ON GREEN-TAO THEOREM

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The last twenty years lavished mathematics with incredible presents: many old-standing problems were solved. The most famous of them are the Fermat's Last Theorem and the Poincare's conjecture. Among other victories is the problem described below. I never thought it would be solved in my lifetime.

A positive prime number is a positive integer greater than 1 whose only positive divisors are 1 and the number itself. The first nine positive primes are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23$$

Questions about properties of primes have fascinated people from the inception of mathematics, and many of them are still not answered. A question which was answered recently relates to primes forming arithmetic progression.

Consider an arithmetic progression (AP):

$$a, a+d, a+2d, \ldots, a+nd, \ldots \;\; n \geq 0.$$

In this note we assume that a and d are positive integers.

Once someone asked: **can all terms of AP be prime numbers?**

It is not hard to argue that one cannot have an *infinite* AP with all terms being prime numbers. Therefore we consider only finite APs. For some small number of terms the answer is yes:

$$2$$
$$2, 3$$
$$3, 5, 7$$
$$5, 11, 17, 23$$
$$5, 11, 17, 23, 29$$
$$7, 37, 67, 97, 127, 157$$
$$7, 157, 307, 457, 607, 757$$
$$\ldots,$$

but longer examples are hard to find:

$$5749146449311 + 26004868890 \cdot n : \quad n = 0, \ldots, 20$$
$$11410337850553 + 4609098694200 \cdot n : \quad n = 0, \ldots, 21$$

(Moran, Pritchard, Thyssen (1995))

$$56211383760397 + 44546738095860 \cdot n : \quad n = 0, \ldots, 22$$

(Frind, Underwood, Jobling (2004))

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At this stage the existence of AP with even 24 prime terms is not clear. Nevertheless, based on very few initial examples and some heuristics, it was conjectured at least a century ago that

Conjecture 1: *There are arbitrarily long AP of primes. In other words, the set of all positive primes contains arbitrarily long APs.*

Conjecture 1 should not be confused with a famous result of Dirichlet (1837): any infinite AP with a and d having the greatest common divisor 1 contains infinitely many primes. Dirichlet's theorem can be considered as a generalization of a great theorem of antiquity: there are infinitely many primes ($a = d = 1$).

Though Conjecture 1 withstood a century, several results on arithmetic progressions in certain subsets of integers were obtained. In order to discuss these results we need the notion of density.

Let A be a subset of positive integers. The density of A, $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} \frac{\mid A \cap [1,n] \mid}{n}$$

if the limit exists. The fraction $\mid A \cap [1,n] \mid /n$ represents the proportion of members of A among the first n integers. For example, if A is the set of all odd (or all even) positive integers, $d(A) = 1/2$. Moreover, if A is an infinite AP with common difference d, then $d(A) = 1/d$. If P is the set of all positive primes, $d(P) = 0$. This nontrivial result was first obtained by Legendre in 1798. About a century later, it was shown by de la Valle-Poussin, and by Hadamard, that for large n, $\mid P \cap [1,n] \mid /n$ behaves like $1/\ln n$, which also implies that $d(P) = 0$.

In 1927 van der Waerden proved that if the integers are coloured using finitely many colors, then one of the color classes must contain arbitrarily long arithmetic progression. Which color class? Though it is reasonable to suspect that it should be one of positive density (such class must exist), van der Waerden's theorem did not imply it. In relation to AP of primes, Van der Corput (1939) proved that there are infinitely many arithmetic progressions of primes of length 3. A theorem of Roth (1956) extended Van der Corput's result to all subsets of integers of positive density, and Szemerédi, using combinatorial ideas, extended Roth's result to APs with 4 terms in 1969. Finally, in 1975, Szemerédi proved that any subsets of integers of positive density contains arbitrarily long APs. This was achieved by an ingenious and complicated extension of his previous combinatorial argument. Unfortunately, as the density of primes is 0, Szemerédi's theorem does not imply Conjecture 1. What needed was the extension of Szemerédi's theorem to sparse sets, in particular to some sets of density 0. Such an extension had to wait until 2004.

Green-Tao Theorem (2004): *The set of all primes contains arbitrarily long arithmetic progressions.*

Actually they proved a more general statement, that not only do the primes contain arbitrarily long APs, but so does every sufficiently dense subset of primes. The proof is hard, and is based on the ideas and results from several areas of mathematics, e.g., analytical number theory, combinatorics, pseudorandomness, harmonic analysis, ergodic theory. The paper was published in *Annals of Math.* 167 (2008), 481-547.

As we all know, the existence of an object in mathematics does not imply that its explicit construction is known. Though Green-Tao theorem does not tell us how to find long APs of primes, it encourages one to keep looking for such a method. So far, the only progress in this direction was a construction of APs of primes of length 24 and 25:

$$468395662504823 + 205619 \cdot 223092870 \cdot n, \quad n = 0, \ldots, 23$$

(Wróblewski (2007)

$$6171054912832631 + 366384 \cdot 223092870 \cdot n, \quad n = 0, \ldots, 24$$

(Wróblewski, Chermoni (2008))

We finish this note with another beautiful conjecture of Paul Erdős of 1936.

Conjecture 2: *Any set of positive integers whose sum of reciprocals diverges contains arbitrarily long arithmetic progressions.*

As the sum of the reciprocal of positive primes diverges (Euler, 1737), this conjecture implies the Green-Tao theorem. At this time Conjecture 2 is totaly open: it is not even known if such a set must contain an arithmetic progression with three terms!