Short Proofs of Two Basic Properties of Central Projections

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In this note we prove two important properties of central projections, stated as Theorems A and B. They can be applied to obtain simple solutions of many hard problems in Euclidean geometry: for numerous examples see Yaglom [1, Ch. 1.3]. The proofs of these properties that I found in the literature were of two kinds: those which were completely elementary, assumed proficiency with Euclidean geometry, and were not easy), and those where the properties followed from more general results of projective geometry. The latter require substantial background.

In my opinion, the proofs presented below are elementary and much easier than the ones I have seen. They can be presented to students familiar with the basics of the analytical geometry in space, namely with equations of circles and lines.

Theorem A. *Given a circle C in plane π and a point D inside of C, one can always find a point S and a plane π' such that the central projection of π to π' from S maps C onto a circle C' and D into its center D'.*

Theorem B. *Given a circle C in plane π and a line l not intersecting C, one can always find a point S and a plane π' such that the central projection of π to π' from S maps C onto a circle C' and l onto l'_∞ – the line at infinity of π'.*

We start by giving a short proof of Theorem A. A very similar argument can be used to prove Theorem B, and it is presented next.

Proof of Theorem A If D is a center of C, the claim is obvious: take $\pi = \pi'$ and S being any point not in π. If D is not a center of C, we introduce a coordinate system $OXYZ$ such that C is a unit circle in XOY and D has coordinates $(d, 0, 0)$, for some d, $0 < d < 1$. Hence π is the OXY-plane and its equation is $z = 0$.

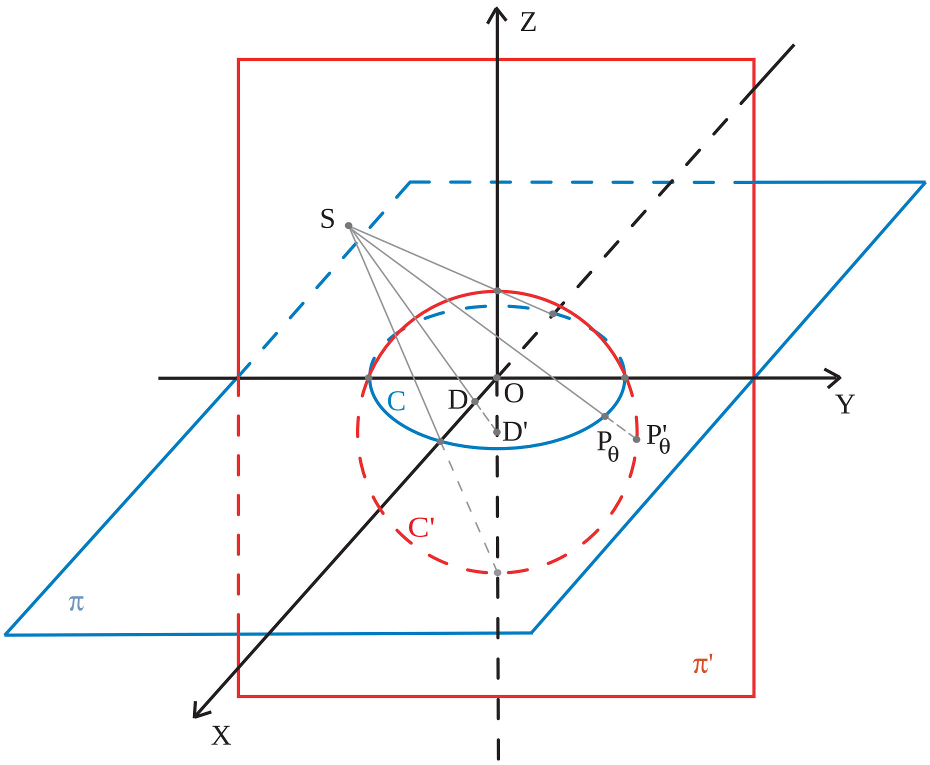

Figure 1: Central projection for Theorem A

Let π' be the OYZ-plane and S be the point with coordinates $(a, 0, b)$, where $a = 1/d$ and $b = \sqrt{1-d^2}/d$. We will show that the central projection of π to π' from S has the required properties.
Indeed, let $P_\theta = (\cos\theta, \sin\theta, 0)$, $0 \leq \theta < 2\pi$, be a parametrization of C. The parametric equation of the line SP_θ is:

$$x = a + (a - \cos\theta)s,\ y = -\sin\theta\, s,\ z = b + bs, \quad s \in \mathbb{R}$$

Plane π' has an equation $x = 0$. Hence point P'_θ – the image of P_θ, and point D' – the image of D, have coordinates

$$P'_\theta \left(0,\ \frac{\sin\theta}{d\cos\theta - 1},\ \frac{\sqrt{1-d^2}\cos\theta}{d\cos\theta - 1}\right)$$

$$D' \left(0,\ 0,\ -\frac{d}{\sqrt{1-d^2}}\right)$$

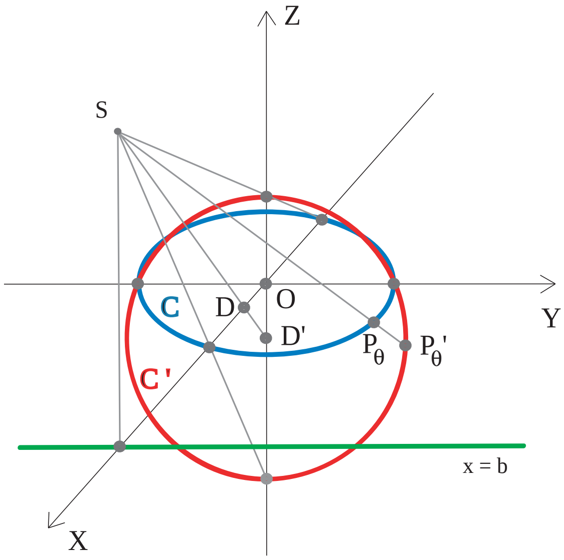

Figure 2: Central projection for Theorem B

Then the square of the distance $D'P'_\theta$ is (check!):

$$D'P'^2_\theta = 0^2 + \left(\frac{\sin\theta}{d\cos\theta - 1} - 0\right)^2 + \left(\frac{\sqrt{1-d^2}\cos\theta}{d\cos\theta - 1} + \frac{d}{\sqrt{1-d^2}}\right)^2 = \frac{1}{1-d^2}.$$

Therefore the distance between D' and P'_θ is $\frac{1}{1-d^2}$ for all θ. Hence C' is a circle (by continuity), and D' is its center. □

Proof of Theorem B We introduce a coordinate system $OXYZ$ such that C becomes a unit circle in the OXY-plane, and l is in OXY-plane having an equation $x = b$, for some b, $b > 1$. Hence π is the OXY-plane and its equation is $z = 0$.
Let π' be the OYZ-plane and S be the point with coordinates $(b, 0, \sqrt{b^2-1})$. As was demonstrated in the proof of Theorem A, set $d = 1/b$, when C is centrally projected from S to π', its image is a circle. Since the plane defined by l and S is parallel to π', l is mapped to the line at infinity of π'. □

The proof of Theorem A leaves an important question unanswered: how did we know how to choose π' and S, i.e., why did we choose $\pi' : \; x = 0$, and $(a, 0, b) = (1/d, 0, \sqrt{1 - d^2}/d)$?

The truth is that these values of a and b were originally found as a result of some more tedious computations using computer and Maple. The details can be found in the Appendix section (see next six pages).

To end this note, I would like to mention a fact which sparked my interest in the topic. This was a sketch of David Hilbert's proof from M. Kac and S.M. Ulam [2] on the impossibility of constructing the center of a circle by using a straight edge only. I invite the reader to find a proof. Having Theorem A makes it an easy exercise.

REFERENCES

1. Yaglom, I.M., Geometric Transformations III, The Mathematical Association of America, New Mathematical Library 24, 1973.

2. Kac M., Ulam S., Mathematics and Logic, Dover Pub. Inc., 1992.

APPENDIX

Below we present the logic behind these computations of the coordinates a and b of S and a transcript of the related Maple session.

We consider a general cone defined by a point $S = (a, 0, b)$ and the unit circle C. The cone is not right, i.e., the orthogonal projection of S onto OXY is not the center of C. We look for a cross section of this cone by a plane α not parallel to its base which is again a circle. We assume that a normal vector of α is $[1, 0, t]$, which is suggested by the symmetry of the cone with respect to OXZ-plane. Clearly this choice of a normal vector implies that α is not parallel to the base of the cone. Also, replacing α with any other plane parallel to it, gives another circle as a cross section. Therefore we may assume that α passes through the origin. Hence $\alpha : x + tz = 0$.
We start with a parametric equation of line SP_θ:

```
> x:= a + (a - cos( theta))*s; y := -s*sin( theta ); z:= b + b*s;
```

$$x := a + (a - \cos(\theta))\, s$$
$$y := -s\sin(\theta)$$
$$z := b + b\, s$$

Let s_0 be the value of s corresponding the intersection PO of SP_θ with α:

```
> s_0 := solve(x + t*z = 0, s); simplify(
> P0 := subs(s = s_0, [x,y,z]));
```

$$s\_0 := -\frac{a + t\, b}{a - \cos(\theta) + t\, b}$$

$$P0 := [\frac{\cos(\theta)\, t\, b}{a - \cos(\theta) + t\, b}, \frac{(a + t\, b)\sin(\theta)}{a - \cos(\theta) + t\, b}, -\frac{b\cos(\theta)}{a - \cos(\theta) + t\, b}]$$

Let $P1, P2, P3$ be the points of intersection of α with $SP_0, SP_{\pi/2}, SP_\pi$, respectively.

Then their coordinates are:

```
> P1 := subs({cos(theta) = 1, sin(theta) =
> 0},P0); P2 := subs({cos(theta) = 0, sin(theta) = 1},P0); P3 :=
> subs({cos(theta) =
> -1, sin(theta) = 0},P0);
```

$$P1 := [\frac{t\,b}{a-1+t\,b},\, 0,\, -\frac{b}{a-1+t\,b}]$$
$$P2 := [0,\, 1,\, 0]$$
$$P3 := [-\frac{t\,b}{a+1+t\,b},\, 0,\, \frac{b}{a+1+t\,b}]$$

Now we find a point (k, m, n) in α which is equidistant from $P1, P2, P3$, i.e., the center of the circle trough these three points :

```
> DistP1 := (P1[1] - k)^2 + (P1[2]-m)^2 +
> (P1[3] - n)^2; DistP2 := (P2[1] - k)^2 + (P2[2]-m)^2 + (P2[3] -
> n)^2; DistP3 := (P3[1] - k)^2 + (P3[2]-m)^2 + (P3[3] - n)^2;
> solve({ DistP1 = DistP2, DistP1 =
> DistP3, DistP2 = DistP3, k+t*n =0, a<>0, b<>0} );
```

$$DistP1 := (\frac{t\,b}{a-1+t\,b} - k)^2 + m^2 + (-\frac{b}{a-1+t\,b} - n)^2$$
$$DistP2 := k^2 + (1-m)^2 + n^2$$
$$DistP3 := (-\frac{t\,b}{a+1+t\,b} - k)^2 + m^2 + (\frac{b}{a+1+t\,b} - n)^2$$

$$\{a = a,\, k = -n\,\mathrm{RootOf}(\_Z^2+1),\, b = b,\, m = \frac{1}{2}, n = n,\, t = \mathrm{RootOf}(\_Z^2+1)\},$$
$$\{k = -n\,t,\, a = -t\,b,\, b = b,\, t = t,\, n = n, m = \frac{1}{2} - \frac{1}{2}\,t^2\,b^2 + n\,t^2\,b + b\,n - \frac{1}{2}\,b^2\},$$
$$\{a = a, n = -\frac{b}{(a-1+t\,b)\,(a+1+t\,b)}, m = \frac{1}{2}\,\frac{a^2 + 2\,a\,t\,b - b^2 - 1}{(a-1+t\,b)\,(a+1+t\,b)},$$
$$b = b,\, t = t, k = \frac{b\,t}{(a-1+t\,b)\,(a+1+t\,b)}\}$$

Since the values of all parameters are reals, we have two sets of values for both t and (k, m, n):

```
> t1 := -a/b; k1 := simplify(subs(t=t1,
> -1/2*(-1+t^2*b^2+b^2)/b/(t^2+1)*t)); m1 := 0; n1 :=
> simplify(subs(t=t1, 1/2*(-1+t^2*b^2+b^2)/b/(t^2+1) ));
```

$$t1 := -\frac{a}{b}$$

$$k1 := \frac{1}{2} \frac{(a^2 - 1 + b^2)\, a}{a^2 + b^2}$$

$$m1 := 0$$

$$n1 := \frac{1}{2} \frac{b\,(a^2 - 1 + b^2)}{a^2 + b^2}$$

```
> t2 := -1/2*(a^2-b^2-1)/a/b; k2 :=
> simplify(subs(t=t2,
> -2*a/(a^2-2*a+b^2+1)/(a^2+2*a+b^2+1)*(a^2-b^2-1))); m2 := 0; n2 :=
> simplify(subs(t=t1, -4*b*a^2/(a^2-2*a+b^2+1)/(a^2+2*a+b^2+1)));
```

$$t2 := -\frac{1}{2} \frac{a^2 - b^2 - 1}{a\, b}$$

$$k2 := -2 \frac{a\,(a^2 - b^2 - 1)}{(a^2 - 2\,a + b^2 + 1)\,(a^2 + 2\,a + b^2 + 1)}$$

$$m2 := 0$$

$$n2 := -4 \frac{b\, a^2}{(a^2 - 2\,a + b^2 + 1)\,(a^2 + 2\,a + b^2 + 1)}$$

Since the cross section should be a circle, we get two values for the square of the radius :

```
> R1sqr := simplify( (subs(t=t1,P0[1]) -
> k1)^2 +  (subs(t = t1, P0[2]))^2 + (subs(t=t1,P0[3]) - n1)^2);
> R2sqr := simplify((subs(t=t2,P0[1]) - k2)^2 +  (subs(t = t2,
> P0[2]))^2 +
> (subs(t=t2,P0[3]) - n2)^2);
```

$$R1sqr := \frac{1}{4} \frac{(a^2 + b^2 + 1)^2}{a^2 + b^2}$$

$$R2sqr := \frac{a^4 + 2\,b^2\,a^2 + 2\,a^2 + b^4 + 2\,b^2 + 1}{(a^2 - 2\,a + b^2 + 1)\,(a^2 + 2\,a + b^2 + 1)}$$

Now we check that for **all** a, b, $a > 0, b > 0$, the cross section of the cone by plane α with the slope $[1, 0, t2]$ is a circle with the center $(k2, m2, n2)$ and radius $R2sqr$. Below we will show that only these second values of the parameters t, k, m, n give the desired cross section, therefore we do not verify same thing for $t1$ and $(k1, m1, n1)$.

```
> P0 := subs(t=t2, P0); (P0[1] - k2)^2
> +(P0[2] - m2)^2 + (P0[3] - n2)^2 -R2sqr;
> is(simplify( %=0));
```

$$P0 := \left[-\frac{1}{2}\frac{\cos(\theta)\,(a^2-b^2-1)}{a\,\%1}, \frac{(a-\frac{1}{2}\frac{a^2-b^2-1}{a})\sin(\theta)}{\%1}, -\frac{b\cos(\theta)}{\%1}\right]$$

$$\%1 := a-\cos(\theta)-\frac{1}{2}\frac{a^2-b^2-1}{a}$$

$$(-\frac{1}{2}\frac{\cos(\theta)\,(a^2-b^2-1)}{a\,\%1}+\frac{2\,a\,(a^2-b^2-1)}{(a^2-2\,a+b^2+1)\,(a^2+2\,a+b^2+1)})^2+$$
$$\frac{(a-\frac{1}{2}\frac{a^2-b^2-1}{a})^2\sin(\theta)^2}{\%1^2}+$$
$$(-\frac{b\cos(\theta)}{\%1}+\frac{4\,b\,a^2}{(a^2-2\,a+b^2+1)\,(a^2+2\,a+b^2+1)})^2-\frac{b^4+2\,a^2\,b^2+2\,b^2+a^4+2\,a^2+1}{(a^2-2\,a+b^2+1)\,(a^2+2\,a+b^2+1)}$$
$$\%1 := a-\cos(\theta)-\frac{1}{2}\frac{a^2-b^2-1}{a}$$

$$\mathit{true}$$

Now we determine the coordinates of (xd, yd, zd) of D' (= imageD):

```
> xd:= a + (a - d)*u; yd := 0; zd:= b +b*u;
> u_0 := solve(xd + t*zd =0, u); imageD := simplify( subs(u = u_0, [xd,yd,zd]));
```

$$xd := a+(a-d)\,u$$

$$yd := 0$$

$$zd := b+b\,u$$

$$u\_0 := -\frac{a+t\,b}{a-d+t\,b}$$

$$imageD := [\frac{d\,t\,b}{a-d+t\,b},\ 0,\ -\frac{b\,d}{a-d+t\,b}]$$

Substituting two found values for t1 and t2, we get:

```
> image1D := simplify( subs(t = t1, imageD));
```

$$image1D := [a,\ 0,\ b]$$

```
> image2D :=simplify( subs(t = t2, imageD));
```

$$image2D := [-\frac{d\,(a^2-b^2-1)}{a^2-2\,a\,d+b^2+1},\ 0, -2\,\frac{b\,d\,a}{a^2-2\,a\,d+b^2+1}]$$

The first point $(a, 0, b)$ has to be rejected, since it coincides with D. Requiring that $image2D = (k2, m2, n2)$, we obtain the possible values for a and b:

```
> solve({image2D[1] = k2, image2D[3] = n2, b<>0}, {a,b});
```

$$\{a = \frac{1}{d}, b = \frac{\mathrm{RootOf}(-1 + d^2 + \_Z^2,\ label = \_L20)}{d}\},\ \{a = a,\ b = \mathrm{RootOf}(\_Z^2 + a^2 + 1)\},$$
$$\{a = a, b = \mathrm{RootOf}(d\,\_Z^2 + d\,a^2 - 2\,a + d)\}$$

Since a and b are reals, we get the coordinates of their image of D and the value of $t2$:

```
> imageD := simplify( subs({a = 1/d,b =
> (1-d^2)^(1/2)/d},image2D) ); t2 := simplify(subs({a = 1/d,b =
> (1-d^2)^(1/2)/d},t2) ); R2sqr :=
> simplify(subs({a = 1/d,b = (1-d^2)^(1/2)/d},R2sqr) );
```

$$imageD := [0,\ 0,\ \frac{\sqrt{1 - d^2}\,d}{-1 + d^2}]$$
$$t2 := 0$$
$$R2sqr := -\frac{1}{-1 + d^2}$$

What is left is just to verify that the image of C is indeed a circle centered at $D'(0, 0, -d/\sqrt{1 - d^2})$ and of radius $1/\sqrt{1 - d^2}$.

This was already done at the end of our short version of the proof, but we repeat it here.

First we compute the image P'_θ for $a = 1/d, b = \sqrt{1 - d^2}/d$ and $t = t2 = 0$:

```
> P0 := simplify( subs({a = 1/d,b
> =(1-d^2)^(1/2)/d, t = 0},P0)) ;
```

$$P0 := [0,\ -\frac{\sin(\theta)}{-1 + \cos(\theta)\,d},\ \frac{\sqrt{1 - d^2}\cos(\theta)}{-1 + \cos(\theta)\,d}]$$

Then we check that P'_θ is on the circle:

```
> simplify((P0[1] - imageD[1])^2 + (P0[2]
> -imageD[2])^2 +
> (P0[3] -imageD[3])^2 - R2sqr);
```

$$0$$

This ends the proof.