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Note

On the uniqueness of some girth eight algebraically defined graphs

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Dedicated to the memory of Vasyl Dmytrenko (1961–2013)

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ABSTRACT

Let $\mathbb{F}$ be a field. For a polynomial $f \in \mathbb{F}[x, y]$, we define a bipartite graph $\Gamma_{\mathbb{F}}(f)$ with vertex partition $P \cup L$, $P = \mathbb{F}^3 = L$, and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if

$$p_2 + l_2 = p_1 l_1 \quad \text{and} \quad p_3 + l_3 = f(p_1, l_1).$$

It is known that the graph $\Gamma_{\mathbb{F}}(xy^2)$ has no cycles of length less than eight. The main result of this paper is that $\Gamma_{\mathbb{F}}(xy^2)$ is the only graph $\Gamma_{\mathbb{F}}(f)$ with this property when $\mathbb{F}$ is an algebraically closed field of characteristic zero; i.e. over such a field $\mathbb{F}$, every graph $\Gamma_{\mathbb{F}}(f)$ with no cycles of length less than eight is isomorphic to $\Gamma_{\mathbb{F}}(xy^2)$. We also prove related uniqueness results for some polynomials f over infinite families of finite fields.

1. Introduction

It is well known that the greatest number of edges in a graph of order n of girth (i.e. the shortest cycle length) eight is of magnitude $n^{1+\frac{1}{3}}$. The upper bound comes from Bondy–Simonovits [2] and the lower bound from generalized quadrangles (which will be defined later) or their subgraphs. The primary motivation for this paper is the desire to prove the uniqueness (in a certain sense) of the existing constructions for the lower bound.

For definitions related to graphs, we refer the reader to Bollobás [1]. Our primary object of study in this paper is defined as follows. For a field $\mathbb{F}$ and two polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, let P and L be two copies of the 3-dimensional vector space $\mathbb{F}^3$. Consider a bipartite graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ with vertex partitions P and L and with edges defined as follows: for every $(p) = (p_1, p_2, p_3) \in P$ and every $[l] = [l_1, l_2, l_3] \in L$, $\{(p), [l]\} = (p)[l]$ is an edge in $\Gamma_{\mathbb{F}}(f_2, f_3)$ if

$$\begin{aligned} p_2 + l_2 &= f_2(p_1, l_1) \\ p_3 + l_3 &= f_3(p_1, l_1). \end{aligned}$$

It turns out that the graph $\Gamma_{\mathbb{F}}(xy, xy^2)$ has girth eight; furthermore, when $\mathbb{F}$ is finite, it is isomorphic to an induced subgraph of a classical generalized quadrangle of order q (see Section 6 for details).

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This brings up a natural question: is $\Gamma_{\mathbb{F}}(xy, xy^2)$ the unique (up to isomorphism) girth eight graph of the form $\Gamma_{\mathbb{F}}(f_2, f_3)$?

For some uniqueness results over finite fields of odd characteristic, see Dmytrenko [4], Dmytrenko, Lazebnik and Williford [5], Kronenthal [8], Hou, Lappano, and Lazebnik [7], and references therein. The approach in these papers was to use properties of polynomials over finite fields. In this paper we use another approach. We let $\mathbb{F}$ be an algebraically closed field of characteristic zero, for example the field of complex numbers $\mathbb{C}$. For such $\mathbb{F}$, we prove the uniqueness of $\Gamma_{\mathbb{F}}(xy, xy^2)$ for all graphs in 'close vicinity', i.e. all graphs of the form $\Gamma_{\mathbb{F}}(xy, f)$. We then prove related uniqueness results for some polynomials f over infinite families of finite fields.

The main results of this paper are as follows.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Suppose $f \in \mathbb{F}[x, y]$ and the graph $\Gamma_{\mathbb{F}}(xy, f)$ has girth at least eight. Then $\Gamma_{\mathbb{F}}(xy, f)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.*

The following theorem is an analog of Theorem 1.1 for finite fields $\mathbb{F}_q$ of odd characteristic p and polynomials of 'small' degree. Let $M = M(p)$ be the least common multiple of the integers $1, 2, \ldots, p-2$. The function M is defined in this way so that all polynomials over $\mathbb{F}_q$ of degree at most $p-2$ have a root in $\mathbb{F}_{q^M}$.

Theorem 1.2. *Let q be a power of a prime p, $p \geq 5$. Suppose $f \in \mathbb{F}_q[x, y]$ has degree at most $p-2$ with respect to each of x and y. Then for all positive integers r, every graph $\Gamma_{q^{Mr}}(xy, f)$ of girth at least eight is isomorphic to $\Gamma_{q^{Mr}}(xy, xy^2)$.*

The prime $p = 3$ (so $M = 1$) is excluded, as it is easy to argue in this case that every $\Gamma_{q^r}(xy, f)$ has girth six.

For a polynomial $f = \sum_{0 \leq i,j \leq n} a_{ij}x^iy^j \in \mathbb{Z}[x, y]$, let $\hat{f} = \sum_{0 \leq i,j \leq n} \hat{a}_{ij}x^iy^j \in \mathbb{F}_p[x, y]$, where $\hat{a}_{ij}$ is the image of a_{ij} with respect to the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Corollary 1.3 (*to* Theorem 1.1). *Suppose $f \in \mathbb{Z}[x, y]$. Then there exists a positive constant $c = c(f)$ such that for every prime $p > c(f)$, there exists an integer $s = s(f, p)$ such that for all positive integers r, and $q = p^{sr}$, every graph $\Gamma_q(xy, \hat{f})$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$.*

We wish to comment that in Theorem 1.1, we consider all polynomials f over a given algebraically closed field of characteristic zero. However, in Corollary 1.3, we fix one polynomial with integer coefficients and state the existence of infinitely many characteristics p, and infinitely many finite fields of characteristic p, over which an analog of Theorem 1.1 holds.

This paper is organized as follows. In Section 2 we provide a description of 4- and 6-cycles in graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ and some isomorphisms between these graphs. In Sections 3 and 4, we present proofs of Theorems 1.1 and 1.2, respectively. In Section 5 we discuss the Lefschetz Principle and prove Corollary 1.3. In Section 6, we explain the relationship between the graphs $\Gamma_q(f_2, f_3)$ and generalized quadrangles, make some concluding remarks, and mention open problems.

2. Cycles and isomorphisms of graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$

Let $\Gamma_{\mathbb{F}}(f_2, f_3)$ be the graph defined in Section 1. If two vertices a, b in a graph are adjacent, we will write $a \sim b$. Let us describe cycles of length four and six in $\Gamma_{\mathbb{F}}(f_2, f_3)$. If the graph contains a 4-cycle

$$(a_1, a_2, a_3) \sim [x_1, x_2, x_3] \sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \sim (a_1, a_2, a_3), \tag{1}$$

then $(a_1, a_2, a_3) \sim [x_1, x_2, x_3]$ implies that $x_i = f_i(a_1, x_1) - a_i$ for $i = 2, 3$. Furthermore, $[x_1, x_2, x_3] \sim (b_1, b_2, b_3)$ implies that $b_i = f_i(b_1, x_1) - x_i = f_i(b_1, x_1) - f_i(a_1, x_1) + a_i$ for $i = 2, 3$. Similarly, we have:

$$\begin{aligned} y_i &= f_i(b_1, y_1) - f_i(b_1, x_1) + f_i(a_1, x_1) - a_i \\ a_i &= f_i(a_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) + a_i. \end{aligned}$$

This implies that in order for this 4-cycle to exist, we must have

$$f_i(a_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) = 0$$

for $i = 2, 3$. Similarly in order for a 6-cycle

$$\begin{aligned} (a_1, a_2, a_3) \sim [x_1, x_2, x_3] &\sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \\ &\sim (c_1, c_2, c_3) \sim [z_1, z_2, z_3] \sim (a_1, a_2, a_3), \end{aligned} \tag{2}$$

to exist in $\Gamma_{\mathbb{F}}(f_2, f_3)$, we must have

$$f_i(a_1, z_1) - f_i(c_1, z_1) + f_i(c_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) = 0.$$

To have a convenient notation for the alternating sums above, we define the following functions on the polynomial rings:

$$\begin{aligned} &\Delta_2 : \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, t_2, t_3, t_4] \\ &\quad f(s_1, s_2) \mapsto f(t_1, t_3) - f(t_2, t_3) + f(t_2, t_4) - f(t_1, t_4), \end{aligned}$$

and

$$\begin{aligned}\Delta_3 : \mathbb{F}[s_1, s_2] &\to \mathbb{F}[t_1, \ldots, t_6]\\ f(s_1, s_2) &\mapsto f(t_1, t_4) - f(t_2, t_4) + f(t_2, t_5) - f(t_3, t_5) + f(t_3, t_6) - f(t_1, t_6).\end{aligned}$$

We are ready to formulate necessary and sufficient conditions for the existence of 4- and 6-cycles in $\Gamma_{\mathbb{F}}(f_2, f_3)$.

Proposition 2.1 ([4]). *Graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ contains a 4-cycle if and only if there exist $a, b, x, y \in \mathbb{F}$ such that the following conditions are satisfied:*

$$\begin{cases}\Delta_2(f_2)(a, b; x, y) = \Delta_2(f_3)(a, b; x, y) = 0\\ a \neq b, \quad x \neq y.\end{cases}$$

Similarly, $\Gamma_{\mathbb{F}}(f_2, f_3)$ contains a 6-cycle if and only if there exist $a, b, c, x, y, z \in \mathbb{F}$ such that the following conditions are satisfied:

$$\begin{cases}\Delta_3(f_2)(a, b, c; x, y, z) = \Delta_3(f_3)(a, b, c; x, y, z) = 0\\ a \neq b,\ b \neq c,\ c \neq a\\ x \neq y,\ y \neq z,\ z \neq x.\end{cases}$$

Note that in the above proposition the existence of a 4- or a 6-cycle depends only on a sequence of the first coordinates $a_1, x_1, \ldots$ of its consecutive points and lines of the cycle. Therefore, we say that a 4-cycle (1) is *of type* $(a_1, b_1; x_1, y_1)$, and a 6-cycle (2) is *of type* $(a_1, b_1, c_1; x_1, y_1, z_1)$. Note that the type of a cycle is defined by its first vertex (always chosen to be an element of the partite set P) and a direction on the cycle. Hence, there can be up to $2n$ distinct types for a $2n$-cycle, $n = 2, 3$, and the values of $\Delta_2(f)$ or $\Delta_3(f)$ on all types of a 4- or a 6-cycle are equal to zero. Therefore, when it is convenient, we will use the notation $\Delta_2(f)(S)$ instead of $\Delta_2(f)(a, b; x, y)$ if S is a 4-cycle of type $(a, b; x, y)$, and similarly for a 6-cycle.

The following known isomorphisms of graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ will be especially useful for us. For a polynomial $h = h(x, y)$, let $h^* = h(y, x)$, and for $\Gamma = \Gamma_{\mathbb{F}}(f_2, f_3)$, let $\Gamma^* = \Gamma_{\mathbb{F}}(f_2^*, f_3^*)$.

Proposition 2.2. *Let $\mathbb{F}$ be a field, and let $f_2, f_3 \in \mathbb{F}[x, y]$. Then the following hold.*

(1) *The point–line isomorphism: $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2^*, f_3^*)$.*
(2) *For any $c \in \mathbb{F} \setminus \{0\}$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, cf_3)$.*
(3) *Let $g \in \mathbb{F}[x]$ and $h \in \mathbb{F}[y]$. Then $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, f_3 + g + h)$.*
(4) *$\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_3, f_2)$.*
(5) *For any $\delta \in \mathbb{F}$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, f_3 + \delta f_2)$.*

Proof. Let us define a mapping ϕ on the vertex sets of the graphs for each part above. The verification that ϕ is indeed an isomorphism is left to the reader.

For (1), take $\phi : (p_1, p_2, p_3) \mapsto [p_1, p_2, p_3]$ and $\phi : [l_1, l_2, l_3] \mapsto (l_1, l_2, l_3)$.
For (2), take $\phi : (p_1, p_2, p_3) \mapsto (p_1, p_2, cp_3)$ and $\phi : [l_1, l_2, l_3] \mapsto [l_1, l_2, cl_3]$.
For (3), take $\phi : (p_1, p_2, p_3) \mapsto (p_1, p_2, p_3 - g(p_1))$ and $\phi : [l_1, l_2, l_3] \mapsto [l_1, l_2, l_3 - h(l_1)]$.
For (4), take $\phi : (p_1, p_2, p_3) \mapsto (p_1, p_3, p_2)$ and $\phi : [l_1, l_2, l_3] \mapsto [l_1, l_3, l_2]$.
For (5), take $\phi : (p_1, p_2, p_3) \mapsto (p_1, p_2, p_3 + \delta p_2)$ and $\phi : [l_1, l_2, l_3] \mapsto [l_1, l_2, l_3 + \delta l_2]$. □

Part (3) of Proposition 2.2 is of particular importance. Indeed, it allows us to assume for the remainder of this paper that for any graph $\Gamma_{\mathbb{F}}(f_2, f_3)$, f_2 and f_3 contain only *mixed terms*, i.e., every monomial summand of each of them has degree at least one with respect to x and with respect to y. In other words, we may assume that $f_i(x, 0)$ and $f_i(0, y)$ are zero polynomials, $i = 2, 3$.

3. Proof of Theorem 1.1

Let n be a positive integer, and suppose

$$f = \sum_{1 \le i,j \le n} \alpha_{i,j} x^i y^j \in \mathbb{F}[x, y]. \tag{3}$$

Let $\Gamma = \Gamma_{\mathbb{F}}(xy, f)$ have girth at least eight.

By part (5) of Proposition 2.2, $\Gamma_{\mathbb{F}}(xy, f) \cong \Gamma_{\mathbb{F}}(xy, f + \delta xy)$ for all $\delta \in \mathbb{F}$. Thus, we may assume that $\alpha_{1,1} = 0$ for the remainder of this paper. Also, as $xy = yx$, part (1) of Proposition 2.2 implies that $\Gamma \cong \Gamma^* = \Gamma_{\mathbb{F}}(xy, f^*)$. Hence, Γ^* is of girth at least eight. Therefore neither Γ nor Γ^* contains a 6-cycle S of type $(0, tc, c; 0, 1, 1-t)$, $c \neq 0$, $t \neq 0, 1$. As $\Delta_3(xy)(S) = 0$ for every c and t, the absence of this particular type of a 6-cycle imposes strong restrictions on f. To begin, we have

$$\Delta_3(f)(S) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \alpha_{i,j} \left(t^i - 1 + (1-t)^j\right) \right] c^i \neq 0$$

and

$$\Delta_3(f^*)(S) = \sum_{j=1}^{n}\left[\sum_{i=1}^{n}\alpha_{i,j}\left(t^j - 1 + (1-t)^i\right)\right]c^j \neq 0.$$

Using the notation

$$g_i = g_i(t) = \sum_{j=1}^{n}\alpha_{i,j}\left(t^i - 1 + (1-t)^j\right) \quad \text{and}$$

$$g_j^* = g_j^*(t) = \sum_{i=1}^{n}\alpha_{i,j}\left(t^j - 1 + (1-t)^i\right),$$

we obtain

$$\Delta_3(f)(S) = \sum_{i=1}^{n} g_i c^i \neq 0 \quad \text{and} \quad \Delta_3(f^*)(S) = \sum_{j=1}^{n} g_j^* c^j \neq 0.$$

If $g_i = 0$ for every $i = 1, \ldots, n$, then $\Delta_3(f)(S) = 0$, and so both Γ and Γ^* contains a 6-cycle. Similarly, if $g_j^* = 0$ for every $j = 1, \ldots, n$, then $\Delta_3(f^*)(S) = 0$, and both Γ^* and Γ contains S.

If at least two of the g_i are nonzero polynomials (of t), say g_r and g_s, $1 \le r \ne s \le n$, then

$$\text{there exists } t' \in \mathbb{F} \setminus \{0, 1\} \text{ such that } g_i(t') \neq 0 \text{ for } i = r, s. \tag{4}$$

We claim that in this case Γ contains a 6-cycle S' of type $(0, t'c, c; 0, 1, 1-t')$ for some $c \neq 0$. Indeed, $\Delta_3(f)(S')$, viewed as a univariate polynomial in c, has the property that the coefficients at c^r and at c^s are nonzero. Therefore,

$$\Delta_3(f)(S') \text{ has a nonzero root } c' \in \mathbb{F}, \tag{5}$$

and so Γ contains a 6-cycle S' of type $(0, t'c', c'; 0, 1, 1-t')$. Similarly, if at least two of the $g_j^*(t)$ are nonzero polynomials, then Γ^* contains a 6-cycle.

Therefore, we conclude that *there exist unique positive integers i' and j', $1 \le i', j' \le n$, such that both $g_{i'}$ and $g_{j'}^*$ are nonzero polynomials.*

Hence, $g_i = 0$ for all $i \neq i'$. Let $i' \neq i < n$; consider the coefficient of t^n in g_i, namely $(-1)^n\alpha_{i,n}$. As $g_i = 0$, $\alpha_{i,n} = 0$. Continuing this logic, the coefficient of t^j for each $j = n-1, n-2, \ldots, i+1$ is $(-1)^j\alpha_{i,j}$, and hence $\alpha_{i,j} = 0$ for every $i \notin \{i', n\}$ and $j = i+1, \ldots, n$. By similar reasoning, suppose $j < n$; $g_j^* = 0$ for every $j \neq j'$ implies that $\alpha_{i,j} = 0$ for every $i = j+1, \ldots, n$ and $j \notin \{j', n\}$. Recalling that $\alpha_{1,1} = 0$, we have proven that

$$\alpha_{i,j} = 0 \quad \text{except possibly when} \begin{cases} i = j > 1, & \text{or} \\ i = i' \text{ and } & \text{either } j = j' \text{ or } j \ge i', \text{ or} \\ j = j' \text{ and } & \text{either } i = i' \text{ or } i \ge j'. \end{cases} \tag{6}$$

Suppose without loss of generality that $j' \le i'$ (applying the point–line isomorphism yields the case $i' \le j'$). We will now prove that $\alpha_{i,i} = 0$ for all $i \notin \{i', 2\}$. If $i \le j'$ (and $i \neq i'$), then

$$0 = g_i = \alpha_{i,i}(t^i - 1 + (1-t)^i), \tag{7}$$

and so $\alpha_{i,i} = 0$ ($\alpha_{1,1} = 0$ was assumed; it is not a direct consequence of (7)). In particular, note that unless $i' = j'$, $\alpha_{j',j'} = 0$. By similar reasoning, considering g_j^* proves that unless $i' = j'$, $\alpha_{i',i'} = 0$. If instead $i > j'$ (and $i \neq i'$), we have

$$0 = g_i = \alpha_{i,j'}\left(t^i - 1 + (1-t)^{j'}\right) + \alpha_{i,i}\left(t^i - 1 + (1-t)^i\right). \tag{8}$$

There are two cases to consider. If $i \ge j'+2$ and $i \neq i'$, then the coefficient of t^{i-1} in g_i, namely

$$(-1)^{i-1} i\alpha_{i,i}, \text{ must be zero; thus, } \alpha_{i,i} = 0. \tag{9}$$

If instead $i = j'+1 \neq i'$, we have

$$0 = g_{j'+1} = \alpha_{j'+1,j'}\left(t^{j'+1} - 1 + (1-t)^{j'}\right) + \alpha_{j'+1,j'+1}\left(t^{j'+1} - 1 + (1-t)^{j'+1}\right).$$

The coefficients of $t^{j'}$ and $t^{j'+1}$, namely $\alpha_{j'+1,j'}(-1)^{j'} + \alpha_{j'+1,j'+1}(-1)^{j'}(j'+1)$ and $\alpha_{j'+1,j'} + \alpha_{j'+1,j'+1}\left(1 + (-1)^{j'+1}\right)$ respectively, must be zero. Therefore, we have

$$\begin{cases} \alpha_{j'+1,j'} = -(j'+1)\alpha_{j'+1,j'+1} & \text{regardless of } j', \\ \alpha_{j'+1,j'} = 0 & \text{if } j' \text{ is even}, \\ \alpha_{j'+1,j'} = -2\alpha_{j'+1,j'+1} & \text{if } j' \text{ is odd}. \end{cases} \tag{10}$$

This implies that

$$\text{if } j' \text{ is even, then } \alpha_{j'+1,j'+1} = 0. \tag{11}$$

If instead $j' > 1$ is odd, then $\alpha_{j'+1,j'} = \alpha_{j'+1,j'+1} = 0$. Therefore, we have proven that $\alpha_{i,i} = 0$ for every $i \notin \{i', 2\}$, as desired. Note that the $i \neq 2$ statement results from the case $j' = 1$. Updating (6), we obtain:

$$\alpha_{i,j} = 0 \quad \text{except possibly when} \begin{cases} i = j = 2 \quad \text{and} \quad j' = 1, \text{ or} \\ i = i' \quad \text{and} \quad \text{either } j = j' \text{ or } j > i', \text{ or} \\ j = j' \quad \text{and} \quad \text{either } i = i' \text{ or } i > j'. \end{cases} \tag{12}$$

Note that the equality $i = i'$ in the third case of (12) is actually superfluous given the assumption that $i' \geq j'$. However, we did not remove it in order to stress the duality of these cases.

Now, let $i > j'$ such that $i \notin \{i', 2\}$. Then (8) becomes

$$0 = g_i = \alpha_{i,j'} \left(t^i - 1 + (1-t)^{j'}\right).$$

Thus, either $\alpha_{i,j'} = 0$ or $t^i - 1 + (1-t)^{j'} = 0$. The latter possibility would imply that $i = j' = 1$, which contradicts $i > j'$. Therefore, we have $\alpha_{i,j'} = 0$ for all $i > j'$ such that $i \notin \{i', 2\}$. By similar reasoning, $\alpha_{i',j} = 0$ for all $j > i'$ such that $j \notin \{j', 2\}$.

This proves that the only possibly non-zero $\alpha_{i,j}$ are $\alpha_{2,2}$ (when $j' = 1$) and $\alpha_{i',j'}$. When $j' = 1$, $g_2^* = 0$; hence, $\alpha_{2,2} = 0$. Thus, $f = \alpha_{i',j'} x^{i'} y^{j'}$ and

$$\Delta_3(f)(S) = \alpha_{i',j'} \left(t^{i'} - 1 + (1-t)^{j'}\right) c^{i'}.$$

As $i' > 1$ ($1 = i' \geq j'$ would imply $i' = j' = 1$, and thus the contradiction $f = 0$), the roots $t = 0$ and $t = 1$ of $t^{i'} - 1 + (1-t)^{j'}$ each have multiplicity one; indeed, neither is a root of

$$\frac{\partial}{\partial t}\left(t^{i'} - 1 + (1-t)^{j'}\right) = i' t^{i'-1} - j'(1-t)^{j'-1}. \tag{13}$$

Thus, we conclude that

$$t^{i'} - 1 + (1-t)^{j'} \text{ has a root } t \in \mathbb{F} \setminus \{0, 1\} \tag{14}$$

whenever its degree is at least three (i.e. except when $i' = j' = 3$, $i' = j' = 2$, or $i' = 2$ and $j' = 1$). Therefore, Γ contains a 6-cycle of type S unless $f = \alpha_{3,3} x^3 y^3 \neq 0$, $f = \alpha_{2,2} x^2 y^2 \neq 0$, or $f = \alpha_{2,1} x^2 y \neq 0$. Note that if we consider $i' \leq j'$ instead of $j' \leq i'$, we could also have $f = \alpha_{1,2} x y^2 \neq 0$.

To conclude, note that the first two options do not yield girth eight graphs, as $\Gamma_{\mathbb{F}}(xy, \alpha_{2,2}x^2y^2)$ and $\Gamma_{\mathbb{F}}(xy, \alpha_{3,3}x^3y^3)$ contain 6-cycles of type $(0, 2, 1; 1, 0, 2)$ for any nonzero $\alpha_{2,2}, \alpha_{3,3} \in \mathbb{F}$. When $f = \alpha_{1,2} x y^2 \neq 0$ or $f = \alpha_{2,1} x^2 y \neq 0$, $\Gamma_{\mathbb{F}}(xy, f) \cong \Gamma_{\mathbb{F}}(xy, xy^2)$. □

4. Proof of Theorem 1.2

Let $\mathbb{F} = \mathbb{F}_{q^{Mr}}$ for some positive integer r. In what follows, we indicate all places in the proof of Theorem 1.1 in Section 3 where it mattered that $\mathbb{F}$ was an algebraically closed field of characteristic zero. In each case, we show how to modify the proof to make it valid over finite fields of odd characteristic.

- In (3), the assumption that f has degree at most $p - 2$ implies that $1 \leq i, j \leq n \leq p - 2$. These inequalities will hold throughout this new proof.
- In (4), there exists $t' \in \mathbb{F} \setminus \{0, 1\}$ such that $g_i(t') \neq 0$ for $i = r, s$, $1 \leq r \neq s \leq p - 2$. Indeed, the polynomials $g_r(t)$ and $g_s(t)$ have degree at most $p - 2$, and $|\mathbb{F}| = q^{Mr}$ with $M \geq 2$ implies that $|\mathbb{F}| \geq p^2 > 2p - 2 = 2 + 2(p-2) \geq 2 + r + s$.
- In (5), $\Delta_3(f)(S')$ has a nonzero root $c' \in \mathbb{F}$ because the degree of f is at most $p - 2$, and so $\mathbb{F}_q(c')$ is isomorphic to a subfield of $\mathbb{F}_{q^M}$.
- In (9), $(-1)^{i-1} i \alpha_{i,i} = 0$ implies $\alpha_{i,i} = 0$ because $1 \leq i \leq p - 2$.
- In (11), $-(j'+1)\alpha_{j'+1,j'+1} = 0$ implies $\alpha_{j'+1,j'+1} = 0$ because $1 \leq j' \leq p - 2$.
- In (13), $1 \leq i' \leq p - 2$ and $1 \leq j' \leq p - 2$. Hence, neither $t = 0$ nor $t = 1$ are roots of $i' t^{i'-1} - j'(1-t)^{j'-1}$.
- In (14), $t^{i'} - 1 + (1-t)^{j'}$ has a root $t \in \mathbb{F} \setminus \{0, 1\}$ because $|\mathbb{F}| \geq 3$.

This concludes the proof of Theorem 1.2. □

5. Proof of Corollary 1.3

It is possible to obtain some additional results about $\Gamma_q(f_2, f_3)$ by using a version of the Lefschetz Principle, see e.g. Marker [10,11], which implies that for any sentence ϕ in the language of rings, the validity of ϕ over an algebraically closed field of characteristic zero $\mathbb{F}$ implies its validity over all algebraically closed fields of sufficiently large characteristic.

The sentences we consider are the claims of existence of either 4- or 6-cycles in graphs $\Gamma_{\mathbb{F}}(xy, f)$, for $f \in \mathbb{Z}[x, y]$. As we have seen in Proposition 2.1, these sentences can be rewritten as the claims that certain systems of polynomial equations have a solution. Note that also having inequalities in our systems for the first coordinates of consecutive points or consecutive lines of the cycle is not a restriction. These inequalities can be replaced by equations which use new variables: the inequality $a \neq b$ is equivalent to the statement that the polynomial equation $1 + t(a - b) = 0$ has a solution for a new variable t. The requirement that the coefficients of f are integers is essential because it allows us to view f as a polynomial both over $\overline{\mathbb{F}_p}$ (as $\hat{f}$) and over $\mathbb{F}$.

The algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, p is a prime, can be thought of as the union of all finite fields of characteristic p, i.e., $\overline{\mathbb{F}}_p = \bigcup_{n \geq 1} \mathbb{F}_{p^n}$. A proof can be found in Dummit and Foote [6]. Hence, the validity of a sentence ϕ over $\overline{\mathbb{F}}_p$ implies the existence of a positive integer s such that ϕ is valid for $\mathbb{F}_{p^s}$, and so for all $\mathbb{F}_{p^{sr}}$, $r \geq 1$. This proves Corollary 1.3. □

6. Concluding remarks

Though generalized quadrangles are traditionally viewed as incidence geometries (see Payne and Thas [14], Payne [13], Thas [16], or Van Maldeghem [20]), they can be presented in purely graph theoretic terms. Indeed, for $r \geq 2$, a finite regular generalized quadrangle GQ(r) is an $(r+1)$-regular bipartite graph of diameter four and girth eight. For every prime power r, GQ(r) exist; no example of GQ(r) is known if r is not a prime power. Moreover, when $r = q$ is an odd prime power, only one GQ(q) is known (which corresponds to two dual geometries). We will denote it by Λ_q. It is known that the automorphism group of Λ_q acts transitively on each partition and on its edge set.

The following idea of constructing a new $GQ(q)$ was suggested by Ustimenko in the 1990s. Consider a subgraph Λ'_q of Λ_q induced by the vertices at distance three from a fixed edge. Then Λ'_q is a bipartite q-regular graph of girth eight and diameter six with each partition containing q^3 vertices (see, e.g., Pralle [15] and Dmytrenko [4]). Deletion of all edges of Λ'_q from Λ_q results in a $(q+1)$-regular spanning tree of Λ_q (all inner vertices have degree $q+1$), and we can say that Λ_q is obtained from Λ'_q by "attaching" the $(q+1)$-tree to it. As the graph Λ_q is edge-transitive, all its induced subgraphs Λ'_q are isomorphic.

Most importantly, it is also known that Λ'_q is isomorphic to the graph $\Gamma_q(xy, xy^2)$. This result is implicit in Ustimenko [17–19], and is more explicit in Payne [12], Lazebnik and Ustimenko [9], and [20], where Λ'_q is presented with slightly different equations. The presentation of Λ'_q as $\Gamma_q(xy, xy^2)$ appears in Viglione [21], and with more details in [4].

Hence, if it were possible to replace the polynomial xy^2 by a polynomial $f \in \mathbb{F}_q[x, y]$ such that the graph $\Gamma_q(xy, f)$ had girth eight, diameter six, and was not isomorphic to the graph Λ'_q, then one could try to attach a $(q+1)$-regular tree to it to obtain a new $GQ(q)$. The results in this paper show that for this construction to succeed for all q, one must go beyond the families of graphs $\Gamma_q(xy, f)$.

We end with a few comments. For odd prime powers q, the girth of the graph $\Gamma_q(f_2, f_3)$ has been studied extensively in the case where f_2 and f_3 are monomials in $\mathbb{F}[x, y]$, see [4,5] and [8]. Recently, this case was resolved in [7].

Theorem 6.1 *([7]). Let $q = p^e$ be an odd prime power, and let f_2 and f_3 be monomials in $\mathbb{F}_q[x, y]$. Then every graph $\Gamma_q(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$.*

Theorem 6.1 is of particular interest because it stands in stark contrast to the situation in which q is a power of 2. In this case, there are examples of both monomial and non-monomial graphs $\Gamma_q(f_2, f_3)$ of girth eight, which do lead to nonisomorphic generalized quadrangles. Hence, the strategy of constructing new generalized quadrangles as described above succeeds in this case. See [13,20], and Cherowitzo [3] for additional information.

It was shown in [4] that for a binomial f, the girth of $\Gamma_q(xy, f)$ is six, provided that the odd prime power q is sufficiently large.

Open Problem 6.2. *Let $q = p^e$ be an odd prime power, and let $f_2, f_3 \in \mathbb{F}_q[x, y]$. Is it true that every graph $\Gamma_q(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$?*

Over algebraically closed fields $\mathbb{F}$ of characteristic zero, we believe that an analog of Theorem 1.1 holds for all $\Gamma_{\mathbb{F}}(f_2, f_3)$:

Conjecture 6.3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, let $f_2, f_3 \in \mathbb{F}[x, y]$, and let $\Gamma_{\mathbb{F}}(f_2, f_3)$ have girth at least eight. Then $\Gamma_{\mathbb{F}}(f_2, f_3)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.*

The authors, together with Jason Williford, have recently found a proof of Conjecture 6.3 for $f_2 = x^m y^n$, where $m, n \geq 2$, and every $f_3 \in \mathbb{F}[x, y]$. It makes use of Theorem 1.1 of this paper, but given the proof's length, it will be addressed elsewhere.

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