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On the uniqueness of some girth eight algebraically defined graphs, Part II

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Dedicated to the memory of Vasyl Dmytrenko (1961-2013)

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ABSTRACT

Let $\mathbb{F}$ be a field. For a polynomial $f \in \mathbb{F}[x, y]$ and positive integers k and m, we define a bipartite graph $\Gamma_{\mathbb{F}}(x^k y^m, f)$ with vertex partition $P \cup L$, where P and L are two copies of $\mathbb{F}^3$, and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if

$$p_2 + l_2 = p_1^k l_1^m \text{ and } p_3 + l_3 = f(p_1, l_1)$$

It is known that $\Gamma_{\mathbb{F}}(xy, xy^2)$ has no cycles of length less than eight. The main result of this paper is that $\Gamma_{\mathbb{F}}(xy, xy^2)$ is the only graph $\Gamma_{\mathbb{F}}(x^k y^m, f)$ with this property when $\mathbb{F}$ is an algebraically closed field of characteristic zero; i.e. over such a field $\mathbb{F}$, every graph $\Gamma_{\mathbb{F}}(x^k y^m, f)$ with no cycles of length less than eight is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$. We also prove related uniqueness results over infinite families of finite fields.

1. Introduction

In this paper, we extend the results of Kronenthal and Lazebnik [5]. Since both papers study similar objects and have the same motivation, we will minimize our discussion in Sections 1, 2 and 5 of this paper and refer the reader to [5] for additional details.

For a field $\mathbb{F}$ and two polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, let P and L be two copies of the 3-dimensional vector space $\mathbb{F}^3$. Consider a bipartite graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ with vertex partitions P and L and with edges defined as follows: for every $(p) = (p_1, p_2, p_3) \in P$ and every $[l] = [l_1, l_2, l_3] \in L$, $\{(p), [l]\} = (p)[l]$ is an edge in $\Gamma_{\mathbb{F}}(f_2, f_3)$ if

$$p_2 + l_2 = f_2(p_1, l_1)$$
$$p_3 + l_3 = f_3(p_1, l_1).$$

It turns out that the graph $\Gamma_{\mathbb{F}}(xy, xy^2)$ has girth eight and for finite fields $\mathbb{F}$, has been studied in the context of finite geometries and extremal graph theory (see [5] for details). This brings up a natural question: is $\Gamma_{\mathbb{F}}(xy, xy^2)$ the unique (up to isomorphism) girth eight graph of the form $\Gamma_{\mathbb{F}}(f_2, f_3)$?

In this paper, as in [5], we let $\mathbb{F}$ be an algebraically closed field of characteristic zero, for example the field of complex numbers $\mathbb{C}$. For such $\mathbb{F}$, it is clear that $\Gamma_{\mathbb{F}}(f_2, f_3)$ is an infinite graph. We prove the uniqueness of $\Gamma_{\mathbb{F}}(xy, xy^2)$ for all graphs in

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'close vicinity', or more precisely all graphs of the form $\Gamma_{\mathbb{F}}(x^k y^m, f)$, $f \in \mathbb{F}[x, y]$. The main result of this paper (which was mentioned in the concluding remarks of [5]) is as follows.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and let k and m be positive integers. Suppose $f \in \mathbb{F}[x, y]$ and the graph $\Gamma_{\mathbb{F}}(x^k y^m, f)$ has girth at least eight. Then $\Gamma_{\mathbb{F}}(x^k y^m, f)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.*

The case $k = m = 1$ was proven in [5], and the proof of Theorem 1.1 will depend on it.

The following theorem is an analog of Theorem 1.1 for finite fields $\mathbb{F} = \mathbb{F}_q$ of characteristic p; its proof will mirror that of Theorem 1.1. When $\mathbb{F} = \mathbb{F}_q$, we simplify the notation $\Gamma_{\mathbb{F}_q}(f_2, f_3)$ to $\Gamma_q(f_2, f_3)$.

Theorem 1.2. *Let q be a power of a prime p, $p \geq 3$. Suppose that integers k and m are relatively prime to p, and that $f \in \mathbb{F}_q[x, y]$ has degree at most $p - 2$ with respect to each of x and y. Then there exists a positive integer $M = M(k, m, q)$ such that for all positive integers r, every graph $\Gamma_{q^{Mr}}(x^k y^m, f)$ of girth at least eight is isomorphic to $\Gamma_{q^{Mr}}(xy, xy^2)$.*

In the following corollary, we obtain some additional results about $\Gamma_q(f_2, f_3)$ by using a version of the Lefschetz Principle, see e.g. Marker [6,7], which implies that for any sentence ϕ in the language of rings, the validity of ϕ over an algebraically closed field of characteristic zero $\mathbb{F}$ implies its validity over all algebraically closed fields of sufficiently large characteristic. For a polynomial $f = \sum_{0 \leq i,j \leq n} a_{ij} x^i y^j \in \mathbb{Z}[x, y]$, let $\hat{f} = \sum_{0 \leq i,j \leq n} \hat{a}_{ij} x^i y^j \in \mathbb{F}_p[x, y]$, where $\hat{a}_{ij}$ is the image of a_{ij} with respect to the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Corollary 1.3 (*to Theorem 1.1*). *Suppose $f \in \mathbb{Z}[x, y]$ and that k and m are positive integers. Then there exists a positive constant $c = c(f)$ such that for every prime $p > c(f)$, there exists an integer $s = s(f, p)$ such that for all positive integers r, and $q = p^{sr}$, every graph $\Gamma_q(x^k y^m, \hat{f})$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$.*

The proof of Corollary 1.3 follows similarly to that of Corollary 1.3 from [5]; please see Section 5 of [5] for details.

We wish to comment that in Theorem 1.1, we consider all polynomials f over an algebraically closed field of characteristic zero. However, in Corollary 1.3, we fix one polynomial with integer coefficients and state the existence of infinitely many characteristics p, and infinitely many finite fields of characteristic p, over which an analog of Theorem 1.1 holds.

This paper is organized as follows. In Section 2 we provide a description of 4- and 6-cycles in graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ and some isomorphisms between these graphs. In Sections 3 and 4, we present proofs of Theorems 1.1 and 1.2, respectively. In Section 5, we make concluding remarks and state an open problem and a conjecture.

2. Cycles and isomorphisms of graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$

Let $\Gamma_{\mathbb{F}}(f_2, f_3)$ be the graph defined in Section 1. Define the functions

$$\begin{aligned} \Delta_2 &: \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, t_2, t_3, t_4] \\ &f(s_1, s_2) \mapsto f(t_1, t_3) - f(t_2, t_3) + f(t_2, t_4) - f(t_1, t_4), \end{aligned}$$

and

$$\begin{aligned} \Delta_3 &: \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, \ldots, t_6] \\ &f(s_1, s_2) \mapsto f(t_1, t_4) - f(t_2, t_4) + f(t_2, t_5) - f(t_3, t_5) + f(t_3, t_6) - f(t_1, t_6). \end{aligned}$$

These functions allow us to formulate necessary and sufficient conditions for the existence of 4- and 6-cycles in $\Gamma_{\mathbb{F}}(f_2, f_3)$; for more details on their derivation, see Dmytrenko [1], [5], and references therein.

Proposition 2.1 ([1]). *Graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ contains a 4-cycle if and only if there exist $a, b, x, y \in \mathbb{F}$ such that the following conditions are satisfied:*

$$\begin{cases} \Delta_2(f_2)(a, b; x, y) = \Delta_2(f_3)(a, b; x, y) = 0 \\ a \neq b, \quad x \neq y. \end{cases}$$

Similarly, $\Gamma_{\mathbb{F}}(f_2, f_3)$ contains a 6-cycle if and only if there exist $a, b, c, x, y, z \in \mathbb{F}$ such that the following conditions are satisfied:

$$\begin{cases} \Delta_3(f_2)(a, b, c; x, y, z) = \Delta_3(f_3)(a, b, c; x, y, z) = 0 \\ a \neq b, \; b \neq c, \; c \neq a \\ x \neq y, \; y \neq z, \; z \neq x. \end{cases}$$

Note that in the above proposition the existence of a 4- or a 6-cycle depends only on a sequence of the first coordinates $a_1, x_1, \ldots$ of its consecutive points and lines of the cycle. Therefore, we say that a 4-cycle is *of type* $(a_1, b_1; x_1, y_1)$, and a 6-cycle is *of type* $(a_1, b_1, c_1; x_1, y_1, z_1)$; note that a_1, b_1, c_1 are the first coordinates of points while x_1, y_1, z_1 are the first coordinates of lines. Note that the type of a cycle is defined by its first vertex (always chosen to be a point) and a direction on the cycle. Hence, there can be up to $2n$ distinct types for a $2n$-cycle, $n = 2,3$, and the values of $\Delta_2(f)$ or $\Delta_3(f)$ on all types of a 4- or a 6-cycle are equal to zero. Therefore, when it is convenient, we will use the notation $\Delta_2(f)(S)$ instead of $\Delta_2(f)(a, b; x, y)$ if S is a 4-cycle of type $(a, b; x, y)$, and similarly for a 6-cycle.

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The following known isomorphisms of graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ will be especially useful for us; we refer the reader to [5] for proofs. For a polynomial $h = h(x, y)$, let $h^* = h(y, x)$.

Proposition 2.2. *Let $\mathbb{F}$ be a field, and let $f_2, f_3 \in \mathbb{F}[x, y]$. Then the following hold.*

(1) *The point-line isomorphism: $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2^*, f_3^*)$.*
(2) *For any $c \in \mathbb{F} \setminus \{0\}$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, cf_3)$.*
(3) *Let $g \in \mathbb{F}[x]$ and $h \in \mathbb{F}[y]$. Then $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, f_3 + g + h)$.*
(4) *$\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_3, f_2)$.*
(5) *For any $\delta \in \mathbb{F}$, $\Gamma_{\mathbb{F}}(f_2, f_3) \cong \Gamma_{\mathbb{F}}(f_2, f_3 + \delta f_2)$.*

Part (3) of Proposition 2.2 is of particular importance. Indeed, it allows us to assume for the remainder of this paper that for any graph $\Gamma_{\mathbb{F}}(f_2, f_3)$, f_2 and f_3 contain only *mixed terms*, i.e., every monomial summand of each of them has degree at least one with respect to x and with respect to y. In other words, we may assume that $f_i(x, 0)$ and $f_i(0, y)$ are zero polynomials, $i = 2,3$.

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1 when $k \geq 2$ or $m \geq 2$. We remind the reader that the case $k = m = 1$ was proven in [5].

Let k and m be positive integers, and define

$$f = \sum_{1 \leq i,j \leq n} \alpha_{i,j} x^i y^j \in \mathbb{F}[x, y]. \tag{1}$$

Note that f only contains mixed terms, as explained at the end of the previous section. The main idea of the proofs is to exhibit a sequence of particular 4- and 6-cycles which impose strong conditions on the parameters k, m, and $\alpha_{i,j}$. Our logic is as follows.

- First, we show that when both $k \geq 2$ and $m \geq 2$, the girth of $\Gamma_{\mathbb{F}}(x^k y^m, f)$ is at most six. An equivalent formulation of this statement is Proposition 3.1. In the proof of this proposition, we also obtain important information about the polynomial f.
- We next use the point-line isomorphism to conclude that every graph $\Gamma_{\mathbb{F}}(x^a y^b, g)$ of girth at least eight is isomorphic to a graph $\Gamma_{\mathbb{F}}(xy^m, f)$.
- Finally, we show that if $\Gamma_{\mathbb{F}}(xy^m, f)$ has girth at least eight, then $m = 1$ or 2 (see Propositions 3.2 and 3.3). We conclude the proof with Proposition 3.5.

Proposition 3.1. *If $\Gamma = \Gamma_{\mathbb{F}}(x^k y^m, f)$ has girth at least eight, then $k = 1$ or $m = 1$.*

The idea of this proof is to use the fact that Γ has neither 4-cycles nor 6-cycles. The absence of cycles of particular types allows us to restrict the values of k and m, as well as the coefficients of f. The cycles we used were found by experimenting with various cycle types and determining which would provide more information about k, m, and f. Recall that the presence of a 4- or 6-cycle can be determined algebraically (see Proposition 2.1).

Proof of Proposition 3.1. Our proof is based on Claims 3.1.1 and 3.1.2.

Claim 3.1.1. *Let f be as in (1). Then:*

(i) *If $k \geq 2$, then for $j \neq j'$ and for each i, either $\alpha_{i,j} = 0$ or $k|i$.*
(ii) *If $m \geq 2$, then for $i \neq i'$ and for each j, either $\alpha_{i,j} = 0$ or $m|j$.*

Proof. (i) Γ contains no 4-cycles, in particular no 4-cycle S_1 of type $(a, \zeta a; 0, d)$, where $a \neq 0$, $d \neq 0$, and ζ is a primitive k^{th} root of unity. Now, we have $\Delta_2(x^k y^m)(S_1) = 0$ and

$$\Delta_2(f)(S_1) = \sum_{1 \leq i,j \leq n} \alpha_{i,j} \left[(\zeta a)^i d^j - a^i d^j \right] = \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \alpha_{i,j} (\zeta^i - 1) a^i \right] d^j.$$

Define

$$g_j = g_j(a) = \sum_{i=1}^{n} \alpha_{i,j} (\zeta^i - 1) a^i,$$

which implies

$$\Delta_2(f)(S_1) = \sum_{j=1}^{n} g_j(a) d^j.$$

If $g_j = 0$ for every $j = 1, \ldots, n$, then $\Delta_2(f)(S_1) = 0$, and so Γ would contain a 4-cycle. If at least two of the g_j are nonzero polynomials (of a), say g_s and g_t, $1 \le s \ne t \le n$, then

$$\text{there exists } a' \in \mathbb{F} \setminus \{0\} \text{ such that } g_j(a') \ne 0 \text{ for } j = s, t. \tag{2}$$

We claim that in this case Γ would contain a 4-cycle S_1' of type $(a', \zeta a'; 0, d)$ for some $d \ne 0$. Indeed, $\Delta_2(f)(S_1')$, viewed as a univariate polynomial in d, has the property that the coefficients at d^r and at d^s are nonzero. Therefore,

$$\Delta_2(f)(S_1') \text{ has a nonzero root } d' \in \mathbb{F}, \tag{3}$$

and so Γ would contain a 4-cycle S' of type $(a', \zeta a'; 0, d')$. Thus, there exists a unique positive integer j', $1 \le j' \le n$, such that $g_{j'}$ is a nonzero polynomial of a. Furthermore,

$$\text{for } j \ne j' \text{ and for each } i, \text{ either } \alpha_{i,j} = 0 \text{ or } \zeta^i = 1 \text{ (i.e. } k|i), \tag{4}$$

as claimed in (i).

(ii) The proof mimics the proof of part (i), replacing the cycle S_1 with a 4-cycle S_2 of type $(0, b; c, \xi c)$, where $b \ne 0$, $c \ne 0$, and ξ is a primitive m^{th} root of unity. □

Remark: Please note that the primitivity requirement is neither important in this step, nor in many other similar instances in this section. However, we will assume primitivity because it is not restrictive, and forces the root to not equal 1.

The results of Claim 3.1.1 imply that f must have one of the following forms:

1. If $k \nmid i'$ and $m \nmid j'$, then
$$f = \sum_{s,t} \alpha_{sk,tm} x^{sk} y^{tm} + \alpha_{i',j'} x^{i'} y^{j'}.$$
2. If $k \nmid i'$ and $m|j'$, then
$$f = \sum_{\substack{s,t \\ t \ne j'/m}} \alpha_{sk,tm} x^{sk} y^{tm} + \sum_{i \ne i'} \alpha_{i,j'} x^{i} y^{j'} + \alpha_{i',j'} x^{i'} y^{j'}.$$
3. If $k|i'$ and $m \nmid j'$, then
$$f = \sum_{\substack{s,t \\ s \ne i'/k}} \alpha_{sk,tm} x^{sk} y^{tm} + \sum_{j \ne j'} \alpha_{i',j} x^{i'} y^{j} + \alpha_{i',j'} x^{i'} y^{j'}.$$
4. If $k|i'$ and $m|j'$, then
$$f = \sum_{\substack{s,t \\ s \ne i'/k \\ t \ne j'/m}} \alpha_{sk,tm} x^{sk} y^{tm} + \sum_{i \ne i'} \alpha_{i,j'} x^{i} y^{j'} + \sum_{j \ne j'} \alpha_{i',j} x^{i'} y^{j} + \alpha_{i',j'} x^{i'} y^{j'}.$$

While bounds on s and t will not be explicitly used, it should be noted that $1 \le s \le \lfloor n/k \rfloor$ and $1 \le t \le \lfloor n/m \rfloor$ in each of the above sums that they appear.

Claim 3.1.2. *If $k \ge 2$ and $m \ge 2$, then the polynomial f is of the form*
$$f = \sum_{s,t} \alpha_{sk,tm} x^{sk} y^{tm} + \alpha_{1,1} xy.$$

Proof. Γ contains no 4-cycles, in particular no 4-cycle S_3 of type $(1, \zeta; 1, \xi)$, where ζ and ξ are primitive k^{th} and m^{th} roots of unity, respectively.

Then for f of the form 2, 3, or 4 above, $\Delta_2(x^k y^m)(S_3) = \Delta_2(f)(S_3) = 0$, and so Γ would have girth four. Hence, we need only consider f of form 1, namely

$$f = \sum_{s,t} \alpha_{sk,tm} x^{sk} y^{tm} + \alpha_{i',j'} x^{i'} y^{j'} \tag{5}$$

with $k \nmid i'$ and $m \nmid j'$. If $j' > 1$, then Γ would contain a 4-cycle S_4 of type $(1, \zeta; 1, \varphi)$, where ζ and φ are primitive k^{th} and j'^{th} roots of unity, respectively. Thus, since $j' \ge 1$, we conclude $j' = 1$.

Similarly, if $i' > 1$, then Γ would contain a 4-cycle S_5 of type $(a, \phi a; c, \xi c)$, where ϕ and ξ are primitive i'^{th} and m^{th} roots of unity, respectively. Thus, since $i' \ge 1$, we conclude $i' = 1$.

Therefore,
$$f = \sum_{s,t} \alpha_{sk,tm} x^{sk} y^{tm} + \alpha_{1,1} xy. \quad \square$$

We are now ready to finish the proof of Proposition 3.1. Suppose $k \geq 2$ and $m \geq 2$. Then f has the form stated in Claim 3.1.2. In this case, if $k > 2$, then Γ would contain a 6-cycle S_6 of type $(1, \zeta_2, \zeta_3; (1-\zeta_3)/(1-\zeta_2), 0, 1)$, where

$$\zeta_2 \text{ and } \zeta_3 \text{ are distinct primitive } k^{\text{th}} \text{ roots of unity.} \tag{6}$$

Thus, $k = 2$. Similarly, if $m > 2$, then Γ would contain a 6-cycle of type $((\zeta_2-\zeta_3)/(1-\zeta_3), 0, 1; 1, \zeta_2, \zeta_3)$, which implies $m = 2$.

This leaves only the case $k = m = 2$, where we have

$$f = \sum_{s,t} \alpha_{2s,2t} x^{2s} y^{2t} + \alpha_{1,1} xy.$$

However, this would imply that Γ contains a 6-cycle of type $(1, -1, 0; 0, 1, -1)$, contradicting the fact that Γ has girth eight. Hence, $k = 1$ or $m = 1$. □

Note that by Proposition 3.1 and the point-line isomorphism, we may assume without loss of generality that $k = 1$. Also note that the forms of f that appeared in the proof of Proposition 3.1 under the assumption that $k \geq 2$ and $m \geq 2$ are no longer applicable. Nevertheless, the conclusion that $k = 1$ is instrumental to our proof of Theorem 1.1. The following proposition allows us to further restrict the form of f from (1).

Proposition 3.2. *If $\Gamma = \Gamma_{\mathbb{F}}(xy^m, f)$ has girth at least eight, then*

$$f = \alpha_{1,j^*} xy^{j^*} + \sum_{i=1}^{n} \sum_{t=1}^{\lfloor n/m \rfloor} \alpha_{i,tm} x^i y^{tm} \tag{7}$$

for some $\alpha_{1,j^*} \neq 0$ such that j^* is not divisible by m.

Proof. We continue using the labels for cycle types introduced in the proof of Proposition 3.1. As the case $k = m = 1$ was accounted for in [5], we assume $m \geq 2$. Note that $\Delta_2(xy^m)(S_2) = 0$, and from (1),

$$\Delta_2(f)(S_2) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \alpha_{i,j} \left(\xi^j - 1\right) c^j \right] b^i.$$

Then, as Γ does not contain S_2, there exists a unique i' such that for $i \neq i'$ and for each j, $\alpha_{i,j} = 0$ or $m | j$. Thus, we need only consider

$$f = \sum_{j=1}^{n} \alpha_{i',j} x^{i'} y^j + \sum_{\substack{1 \leq i \leq n \\ i \neq i'}} \sum_{t=1}^{\lfloor n/m \rfloor} \alpha_{i,tm} x^i y^{tm}.$$

If $i' > 1$, then $\Delta_2(xy^m)(S_5) = \Delta_2(f)(S_5) = 0$, and so Γ would contain a 4-cycle S_5. Hence, $i' = 1$, and so

$$f = \sum_{j=1}^{n} \alpha_{1,j} xy^j + \sum_{i=2}^{n} \sum_{t=1}^{\lfloor n/m \rfloor} \alpha_{i,tm} x^i y^{tm}.$$

Finally, let S_7 denote a 4-cycle of type $(a, b; c, \xi c)$, where $a \neq b$, $c \neq 0$, and ξ is a primitive m^{th} root of unity. Then $\Delta_2(xy^m)(S_7) = 0$ and

$$\Delta_2(f)(S_7) = (a-b) \sum_{j=1}^{n} \alpha_{1,j} \left(1 - \xi^j\right) c^j.$$

Since Γ does not contain S_7, there must exist a unique j^* such that $m \nmid j^*$ and for each $j \neq j^*$, $\alpha_{1,j} = 0$ or $m | j$. This implies that $\alpha_{1,j^*} \neq 0$ and

$$f = \alpha_{1,j^*} xy^{j^*} + \sum_{i=1}^{n} \sum_{t=1}^{\lfloor n/m \rfloor} \alpha_{i,tm} x^i y^{tm}. \quad \square$$

The following proposition provides a further reduction: $m = 1$ or 2.

Proposition 3.3. *If $\Gamma = \Gamma_{\mathbb{F}}(xy^m, f)$ has girth at least eight, then $m = 1$ or 2.*

Proof. Suppose $m \geq 3$. By Proposition 3.2, it suffices to consider f of the form (7). If $m = 4$ and $j^* = 2$, then Γ would contain a 4-cycle of type $(0, 1; 1, -1)$. Otherwise, let S_8 denote a 6-cycle of type $(a, b, c; \xi_1, \xi_2, \xi_3)$, where a, b, c are all distinct, and ξ_1, ξ_2, ξ_3 are distinct m^{th} roots of unity. Then

$$\Delta_3(f)(S_8) = \alpha_{1,j^*} \left[a\left(\xi_1^{j^*} - \xi_3^{j^*}\right) + b\left(\xi_2^{j^*} - \xi_1^{j^*}\right) + c\left(\xi_3^{j^*} - \xi_2^{j^*}\right) \right].$$

If $\xi_1^{j^*} = \xi_2^{j^*} = \xi_3^{j^*}$, then $\Delta_3(xy^m)(S_8) = \Delta_3(f)(S_8) = 0$, and S_8 would be a 6-cycle in Γ. If instead $\xi_1^{j^*}, \xi_2^{j^*}$, and $\xi_3^{j^*}$ are distinct, we let $a = \frac{\xi_1^{j^*} - \xi_2^{j^*}}{\xi_1^{j^*} - \xi_3^{j^*}}$, $b = 1$, and $c = 0$. Then a, b, and c are distinct, $\Delta_3(xy^m)(S_8) = \Delta_3(f)(S_8) = 0$, and S_8 would be a 6-cycle in Γ. The existence of the required ξ_1, ξ_2, ξ_3 results from the following lemma.

Lemma 3.4. *Let j and m be positive integers such that $m \geq 3$. Then unless $m = 4$ and $j \equiv 2 \pmod 4$, there exist distinct m^{th} roots of unity ξ_1, ξ_2, and ξ_3 such that ξ_1^j, ξ_2^j, and ξ_3^j are either all equal or all distinct.*

Proof. Let R denote the multiplicative group of all m^{th} roots of unity in $\mathbb{F}$, and let $R^j = \{r^j \mid r \in R\}$. Consider a surjective homomorphism from R to R^j that maps r to r^j. Let $K = \{r \in R \mid r^j = 1\}$ denote its kernel. We wish to show that unless $m = 4$ and $j \equiv 2 \pmod 4$, we have $|K| \geq 3$ or $|R^j| \geq 3$. As $R/K \cong R^j$, we have $m = |R| = |K||R^j|$. If $m = 4$, then either $|K| = 4$, $|R^j| = 4$, or $|K| = 2 = |R^j|$; this last case occurs if and only if $j \equiv 2 \pmod 4$. If $m = 3$ or $m > 4$, then either $|K| \geq 3$ or $|R^j| \geq 3$. □

This completes the proof of Proposition 3.3. □

If $m = 1$, then $\Gamma_{\mathbb{F}}(xy, f)$ of girth at least eight is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$ by the main result of [5]. Therefore, proving the following proposition will conclude the proof of Theorem 1.1.

Proposition 3.5. *If $\Gamma = \Gamma_{\mathbb{F}}(xy^2, f)$ has girth at least eight, then $\Gamma \cong \Gamma_{\mathbb{F}}(xy, xy^2)$.*

Proof. As Γ has girth at least eight, by Proposition 3.2, we have

$$f = \alpha_{1,j^*} xy^{j^*} + \sum_{i=1}^{n} \sum_{t=1}^{\lfloor n/2 \rfloor} \alpha_{i,2t} x^i y^{2t},$$

where j^* is odd (because $2 = m \nmid j^*$) and $\alpha_{1,j^*} \neq 0$. Let S_9 denote a 6-cycle of type $(0, 1-d^2, 1; 0, c, cd)$, where $c \neq 0$ and $d \neq -1, 0, 1$. Then

$$\Delta_3(xy^2)(S_9) = c^2(1-d^2) - c^2 + c^2 d^2 = 0$$

and

$$\begin{aligned}\Delta_3(f)(S_9) &= \alpha_{1,j^*}\left[(1-d^2)c^{j^*} - c^{j^*} + d^{j^*} c^{j^*}\right] + \sum_{i=1}^{n} \sum_{t=1}^{\lfloor n/2 \rfloor} \alpha_{i,2t}\left[(1-d^2)^i c^{2t} - c^{2t} + d^{2t} c^{2t}\right] \\ &= \alpha_{1,j^*}\left[d^{j^*} - d^2\right] c^{j^*} + \sum_{t=1}^{\lfloor n/2 \rfloor}\left[\sum_{i=1}^{n} \alpha_{i,2t}\left(\left(1-d^2\right)^i - 1 + d^{2t}\right)\right] c^{2t}.\end{aligned}$$

For $i = 1, \ldots, n$, and $t = 1, \ldots, \lfloor n/2 \rfloor$, define $\varphi_{i,t} = \left(1-d^2\right)^i - 1 + d^{2t}$, and

$$g_t = g_t(d) = \sum_{i=1}^{n} \alpha_{i,2t}\left(\left(1-d^2\right)^i - 1 + d^{2t}\right) = \sum_{i=1}^{n} \alpha_{i,2t}\, \varphi_{i,t}.$$

Then

$$\Delta_3(f)(S_9) = \alpha_{1,j^*}\left[d^{j^*} - d^2\right] c^{j^*} + \sum_{t=1}^{\lfloor n/2 \rfloor} g_t c^{2t}.$$

If at least one g_t, say $g_{t'}$, is a nonzero polynomial of d, then

$$\text{there exists } d' \neq -1, 0, 1 \text{ such that both } (d')^{j^*} - (d')^2 \text{ and } g_{t'}(d') \text{ are not zero.} \tag{8}$$

Then $\Delta_3(f)(S_9)$, considered as a polynomial of c, contains one term of odd degree, namely $\alpha_{1,j^*}\left[d^{j^*} - d^2\right] c^{j^*}$, and at least one term of even degree, namely $g_{t'}(d')c^{2t'}$. Hence,

$$\Delta_3(f)(S_9) \text{ has a nonzero root } c', \tag{9}$$

and Γ would contain a 6-cycle S_9 of type $(0, 1-(d')^2, 1; 0, c', c'd')$, a contradiction. Therefore g_t is the zero polynomial of d for all t, and

$$\Delta_3(f)(S_9) = \alpha_{1,j^*}\left[d^{j^*} - d^2\right] c^{j^*}.$$

If $j^* \geq 5$, then there exists $d' \neq -1, 0, 1$, such that $(d')^{j^*} - (d')^2 = 0$, and Γ would contain a 6-cycle of type $(0, 1-(d')^2, 1; 0, c, cd')$ for any $c \neq 0$, a contradiction. Therefore $1 \leq j^* \leq 4$, and being odd, $j^* = 1$ or 3.

To continue the proof, we observe that the requirement that g_t is the zero polynomial of d for all t allows us to describe the coefficients $\alpha_{i,2j}$.

Lemma 3.6. *Let $2 \le t \le \lfloor n/2 \rfloor$. Then g_t is the zero polynomial if and only if $\alpha_{i,2t} = (-1)^{i+t}\binom{t}{i}\alpha_{t,2t}$ for $1 \le i \le t$, and $\alpha_{i,2t} = 0$ for $t+1 \le i \le n$.*

Proof. Let $t \ge 2$, and suppose that

$$g_t = \sum_{i=1}^{n} \alpha_{i,2t}\,\varphi_{i,t} = \sum_{i=1}^{n} \alpha_{i,2t}\left[(1-d^2)^i - 1 + d^{2t}\right]$$

is the zero polynomial. Equating the coefficient at d^{2j} above to zero for $j = 1, \ldots, t$, we obtain the following system of linear equations with respect to $\alpha_{i,2t}$:

$$E_j: \quad (-1)^j \sum_{i=j}^{t} \binom{i}{j} \alpha_{i,2t} = 0, \quad j = 1, 2, \ldots, t-1,$$

and

$$E_t: \quad (-1)^t \alpha_{t,2t} + \sum_{i=1}^{t} \alpha_{i,2t} = 0.$$

Once $\alpha_{t,2t}$ is assigned an arbitrary nonzero value, the linear system of equations $E_1, \ldots, E_{t-1}$ becomes a triangular one with no zero entries on the diagonal. So it has a unique solution. We claim that this solution is of the form $\alpha_{i,2t} = (-1)^{i+t}\binom{t}{i}\alpha_{t,2t}$, and that it also satisfies the last equation E_t. Both claims are easily verified by direct substitution and by using the identity $\sum_{i=0}^{t}(-1)^i\binom{t}{i} = 0$.

Equating the coefficient at d^{2j} above to zero for $j = t+1, \ldots, \lfloor n/2 \rfloor$, we obtain a linear triangular homogeneous system of equations with respect to $\alpha_{i,2t}$ with no zero entries on the diagonal. Hence, its only solution is the trivial one. □

Therefore, if $\Gamma_{\mathbb{F}}(xy^2, f)$ has girth at least eight, then the polynomial f must have one of the following two forms:

$$f_1 = \alpha_{1,1}xy + \sum_{\substack{1 \le i \le n \\ 1 \le t \le \lfloor n/2 \rfloor}} \alpha_{i,2t}x^i y^{2t} = \alpha_{1,1}xy + \sum_{t=1}^{\lfloor n/2 \rfloor} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t}\binom{t}{i} x^i y^{2t},$$

where $\alpha_{1,1} \neq 0$, or

$$f_2 = \alpha_{1,3}xy^3 + \sum_{\substack{1 \le i \le n \\ 1 \le t \le \lfloor n/2 \rfloor}} \alpha_{i,2t}x^i y^{2t} = \alpha_{1,3}xy^3 + \sum_{t=1}^{\lfloor n/2 \rfloor} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t}\binom{t}{i} x^i y^{2t},$$

where $\alpha_{1,3} \neq 0$. We consider three cases. First, if $f = \alpha_{1,1}xy$, then Γ has girth eight and is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$ by parts (2) and (4) of Proposition 2.2, as desired. Second, if $f = \alpha_{1,3}xy^3$, then Γ would contain a 6-cycle of type $(1, 0, 32/27; -4/3, 2, 4)$, a contradiction. We use the remainder of this proof to analyze the third case, namely when $f \neq \alpha_{1,1}xy$ and $f \neq \alpha_{1,3}xy^3$. In this situation, note that $T = \max\{t : \alpha_{t,2t} \neq 0\}$ is well-defined. Furthermore, by part (5) of Proposition 2.2, we may assume that $\alpha_{1,2} = 0$. Thus, we have $T \ge 2$ and we rewrite f_1 and f_2 as follows:

$$f_1 = \alpha_{1,1}xy + \sum_{t=2}^{T} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t}\binom{t}{i} x^i y^{2t},$$

and

$$f_2 = \alpha_{1,3}xy^3 + \sum_{t=2}^{T} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t}\binom{t}{i} x^i y^{2t}.$$

Let S_{10} denote a 6-cycle of type $(z^2, 0, z^6; z^3, z, 0)$, where $z \neq 0$, $z \neq \pm 1$, and $z \neq \pm \mathrm{i}$, and

$$\mathrm{i} \text{ denotes a primitive fourth root of unity.} \tag{10}$$

Note that

$$\Delta_3(xy^2)(S_{10}) = z^8 - z^8 = 0.$$

We will examine $\Delta_3(f_1)(S_{10})$ and $\Delta_3(f_2)(S_{10})$, starting with $\Delta_3(f_1)(S_{10})$. We have

$$\Delta_3(f_1)(S_{10}) = \alpha_{1,1}(z^5 - z^7) + \sum_{t=2}^{T} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t}\binom{t}{i}\left(z^{2i+6t} - z^{6i+2t}\right), \tag{11}$$

which has degree at least 14. Note that i and $-$i are not roots of $\Delta_3(f_1)(S_{10})$ because $\Delta_3(f_1)(S_{10})(\mathrm{i}) = 2\mathrm{i}\alpha_{1,1} \neq 0$ and $\Delta_3(f_1)(S_{10})(-\mathrm{i}) = -2\mathrm{i}\alpha_{1,1} \neq 0$. Hence, to prove that $\Gamma_{\mathbb{F}}(xy^2, f_1)$ has a 6-cycle S_{10}, thereby proving that $f \neq f_1$ unless $f = \alpha_{1,1}xy$, we need only show that $\Delta_3(f_1)(S_{10})$ has a root besides 1, -1, and 0. Suppose otherwise. Then $\Delta_3(f_1)(S_{10}) = K(z-1)^a(z+1)^b z^c$, where K is a constant and a, b, and c are positive integers. As it is clear from (11) that $\Delta_3(f_1)(S_{10})$ has no terms of degree less than 5 and the coefficient at z^5 is $\alpha_{1,1} \neq 0$, we must have $c = 5$ and

$$\Delta_3(f_1)(S_{10}) = K(z-1)^a(z+1)^b z^5. \tag{12}$$

From (12) we see that the degree of $\Delta_3(f_1)(S_{10})$ is $a+b+5$. Since (11) implies the degree of $\Delta_3(f_1)(S_{10})$ is even, it must be the case that $a \neq b$. Now we consider the coefficient of z^{a+b+4} in $\Delta_3(f_1)(S_{10})$. From (11) the highest term of odd degree in $\Delta_3(f_1)(S_{10})$ is of degree 7. Since the degree of $\Delta_3(f_1)(S_{10})$ is at least 14, the coefficient of z^{a+b+4} must be zero. Equating this with the same coefficient in (12), we get

$$K(-a+b) = 0 \tag{13}$$

forcing $a = b$, a contradiction. Hence, $\Delta_3(f_1)(S_{10})$ has a root besides 0, 1, and -1, and so $\Gamma_{\mathbb{F}}(xy^2, f_1)$ has a 6-cycle S_{10}.

We now use a similar argument to analyze

$$\Delta_3(f_2)(S_{10}) = \alpha_{1,3}(z^{11} - z^9) + \sum_{t=2}^{T} \alpha_{t,2t} \sum_{i=1}^{t} (-1)^{i+t} \binom{t}{i} \left(z^{2i+6t} - z^{6i+2t}\right), \tag{14}$$

which also has degree at least 14. Note that i and $-$i are not roots of $\Delta_3(f_2)(S_{10})$ because $\Delta_3(f_2)(S_{10})(\mathrm{i}) = -2\mathrm{i}\alpha_{1,3} \neq 0$ and $\Delta_3(f_2)(S_{10})(-\mathrm{i}) = 2\mathrm{i}\alpha_{1,3} \neq 0$. Hence, to prove that $\Gamma_{\mathbb{F}}(xy^2, f_2)$ has a 6-cycle S_{10}, thereby proving that $f \neq f_2$ unless $f = \alpha_{1,3}xy^3$ (which we already rejected), we need only show that $\Delta_3(f_2)(S_{10})$ has a root besides 1, -1, and 0. Suppose otherwise. Then $\Delta_3(f_2)(S_{10}) = K(z-1)^a(z+1)^b z^c$, where K is a constant and a, b, and c are positive integers. As it is clear from (14) that $\Delta_3(f_2)(S_{10})$ has no terms of degree less than 9 and the coefficient at z^9 is $-\alpha_{1,3} \neq 0$, we must have $c = 9$ and

$$\Delta_3(f_2)(S_{10}) = K(z-1)^a(z+1)^b z^9. \tag{15}$$

Similar to the previous case, the degree of $\Delta_3(f_2)(S_{10})$ is $a+b+9$ and must be even, so $a \neq b$. Also, the coefficient of z^{a+b+8} must be zero, forcing

$$K(-a+b) = 0, \tag{16}$$

a contradiction.

This contradiction implies that $\Delta_3(f_2)(S_{10})$ has a root besides 0, 1, and -1, and so $\Gamma_{\mathbb{F}}(xy^2, f_2)$ has a 6-cycle S_{10}.

Hence, if Γ has girth at least eight, then $f = \alpha_{1,1}xy$ and $\Gamma \cong \Gamma_{\mathbb{F}}(xy, xy^2)$. □

4. Proof of Theorem 1.2

When $p = 3$, Theorem 1.2 reduces to analyzing graphs of the form $\Gamma_q(x^k y^m, xy)$. When $\{k, m\} = \{1, 2\}$, $\Gamma_q(x^k y^m, xy)$ has girth eight and is isomorphic to $\Gamma_q(xy, xy^2)$. When $k = m = 1$ or $k = m = 2$, $\Gamma_q(x^k y^m, xy)$ contains a 6-cycle of type $(1, 0, 2; 2, 1, 0)$. Otherwise, suppose without loss of generality that $k \geq 4$ (recall that k and m are relatively prime to $p = 3$). Let $M = \phi(k)$; then $\mathbb{F}_{q^M}$ contains a primitive k^{th} root of unity ζ, and $\Gamma_{q^M}(x^k y^m, xy)$ contains a 6-cycle of type $\left(1, \zeta, \zeta^2; 0, \frac{1-\zeta^2}{\zeta-\zeta^2}, 1\right)$.

For the remainder of this section, let $p \geq 5$. Define $M = M(k, m, q)$ to be the least common multiple of the integers $\phi(k)$, $\phi(m)$, 2, 3, …, and $4p-15$, where ϕ is Euler's totient function. The function M is chosen in this way so that all polynomials over $\mathbb{F}_q$ of degree at most $4p-15$ have all their roots in $\mathbb{F}_{q^M}$. Also, our choice of M guarantees the existence of primitive k^{th}, m^{th}, and i^{th} roots of unity in $\mathbb{F}_{q^M}$ when k and m are relatively prime to p and i satisfies $1 \leq i \leq p-2$.

In what follows, we indicate all places in the proof of Theorem 1.1 in Section 3 where it mattered that $\mathbb{F}$ was an algebraically closed field of characteristic zero. In each case, we show how to modify the proof to make it valid over an infinite family of finite fields of odd characteristic. Let $\mathbb{F} = \mathbb{F}_{q^{Mr}}$ for some positive integer r.

- In (1), the assumption that f has degree at most $p-2$ implies that $1 \leq i, j \leq n \leq p-2$. These inequalities will hold throughout this new proof.
- In Steps 1 and 2 in the proof of Proposition 3.1, recall that k and m are relatively prime to p. Therefore, the existence of a primitive k^{th} root of unity ζ is assured because the degree of the minimal polynomial for ζ, namely $\phi(k)$, is a divisor of M, and so $\mathbb{F}_q(\zeta)$ is isomorphic to a subfield of $\mathbb{F}_{q^M}$. The existence of a primitive m^{th} root of unity ξ follows similarly.
- In (2), there exists $a' \in \mathbb{F} \setminus \{0, 1\}$ such that $g_j(a') \neq 0$ for $j = s, t$, $1 \leq s \neq t \leq p-2$. Indeed, the polynomials $g_s(a)$ and $g_t(a)$ have degree at most $p-2$, and $|\mathbb{F}| = q^{Mr}$ with $M \geq 2$ implies that $|\mathbb{F}| \geq p^2 > 2p-2 = 2+2(p-2) \geq 2+s+t$.
- In (3), the polynomial $\Delta_2(f)(S_1')$ of d has a nonzero root $d' \in \mathbb{F}$ because its degree is at most $p-2$ (and so it divides M), and at least two of its coefficients are not zero.

- In Claim 3.1.2, since $i', j' \leq p-2$, we have $\phi(i'), \phi(j') \leq p-2$ and so they divide M. The existence of primitive i'^{th} and j'^{th} roots of unity then follows by similar reasoning to what we used above for Steps 1 and 2 in the proof of Proposition 3.1.
- In (6), the existence of a primitive k^{th} root of unity in $\mathbb{F}_{q^M}$ implies there are k distinct k^{th} roots of unity in $\mathbb{F}_{q^M}$. Therefore, since $k \geq 3$, ζ_2 and ζ_3 exist as claimed.
- In the proof of Lemma 3.4, let R denote the multiplicative group of all m^{th} roots of unity in $\mathbb{F}_{q^M}$. They exist since $\mathbb{F}_{q^M}$ contains a primitive m^{th} root of unity.
- In (8), the existence of d' is guaranteed by the fact that the order of $\mathbb{F}_{q^M}$ is greater than the sum of the degrees of $d^{j^*} - d^2$ and $g_{t'}(d)$. Indeed, the sum of the degrees is at most $2n + n = 3n \leq 3p - 6$. As $p \geq 5$ is odd, $M \geq 2(p-2)$. Hence, $q^M \geq 5^{2(p-2)} > 3p - 6$.
- In (9), the polynomial $\Delta_3(f)(S_9)$ of c has a nonzero root $c' \in \mathbb{F}$ because its degree is at most $p-2$ (and so it divides M), and at least two of its coefficients are not zero.
- In Lemma 3.6 and its proof, note that $1 \leq j \leq i \leq t < n < p$ implies that $\binom{t}{i}$ and $\binom{i}{j}$ are nonzero modulo p.
- In the argument following Lemma 3.6, the case $f = \alpha_{1,3}xy^3$ was resolved by exhibiting the cycle $(1, 0, 32/27; -4/3, 2, 4)$. As $p \geq 5$, this cycle type is invalid only for $p = 5$ (because $32/27 \equiv 1 \pmod 5$). In the case $q = p = 5$, it is easy to show that $\Gamma_5(xy^2, \alpha_{1,3}xy^3)$ is isomorphic to $\Gamma_5(xy, xy^2)$, and so has girth eight. When $p = 5$ and $q > 5$, an explicit 6-cycle can easily be found. However, both cases are covered by Theorem 5.1 (see Section 5).
- In (10), the existence of a primitive fourth root of unity i is assured because the degree of the minimal polynomial for i, namely $\phi(\mathrm{i}) = 2$, is a divisor of M, and so $\mathbb{F}_q(\mathrm{i})$ is isomorphic to a subfield of $\mathbb{F}_{q^M}$.
- In (12), we must explain why $\Delta_3(f_1)(S_{10})$ may be factored linearly. From (11), the assumption that f_1 has degree at most $n \leq p-2$, and the bound $1 \leq i \leq t \leq T \leq \lfloor n/2 \rfloor$, the degree of $\Delta_3(f_1)(S_{10})$ is $8T - 2 \leq 8\lfloor n/2 \rfloor - 2 \leq 4n - 2 \leq 4(p-2) - 2 = 4p - 10$ (note that we used $8T - 2$ instead of $8T$ because the $i = t = T$ term in (11) is zero). Furthermore, since $i \geq 1$ and $t \geq 2$, every term of $\Delta_3(f_1)(S_{10})$ has degree at least five. Hence, we may factor out z^5 from $\Delta_3(f_1)(S_{10})$, leaving a polynomial of degree at most $4p - 15$. Since we are working in $\mathbb{F}_{q^{Mr}}$, where M is divisible by $1, 2, \ldots,$ and $4p - 15$, $\Delta_3(f_1)(S_{10})$ factors linearly.
- From (13), we have $a \equiv b \pmod p$. Let us show that this results in a contradiction. First, note that since we are working in $\mathbb{F}_{q^{Mr}}$, and the degree of $\Delta_3(f_1)(S_{10})$ is at most $4p - 10 < q^{Mr}$, reducing modulo $z^{q^{Mr}} - z$ will have no effect.

  Since (11) implies that the coefficients at z^5 and z^7 are additive inverses, (12) yields
$$K\left[(-1)^{a-2}\binom{a}{2} + (-1)^{a-1}\binom{a}{1}\binom{b}{1} + \binom{b}{2}\right] - K(-1)^a \equiv 0 \pmod p.$$
  Using $a \equiv b \pmod p$, we find $a \equiv b \equiv 1 \pmod p$. Hence, $a = 1 + pu$ and $b = 1 + pv$ for some nonnegative integers u and v. Then from (12),
$$\begin{aligned}\Delta_3(f_1)(S_{10}) &= K(z-1)^{1+pu}(z+1)^{1+pv}z^5\\ \Delta_3(f_1)(S_{10}) &= K(z^p-1)^u(z^p+1)^v(z^7-z^5).\end{aligned} \tag{17}$$
  Recall that the degree of $\Delta_3(f_1)(S_{10})$ is at most $4p - 10$. Combined with (17), this implies that $pu + pv + 7 \leq 4p - 10$, and so $17 \leq p(4 - u - v) \leq 4p$. This implies $0 \leq u + v \leq 3$ (recall that $p \geq 5$ by assumption).

  We then substitute the ten (u, v) pairs of nonnegative integers that satisfy $0 \leq u + v \leq 3$ into (17) and expand; we will prove that none of them can equal $\Delta_3(f_1)(S_{10})$ from (11), which is a contradiction. Since $p \geq 5$, all but two of the ten polynomials either have terms of odd degree greater than 7 or have no terms of degree greater than 7, and therefore contradict (11). The remaining two polynomials, corresponding to $(u, v) = (1, 0)$ or $(0, 1)$, are $z^{p+7} - z^{p+5} + z^7 - z^5$ and $z^{p+7} - z^{p+5} - z^7 + z^5$. From (11), the degree of $\Delta_3(f_1)(S_{10})$ is $8T - 2$ (with leading coefficient $-\alpha_{T,2T}T$) and the term with second-largest degree has exponent $8T - 4$ (and coefficient $\alpha_{T,2T}\binom{T}{2}$).

  Comparing coefficients of the terms having the two largest degrees yields $-\alpha_{T,2T}T \equiv 1 \pmod p$ and $\alpha_{T,2T}\binom{T}{2} \equiv -1 \pmod p$, and so $T \equiv 3 \pmod p$. Since $0 \leq T \leq \frac{p-2}{2}$, $T = 3$. Furthermore, $p + 7 = 8T - 2 = 22$; hence, $p = 15$, a contradiction.
- The following analysis of $\Delta_3(f_2)(S_{10})$ will be similar to the one we just presented for $\Delta_3(f_1)(S_{10})$. In (15), $\Delta_3(f_2)(S_{10})$ may be factored linearly. From (16), we have $a \equiv b \pmod p$, and we will show that this leads to a contradiction. Reducing $\Delta_3(f_2)(S_{10})$ modulo $z^{q^{Mr}} - z$ will again have no effect.

  Since (14) implies that the coefficients at z^{11} and z^9 are additive inverses, (15) yields
$$K\left[(-1)^{a-2}\binom{a}{2} + (-1)^{a-1}\binom{a}{1}\binom{b}{1} + \binom{b}{2}\right] - K(-1)^a \equiv 0 \pmod p.$$
  Using $a \equiv b \pmod p$, we find $a \equiv b \equiv 1 \pmod p$. Hence, $a = 1 + pu$ and $b = 1 + pv$ for some nonnegative integers u and v. Then from (15),
$$\begin{aligned}\Delta_3(f_2)(S_{10}) &= K(z-1)^{1+pu}(z+1)^{1+pv}z^9\\ \Delta_3(f_2)(S_{10}) &= K(z^p-1)^u(z^p+1)^v(z^{11}-z^9).\end{aligned} \tag{18}$$

Recall that the degree of $\Delta_3(f_2)(S_{10})$ is at most $4p - 10$. Combined with (18), this implies that $pu + pv + 11 \leq 4p - 10$, and so $21 \leq p(4 - u - v) \leq 4p$. This implies $0 \leq u + v \leq 3$ (as well as $p \geq 7$, but this will not be used). We then substitute the ten (u, v) pairs of nonnegative integers that satisfy $0 \leq u + v \leq 3$ into (18) and expand; as was the case for $\Delta_3(f_1)(S_{10})$, eight out of the ten polynomials cannot equal $\Delta_3(f_1)(S_{10})$ based on the degrees of their terms. The remaining two polynomials are $z^{p+11} - z^{p+9} + z^{11} - z^9$ and $z^{p+11} - z^{p+9} - z^{11} + z^9$. As before, comparing coefficients of the terms having the two largest degrees yields the contradiction $p = 15$.

This concludes the proof of Theorem 1.2. □

5. Concluding remarks

The girth of $\Gamma_q(f_2, f_3)$ has been studied extensively in the case where q is an odd prime power and $f_2, f_3 \in \mathbb{F}[x, y]$ are monomials. Progress on this monomial case was made in [1], Dmytrenko, Lazebnik, and Williford [2], and Kronenthal [4], and it was resolved in Hou, Lappano, and Lazebnik [3].

Theorem 5.1 ([3]). *Let $q = p^e$ be an odd prime power, and let f_2 and f_3 be monomials in $\mathbb{F}_q[x, y]$. Then every graph $\Gamma_q(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$.*

It was shown in [1] that for a binomial f, the girth of $\Gamma_q(xy, f)$ is six, provided that the odd prime power q is sufficiently large. However, little is known when $f_2, f_3 \in \mathbb{F}_q[x, y]$ are arbitrary polynomials:

Open Problem 5.2. *Let $q = p^e$ be an odd prime power, and let $f_2, f_3 \in \mathbb{F}_q[x, y]$. Is it true, perhaps with finitely many exceptions, that every graph $\Gamma_q(f_2, f_3)$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$?*

Over algebraically closed fields $\mathbb{F}$ of characteristic zero, we believe that an analog of Theorem 1.1 holds for all $\Gamma_{\mathbb{F}}(f_2, f_3)$:

Conjecture 5.3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, let $f_2, f_3 \in \mathbb{F}[x, y]$, and let $\Gamma_{\mathbb{F}}(f_2, f_3)$ have girth at least eight. Then $\Gamma_{\mathbb{F}}(f_2, f_3)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.*

Finally, we note that the girth eight algebraically defined graphs discussed in this paper have connections to generalized quadrangles; for details, including how Theorem 5.1 stands in contrast to the case in which q is a power of 2, please see [5] and references therein.

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