]);

(ii) in all those cases where the irregularity strength of a regular graph is known, its value is very close to $\lceil (n-1)/d \rceil + 1$.

In [13], Faudree and Lehel proved that $s(G) \leq n/2 + 9$ for every d-regular graph G, $d \geq 2$, and they expressed a belief that the answer to **Q** is affirmative. Since then not much progress was made towards understanding the irregularity strength of regular graphs until Frieze, Gould, Karoński, and Pfender provided several impressive new upper bounds on $s(G)$ in [15]. In particular, their results offer further support to the affirmative answer to **Q**. We state them in the following two theorems. Throughout the paper, log denotes the natural logarithm.

Theorem 1 [15] *Let G be a graph of order n with no isolated vertices or edges.*

(a) If $\Delta \leq \lfloor (n/\log n)^{1/4} \rfloor$, then $s(G) \leq 7n(1/\delta + 1/\Delta)$,

(b) If $\lfloor (n/\log n)^{1/4} \rfloor + 1 \leq \Delta \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 60(n/\delta)$,

(c) If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$ and $\delta \geq \lceil 6 \log n \rceil$, then $s(G) \leq 336(\log n)(n/\delta)$.

Theorem 2 [15] *Let G be a d-regular graph of order n, $d \geq 2$.*

(a) If $d \leq \lfloor (n/\log n)^{1/4} \rfloor$, then $s(G) \leq 10(n/d)$,

(b) If $\lfloor (n/\log n)^{1/4} \rfloor + 1 \leq d \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 48(n/d) + 1$,

(c) If $d \geq \lfloor n^{1/2} \rfloor + 1$, then $s(G) \leq 240(\log n)(n/d) + 1$.

Observe that in regard to **Q**, both (a) and (b) parts of Theorem 2 give the upper bounds of correct order n/d, and part (c) contains an extra $\log n$ factor. In this paper we prove that increasing the minimum degree one can remove the logarithmic factor in (c). Our main results are as follows.

Theorem 3 *Let G be a graph of order n.*

(a) If $\delta \geq 10 n^{3/4} \log^{1/4} n$, then, $s(G) \leq 48(n/\delta) + 6$.

(b) If G is d-regular with $d \geq 10^{4/3} n^{2/3} \log^{1/3} n$, then, $s(G) \leq 48(n/d) + 6$.

As an immediate corollary, we obtain an affirmative answer to a question of Lehel [22], Problem 26, whether for every $\alpha \in (0, 1)$, there exists a constant $c = c(\alpha)$ such that $s(G) \leq c$ for every graph G of order n with minimum degree $\delta(G) \geq (1 - \alpha)n$.

3 Outline of the proof of Theorem 3.

Our approach follows closely the one in [15], especially the proof of their Lemma 9. The main difference is that for a random assignment, we bound the probability of many vertices having their weights within a certain interval, rather than having a given weight, as was done in [15]. Another main difference is the way we modify a "good" random assignment into an irregular one: instead of using a factor of G consisting of generalized stars, we use a special collection of subgraphs, all isomorphic to $K_{2,a}$ for some $a = a(n)$. More details of our proof are provided below.

For an irregular assignment on G of strength s, every vertex weight is in $[\delta, s\Delta]$, and it appears there at most once.

1. We begin by showing that there exists an $f : E(G) \to \{1, 2\}$ with the property that for every subinterval I_1 of length l inside $I = \{\delta, \dots, 2\Delta\}$, the number of vertices of G which have weights in I_1 is not too large. The value of l will be proportional to $\log n$. The existence of such an assignment f is proved in Lemma 4 by using the same random assignment as in the proof of Lemma 9 of [15]. Clearly, this f could be very far from being irregular.

2. Our plan is to modify f from above in several stages, and to end up with an irregular assignment.

   (a) First, we will re-scale f to an assignment f' by simply multiplying each label used in f by a number proportional to n/δ. Although the strength of the assignment goes up, this will give us more "room" to modify this more "rarified" assignment f' into an irregular one.

   (b) Next we modify f' to make it irregular by using the following approach. First, we notice that for any path of length 2 in G, we can change the weights of its endpoints without changing the weight of any other vertex of the graph by simply adding a constant to the label of one of its edges and subtracting the same constant from the label of the other edge. Suppose $K_{2,a}$, $a \geq 3$, is a subgraph of G with $\{x, y\}$ being the smaller partition. Applying the above idea to every xy-path of length 2 in this $K_{2,a}$ changes the weights of x and y without changing the weight of any other vertex of G. In this way, taking $a = a(n)$ sufficiently large, we can change the weights of x and y substantially, applying only small changes to the edge labels.

   In order to make the idea of such a modification of f' work, we show that if $\delta(G)$ is sufficiently large, then G contains a family $\mathcal{F}$ of edge-disjoint subgraphs, each isomorphic to $K_{2,a}$, such that every vertex of G is special in a copy of $K_{2,a}$. See Lemma 7.

3. Having a family $\mathcal{F}$ described above, we modify f' to an irregular assignment via a simple algorithm. This proves our main Theorem 3.

4 Proofs

In order to simplify the exposition, we omit floors and ceilings in expressions that denote integer-valued graph parameters, provided that their presence doesn't affect the proof.

For each integer y, and assignment h on G, let $Z_y(h)$ denote the number of vertices of G having weight y in h, i.e.,

$$Z_y(h) = |\{v \in V \ : \ h(v) = y\}|.$$

The following technical lemma will be used later. It estimates the number of vertices having weight inside a subinterval of length l of $\{\delta, \ldots, 2\Delta\}$. The proof is quite similar to the proof of Lemma 9 from [15].

Lemma 4 *Let $l = 300 \log n$, and $\delta(G) \geq l$. Then, there exists an assignment $f : E \to \{1, 2\}$, such that for all i, $\delta(G) \leq i \leq 2\Delta$,*

$$\sum_{j=i}^{i+l-1} Z_j(f) \leq \frac{800 n \log n}{\delta}. \tag{1}$$

Proof For each $v \in V$, define X_v to be a uniformly distributed random variable on $[0, 1]$, all X_v independent. Define $g : E(G) \to \{1, 2\}$ such that for $e = uv \in E$,

$$g(e) = \begin{cases} 1 & \text{if } X_u + X_v < 1, \\ 2 & \text{otherwise.} \end{cases}$$

We will prove that the random g defined above will, with positive probability, satisfy (1). For each positive integer y, let $Z^y(g) = Z_y(g) + Z_{y+1}(g) + \ldots + Z_{y+l-1}(g)$, $k = l/3$ and $t = 8nk/\delta$. In the future, we omit the references to g or other assignments, and simply write Z^y and Z_y when its clear from the context which assignment is considered. To prove the lemma, we will bound $\Pr(Z^y \geq t)$ by using Markov's inequality. For an arbitrary event ε (specified later), it gives

$$\Pr(Z^y \geq t|\varepsilon) \leq \frac{E(\binom{Z^y}{k}|\varepsilon)}{\binom{t}{k}}. \tag{2}$$

Therefore, we want to bound $E\Big(\binom{Z^y}{k}|\varepsilon\Big)$. As

$$E\Big(\binom{Z^y}{k}|\varepsilon\Big) = \sum_{S \subseteq V,\ |S|=k} \Pr(g(v) \in \{y, y+1, \ldots, y+3k-1\}, v \in S|\varepsilon), \tag{3}$$

we fix $S = \{v_1, v_2, \ldots, v_k\}$.

For $v \in S$, let $\mu(v) = |N(v) \setminus S|$. Note that $d(v) - \mu(v) \le k - 1$. For $v \in S$, let $\xi_1 < \xi_2 < \ldots < \xi_{d(v)}$ be the values of $X_u, u \in N(v)$, sorted in increasing order and let $\eta_1 < \eta_2 < \ldots < \eta_{\mu(v)}$ be the values of $X_u, u \in N(v) \setminus S$, also sorted in increasing order. Clearly, for every i, $1 \le i \le \mu(v)$,

$$\xi_i \le \eta_i \le \xi_{i+k-1} \tag{4}$$

We will make use of the following Lemma, which was proved in [15].

Lemma 5 [15] *Let $Y_1, \ldots, Y_s$ be a sequence of independent random variables, each having exponential distribution with mean one. Then for any real $a > 0$, $0 < b < 1$ we have*

$$\Pr(Y_1 + \ldots + Y_s \ge (1+a)s) \le ((1+a)e^{-a})^s$$

$$\Pr(Y_1 + \ldots + Y_s \le (1-b)s) \le ((1-b)e^{b})^s.$$

□

We use the convention that $\xi_i = 0$ for $i \le 0$ and $\xi_{d(v)+j} = 1$ for $j > 0$. Let

$$\Theta_v = \max_{0 \le i \le d(v)-5k+1} \{\xi_{i+5k-1} - \xi_i\}, \qquad \Theta = \max_{v \in V} \Theta_v.$$

Since labels on edges of G are only 1 or 2, and $g(uv) = 2$ if $X_u + X_v \ge 1$, then $g(v) = d(v) + |\{u \in N(v) : X_u \ge 1 - X_v\}|$. This implies that if $g(v) = z$, then $|\{u \in N(v) : X_u \ge 1 - X_v\}| = z - d(v)$, and so there are $z - d(v)$ of our ξ_i's which are at least $1 - X_v$. Due to our ordering, the least one with this property is $\xi_{d(v)-(z-d(v)-1)} = \xi_{2d(v)-z+1}$. Using (4), we conclude that $g(v) \in \{y, y+1, \ldots, y+3k-1\}$ implies

$$1 - X_v \in [\xi_{2d(v)-y-3k+1}, \xi_{2d(v)-y+1}] \subseteq [\eta_{2d(v)-y-4k+2}, \eta_{2d(v)-y+1}] \subseteq [\xi_{2d(v)-y-4k+2}, \xi_{2d(v)-y+k}]. \tag{5}$$

Let

$$\varepsilon = \{\Theta < 6k/\delta\}. \tag{6}$$

As in [15], we use the fact that for a sequence $\xi_1 < \xi_2 < \cdots < \xi_s$ of order statistics from the uniform distribution over $[0, 1]$, ξ_i has the same distribution as $(Y_1 + \cdots + Y_i)/(Y_1 + \cdots + Y_{s+1})$, where $Y_1, Y_2, \ldots, Y_{s+1}$ is a sequence of independent random variables, each having exponential distribution with mean one. See, e.g., [24].

If $n \le 800 \log n$, which is the case for $n \le 7050$, then $n \le 800n(\log n)/\delta$ and therefore (1) is true. Therefore, we may assume that $n > 7050$. Applying Lemma 5 to the ordered statistics defining Θ, and using $n > 7050$ at the end, we see that

$$\begin{aligned}
\Pr(\neg\varepsilon) &= \Pr(\exists\, v \in V : \Theta_v \geq \frac{6k}{\delta}) \\
&\leq n \Pr(\exists\, i :\ 1 \leq i \leq \Delta - 5k + 1,\ \frac{Y_i + \cdots + Y_{i+5k-1}}{Y_1 + \cdots + Y_{\delta+1}} \geq \frac{6k}{\delta}) \\
&\leq n \Pr(Y_1 + \cdots + Y_{\delta+1} \leq (11/12)\delta) + n^2 \Pr(Y_1 + \cdots + Y_{5k} \geq (11/2)k) \\
&\leq n(11/12 \cdot e^{1/12})^{\delta} + n^2(11/10 \cdot e^{-1/10})^{5k} \\
&< 1/2. \qquad (7)
\end{aligned}$$

Hence, $\Pr(\varepsilon) > 1/2$. Further,

$$\begin{aligned}
& \Pr(g(v) \in \{y, y+1, \ldots, y+3k-1\}, v \in S|\varepsilon) \qquad (8) \\
\leq\ & \Pr(1 - X_{v_i} \in [\eta_{2d_{v_i}-y-4k+2}, \eta_{2d_{v_i}-y+1}], i = 1, 2, \ldots, k|\varepsilon) \\
\leq\ & 2 \Pr(1 - X_{v_i} \in [\eta_{2d_{v_i}-y-4k+2}, \eta_{2d_{v_i}-y-4k+2} + 6k/\delta], i = 1, 2, \ldots, k) \\
\leq\ & 2\big(\frac{6k}{\delta}\big)^k.
\end{aligned}$$

From Equation (2) and (3), we obtain

$$\Pr(\exists y\ : Z^y > t|\varepsilon) \leq 2n \binom{t}{k}^{-1} \binom{n}{k} \Big(\frac{6k}{\delta}\Big)^k.$$

We use the following elementary estimates, valid for all natural numbers m. They follow immediately from the ones in Bollobás [7], page 4.

$$\left(\frac{m}{e}\right)^m \leq m! \leq e(m+1)\left(\frac{m}{e}\right)^m, \qquad \binom{n}{k} \leq n^k/k! \leq \left(\frac{ne}{k}\right)^k.$$

Since $t \geq 8k$,

$$\binom{t}{k} \geq \frac{(t-k)^k}{k!} \geq \frac{(7t/8)^k}{k!} \geq \frac{1}{e(k+1)} \left(\frac{7te}{8k}\right)^k.$$

It follows that

$$\Pr(\exists y\ : Z^y > t|\varepsilon) \leq 2ne(k+1)\Big(\frac{8k}{7te}\Big)^k \cdot \Big(\frac{ne}{k}\Big)^k \cdot \Big(\frac{6k}{\delta}\Big)^k = 2ne(k+1)\Big(\frac{6}{7}\Big)^k < \frac{1}{10}.$$

Using Equation (7) establishes

$$\Pr(\exists y\ : Z^y > t) \leq \Pr(\exists y : Z^y > t|\varepsilon) + \Pr(\neg\varepsilon) < 1,$$

proving the Lemma. □

Now we prove the existence of a family $\mathcal{F}$ described in Section 3. The proof proceeds in three steps, which are presented below. In the first two, we show the existence of two related families $\mathcal{F}_0$ and $\mathcal{F}_1$, where $\mathcal{F}_1 \subseteq \mathcal{F}_0$, and in the third step, we find $\mathcal{F}$ as a subfamily of $\mathcal{F}_1$.

It is known, that if a graph of order n has size greater than $(1/2)(a-1)^{1/2}(n-1)n^{1/2} + n/2$, then it must contain a copy of $K_{2,a}$ as a subgraph. See, e.g., [6], page 310. Therefore, if $\delta(G)$ is large enough, say $\delta > (n \log n)^{1/2}$, then G contains a copy of $K_{2,\log n}$. For larger values of δ, we can get quite many copies of $K_{2,\log n}$ in G.

We begin by considering a family $\mathcal{F}_0$ of all subgraphs of G isomorphic to $K_{2,A}$, where $A = \delta^2/(10n)$. Next we explain that $\mathcal{F}_0$ contains a subfamily $\mathcal{F}_1$ with the property that every vertex of G is special in a copy of $K_{2,A}$ from $\mathcal{F}_1$ and no vertex of G is special in "many" such copies. Members of $\mathcal{F}_1$ are not necessarily edge-disjoint. In order to obtain $\mathcal{F}_1$, we apply a graph-theoretic Lemma 6 to an auxiliary graph of G. This is done at the beginning of the proof of Lemma 7.

The rest of the proof of Lemma 7 is concerned with extracting the desired edge-disjoint subfamily $\mathcal{F}$ from $\mathcal{F}_1$.

We need one more definition: a forest whose components are stars is called a *star forest.*

Lemma 6 *Let H be a graph of order $n \geq 2$ without isolated vertices. Then there exists a spanning star forest F of H without isolated vertices and with $\Delta = \Delta(F) \leq \max\{n/\delta, 2\}$.*

Proof Let $\delta = \delta(H)$. Then $\delta \geq 1$, and so $n \geq 2$.

We first show that H has a spanning star forest without isolated vertices. It is obvious that the set of spanning forests of H without isolated vertices is not empty. Let F' be one of them with the fewest number of edges. If some edge e of F' had two non-leaf end vertices, then we could delete e to obtain a spanning forest without isolated vertices and fewer edges, contradicting our choice of F'. Therefore, F' is a spanning star forest.

Now we modify F', if needed, to satisfy the maximum degree requirement. Among all spanning star forests of H without isolated vertices (whose existence was proven above), we choose the ones with the smallest possible largest degree and among these pick one with the smallest number of vertices of the largest degree. Let us call it F. We show that $\Delta(F) \leq \max\{n/\delta, 2\}$.

Let us assume the contrary (so $\Delta \geq 3$), and let v_1 be a vertex with $d_F(v_1) = \Delta(F) = \Delta$. Let w be a vertex adjacent to v_1 in F (so $d_F(w) = 1$). Since $d_H(w) \geq \delta \geq 1$, let $\{v_1, v_2, \ldots, v_\delta\}$ be some δ neighbors of w in H having v_1 among them. If v_1 is the only neighbor of w, then $\delta = 1$ and the inequality we wish to prove is obviously true.

If $\delta \geq 2$, then we show that each v_i, $i > 1$, is also a center of a star of F ("a", because K_2 has two centers). If not, let u be the center of the star S of F containing v_i. Then, $|S| \geq 3$, (otherwise v_i is a center of S), and $F'' := F - v_1w - uv_i + wv_i$ is a spanning star forest of H with no isolated vertices. Then either $\Delta(F'') < \Delta(F)$ or

$\Delta(F'') = \Delta(F)$ but the number of vertices of the largest degree in F'' is less than in F. This contradicts our choice of F. Therefore, each v_i is a center of a star of F.

Now, we show that $d_F(v_i) \geq \Delta - 1$ for all $i > 1$. Assume the contrary, i.e. $d_F(v_i) \leq \Delta - 2$ for some $i > 1$. Then, $F - v_1w + wv_i$ is a spanning star forest of H with no isolated vertices and it has either smaller maximum degree than F or fewer vertices of maximum degree than F. Therefore, $d_F(v_i) \geq \Delta - 1$, so, in particular, $d_F(v_i) \geq 2$ for all i.

Hence the stars centered at v_i, $i = 1, \ldots, \delta$, are vertex disjoint. This implies that

$$n \geq \sum_{i=1}^{\delta}(d_F(v_i) + 1) = (\Delta + 1) + \sum_{i=2}^{\delta}(d_F(v_i) + 1) \geq \Delta + 1 + (\delta - 1)\Delta > \delta\Delta,$$

and the required inequality follows. □

For a collection $\mathcal{F}$ of subgraphs of G isomorphic to $K_{2,m}$, $m \geq 3$, we define an auxiliary graph $G_{\mathcal{F}} = (V_{\mathcal{F}}, E_{\mathcal{F}})$ where $V_{\mathcal{F}} = V$ and $xy \in E_{\mathcal{F}}$ if x and y are the special vertices of some element of $\mathcal{F}$.

Lemma 7 *For every $r \geq 3$, and every graph G with*

$$\delta = \delta(G) > \sqrt{10}n^{3/4}r^{1/4}, \tag{9}$$

there is a collection $\mathcal{F}$ of edge-disjoint subgraphs, each isomorphic to $K_{2,r}$, such that $G_{\mathcal{F}}$ is a star forest without isolated vertices. The same conclusion also holds for every d-regular graph G with

$$d > 10^{2/3}n^{2/3}r^{1/3}. \tag{10}$$

Proof The proof of the second statement is almost identical to the first. We begin with the proof of the first statement.

Before introducing the collection $\mathcal{F}$, we first take an orientation of G that will later be useful for showing that the collection is edge-disjoint. We define an orientation of G such that, in the obtained digraph D, for all v, $|d^+(v) - d^-(v)| \leq 1$. Such an orientation can be produced in the following way. If each component of G is Eulerian, following an Eulerian circuit within each component yields the orientation. Otherwise, we introduce an additional vertex and join it to every vertex of G which has odd degree. Now, the degree of every vertex of the obtained graph is even, so each of its component is Eulerian. Following an Eulerian cycle in each component, and deleting the additional vertex, we obtain the desired orientation of G.

If D_1 is a digraph with the underlying graph $G_1 \cong K_{2,a}$, we say the special vertices of G_1 are special vertices of D_1.

Let

$$A = \frac{\delta^2}{10n}.$$

We say that a subdigraph D_1 of D is *important* if its underlying graph is isomorphic to $K_{2,A}$ with special vertices u and v such that u and v are sinks in D_1 (i.e. $d^+(u) = d^+(v) = 0$). Note that if D_1 is important and $\vec{xy}$ is an arc in D_1, then

$$y \text{ is special in } D_1. \tag{11}$$

For the given D, we define an auxiliary graph $G' = (V', E')$ where $V' = V$ and $xy \in E'$ if x and y are the special vertices of some important subdigraph of D. For $v \in V(D)$, let $d'(v)$ be the degree of v in G' and $\delta' = \delta(G')$.

We will show that $\delta' > A$. Fix $v \in V'$. Inequality (9) implies that $\delta \geq 20$. For such δ, there are at least

$$\frac{(\delta - 1)}{2} \cdot \frac{(\delta - 3)}{2} > \frac{\delta^2}{5}$$

(v, b)-paths in G of length 2 of the form (v, a, b) such that $\vec{av}$ and $\vec{ab}$ are arcs in D. Let N_v be the set of those vertices $b \in V$ for which there are at least A such (v, b)-paths of length 2. Then, partitioning these paths starting at v into those which end in N_v (there are at most $n|N_v|$ of them) or $V - N_v$ (there are at most $A(n - |N_v|)$ of them), we obtain

$$n|N_v| + A(n - |N_v|) \geq \delta^2/5, \tag{12}$$

so $|N_v| > A$. But N_v is the set of neighbors of v in G', so we have $d'(v) > A$. This gives $\delta' > A$.

By Lemma 6 , G' has a spanning star forest F such that $\delta(F) \geq 1$ and

$$\Delta(F) \leq \max\{n/A, 2\}. \tag{13}$$

For $uv \in E(F)$, let $D(u, v)$ be a fixed important subgraph of D such that u and v are the special vertices, and let $G(u, v)$ be the underlying graph of $D(u, v)$ (so $G(u, v)$ is a subgraph of G isomorphic to $K_{2,A}$ with special vertices u and v).

If $G(u, v)$ and $G(x, y)$, $uv, xy \in E(F)$, $uv \neq xy$, share a common edge ab with $\vec{ab}$ an arc in D, then

$$b \in \{u, v\} \cap \{x, y\}. \tag{14}$$

This is because $\vec{ab}$ an arc in D gives that b must be special in $D(u, v)$ and $D(x, y)$ by (11). Because F is a star forest, uv and xy must be in the star of F with center b. Every edge in both $G(u, v)$ and $G(x, y)$ has b as an endpoint.

For every $uv \in E(F)$, we will construct a subgraph $H(u, v) \cong K_{2,r}$ of $G(u, v)$ such that every two of these subgraphs are edge-disjoint. They will form the desired family $\mathcal{F}$. Here is the construction. For any star of F, with, say, center b, and leaves $\{v_1, \ldots, v_m\}$, we describe how to obtain an edge-disjoint collection of $\{H(b, v_1), \ldots, H(b, v_m)\}$. If we do this for all stars of F, then (by the remarks in the preceding paragraph), the collection $\{H(u, v) : uv \in E(F)\}$ will be edge-disjoint.

Let each $H(b, v_i)$ be an induced subgraph of $G(b, v_i)$ with the same special vertices, and with L_i being the larger partite set of $H(b, v_i)$. Hence, $|L_i| = r$. If the L_i's are pairwise disjoint, then the $H(b, v_i)$'s will be pairwise edge-disjoint.

The sequence of L_i's is constructed inductively. We pick L_1 to be an arbitrary subset of size r from the non-special vertices of $G(b, v_1)$. Suppose that all L_j's, $j < i \leq m$, have been chosen. Since $m \leq n/A$ by (13), and (9) implies $nr/A \leq A$, we have

$$|\cup_{j<i} L_j| \leq (m-1)r \leq nr/A - r \leq A - r.$$

Therefore, we can always find a set L_i of r non-special vertices in $G(b, v_i)$ whose elements have not been chosen. This proves the first statement of the theorem.

In order to prove the second statement, we replace all occurrences of δ by d. Next, we note that the argument leading to (12) allows us to replace n in (12) by d. We note that $\delta' > d/10$. This is because we can replace the n by d in (12). Next, setting $A_1 = d^2/(10d) = d/10$, we decrease the right hand side of (13) to $\max\{n/A_1, 2\}$. Everything else remains the same until the last paragraph of the proof of the first statement. We write the modification of this paragraph for the second statement.

Finally, since $m \leq n/A_1$ by (13), and (10) implies that $nr/A_1 \leq A$, we have

$$|\cup_{j<i} L_j| \leq (m-1)r \leq nr/A_1 - r \leq A - r.$$

As before, this implies that we can find a set L_i of r non-special vertices in $G(v, v_i)$ whose elements have not been chosen. $\square$

Proof of Theorem 3

Because the proofs of statement (a) and (b) are quite similar, we only explain the proof of statement (a). Let k, l and t be defined as in Lemma 4, namely $k = 100\log n$, $l = 3k$ and $t = 800n\log n/\delta$. Let $\lambda = 16n/\delta + 2$. Since our hypothesis implies that $10n^{3/4}\log^{1/4} n > 300\log n$, we can consider an assignment $f : E(G) \to \{1, 2\}$ satisfying the conclusion of Lemma 4. Let

$$f' : E(G) \to \{\lambda + 1, 2\lambda + 2\} \tag{15}$$

be defined by $f'(e) = (\lambda + 1)f(e)$ for all $e \in E(G)$.

Then, as $Z_{(\lambda+1)i}(f') = Z_i(f)$ for all $i \in \{\delta, \ldots, 2\Delta\}$, we have

$$\sum_{j=i}^{i+3k\lambda-1} Z_j(f') \leq \sum_{j=i}^{i+l(\lambda+1)-1} Z_j(f') \leq \sum_{j=\lceil i/(\lambda+1)\rceil}^{\lceil i/(\lambda+1)\rceil+l-1} Z_j(f) \leq t \tag{16}$$

for each $i \in \{(\lambda + 1)\delta, \ldots, (2\lambda + 2)\Delta\}$.

If $r = k$, then the lower bounds on δ and d in Lemma 7 are the same as in our case. Therefore, there exists a family $\mathcal{F} = \{H_1, H_2, \ldots, H_q\}$ of edge-disjoint subgraphs of

G, each isomorphic to $K_{2,k}$, such that $G_{\mathcal{F}}$ is a spanning star forest without isolated vertices. Let v_i and w_i denote the special vertices of H_i with w_i being a leaf in $G_{\mathcal{F}}$ (as $v_iw_i \in E(G_{\mathcal{F}})$, and $G_{\mathcal{F}}$ is a star forest, such a labelling is possible). Therefore all vertices of G appear in the sequence $v_1, w_1, v_2, w_2, \ldots, v_q, w_q$, where all w_i are distinct, no v_i is equal to any w_j, but v_i are not necessarily all distinct.

Our goal is to show that f' can be modified to an irregular assignment g. It is done by constructing a special sequence of assignments on G, $g_0 = f', g_1, \ldots, g_{q-1}, g_q = g$ in, roughly, the following way. For each $v \in V$, and each assignment h on G, let

$$C_v(h) = |\{u \in V : h(u) = h(v)\}|.$$

The assignment h is irregular if $C_v(h) = 1$ for all $v \in V$. If $C_{v_1}(g_0) = C_{w_1}(g_0) = 1$, then $g_1 = g_0$. Otherwise, we modify labels of edges of H_1 using the procedure described in 2 (b) of Section 3, such that, when we finish, $C_{v_1}(g_1) = C_{w_1}(g_1) = 1$, and no vertex of G other than v_1 or w_1 changed its weight. Now we consider vertices v_2 and w_2. If $C_{v_2}(g_1) = C_{w_2}(g_1) = 1$, then $g_2 = g_1$. Otherwise, we modify labels of edges of H_2 using the same procedure, such that, when we finish, $C_{v_2}(g_2) = C_{w_2}(g_2) = 1$, and no vertex of G other than v_2 or w_2 changed its weight (if $v_1 = v_2$, then v_1 may alter its weight, but $C_{v_1}(g_2) = 1$). Then we consider v_3 and w_3, and, if needed, we modify labels of edges of H_3 such that in the new assignment g_3, $C_{v_i}(g_3) = C_{w_i}(g_3) = 1$ for $i = 1, 2, 3$. And so on. Before we prove that the desired modifications are always possible, we state more formally the properties of g_i we wish to have. Some of them repeat what was described above, while some others are used to show that our algorithm works.

We require each g_i, $i \geq 1$, to satisfy the following properties:

(a) $g_i(e) = g_{i-1}(e)$ for all edges not in H_i;

(b) $|g_i(e) - g_{i-1}(e)| \leq \lambda$ for all $e \in E(G)$;

(c) $g_i(v) = g_{i-1}(v)$ for all v except possibly v_i, w_i;

(d) If $g_{i-1}(v_i) > g_0(v_i)$ (resp. $<$), then $g_i(v_i) \leq g_{i-1}(v_i)$ (resp. $\geq$);

(e) for all $v \in V$, $|g_i(v) - g_0(v)| \leq \lambda k$;

(f) $C_{v_i}(g_i) = C_{w_i}(g_i) = 1$.

We first describe how to construct g_1. As $g_0(v)$ is a positive multiple of $\lambda + 1$ for all v, the numbers $g_0(v_1) + 1$ and $g_0(w_1) - 1$ are different and not equal to $g_0(v)$ for any v. In the graph H_1, we pick a non-special vertex u, and we alter g_0 by adding 1 to $g_0(v_1u)$ and subtracting 1 from $g_0(w_1u)$. It is clear that g_1 satisfies properties (a)-(f).

Suppose we have constructed g_j, for all j, $1 \leq j < i$, such that each satisfies properties (a)-(f). We will describe how to construct g_i.

If $C_{v_i}(g_{i-1}) = C_{w_i}(g_{i-1}) = 1$, then we let $g_i = g_{i-1}$. Otherwise, let $a = g_{i-1}(v_i)$ and $b = g_{i-1}(w_i)$. We assume $a \leq g_0(v_i)$ (the other case being similar).

We show that there is an $r \in \{1, \ldots, \lambda k\}$ such that

$$Z_{a+r}(g_{i-1}) = Z_{b-r}(g_{i-1}) = 0, \quad a + r \neq b - r. \tag{17}$$

Note that the induction hypothesis (property (e)) gives $|g_{i-1}(v) - g_0(v)| \leq \lambda k$ for all $v \in V$. Therefore, for a vertex x such that $g_{i-1}(x) \in \{a + 1, \ldots, a + \lambda k\}$, we have $g_0(x) \in \{a + 1 - \lambda k, \ldots, a + 2\lambda k\}$. This implies that

$$\sum_{r=1}^{\lambda k} Z_{a+r}(g_{i-1}) \leq \sum_{r=1-\lambda k}^{2\lambda k} Z_{a+r}(g_0).$$

Since the second summation is over an interval of $3k\lambda$ integers and $g_0 = f'$, (16) gives that it is at most t, i.e.,

$$\sum_{r=1}^{\lambda k} Z_{a+r}(g_{i-1}) \leq t.$$

In a similar way, we can show that

$$\sum_{r=1}^{\lambda k} Z_{b-r}(g_{i-1}) \leq t.$$

Therefore,

$$|\{r \ : \ 1 \leq r \leq \lambda k, \quad Z_{a+r}(g_{i-1}) \geq 1 \text{ or } Z_{b-r}(g_{i-1}) \geq 1\}| \leq 2t \tag{18}$$

As there is at most one $r \in \{1, \ldots, \lambda k\}$ with $a + r = b - r$, and since $2t + 1 < \lambda k$, there is an r satisfying (17).

Let $u_1, \ldots, u_k$ be the non-special vertices of H_i ($\cong K_{2,k}$). Let x_m, $1 \leq m \leq k$, be non-negative integers, each at most λ, which add to r. As $r \in \{1, \ldots, \lambda k\}$, such x_m obviously exist. We define $g_i(v_i u_m) = g_{i-1}(v_i u_m) + x_m$ and $g_i(u_m w_i) = g_{i-1}(u_m w_i) - x_m$. Then, $g_i(v_i) = a + r$ and $g_i(w_i) = b - r$. By the choice of r, g_i satisfies property (f). Obviously, g_i satisfies (a)-(d). To verify that g_i satisfies (e), we note that $|g_i(v) - g_{i-1}(v)| = r \leq \lambda k$ for $v = v_i, w_i$, and $g_{i-1}(w_i) = g_0(w_i)$. Therefore, $|g_i(w_i) - g_0(w_i)| \leq \lambda k$, and (e) is established for $v = w_i$. Hence, we only need to establish (e) for $v = v_i$. Since g_{i-1} satisfies (e) by the induction hypothesis, and $g_{i-1}(v_i) \leq g_0(v_i)$ by our supposition, we have

$$g_0(v_i) - \lambda k \leq g_{i-1}(v_i) < g_i(v_i) = g_{i-1}(v_i) + r \leq g_0(v_i) + r \leq g_0(v_i) + \lambda k.$$

Therefore, $|g_i(v_i) - g_0(v_i)| \leq \lambda k$, and g_i satisfies property (e). This completes the induction.

We now show that the assignment $g = g_q$ is as desired in the theorem. In our construction, each edge changes its label at most once due to property (a) and the fact that the members of $\mathcal{F}$ are pairwise edge-disjoint. By (b), the change is at most λ. Therefore, as $g_0 = f'$, by (15), we obtain that for all $e \in E(G)$,

$$1 \leq g(e) \leq g_0(e) + \lambda \leq (2\lambda + 2) + \lambda = 3\lambda + 2.$$

Therefore, the strength of g is at most $3\lambda + 2$. By property (c), (f) and the fact that every vertex of V is special in at least one member of $\mathcal{F}$, g is an irregular assignment. □

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