Jensen's inequality (1906).

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Let f be a function concave up on (a,b), i.e., for any $x_1, x_2 \in (a,b)$,

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}.$$

Prove that for any n numbers $x_i \in (a,b)$,

$$f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \frac{\sum_{i=1}^n f(x_i)}{n},$$

and that the equality is attained if and only if f is linear or all x_i are equal.

For concave down functions, the signs of all inequalities change to $\ge$.

Proof. This proof uses the idea of 'up' and 'down' induction which is credited to Cauchy. This proof also shows that the arguments related to limits are not needed.

First we use induction on k to show that the inequality is satisfied for all $n = 2^k$ (the 'up' part). This follows immediately from the fact that it is correct for $k = 1$ and

$$\frac{x_1 + \ldots + x_{2^{k+1}}}{2^{k+1}} = \frac{1}{2}\left(\frac{x_1 + \ldots + x_{2^k}}{2^k} + \frac{x_{2^k+1} + \ldots + x_{2^{k+1}}}{2^k}\right)$$

for all $k \ge 2$.

Now we suppose that the statement is proven for $n \ge 3$ values of x_i's and show that it implies the statement for $n-1$ values of x (the 'down' part). Let $x_i \in (a,b)$, $i \in [n-1]$, be arbitrary $n-1$ numbers on (a,b). Apply the inequality to the following n numbers: $x_1, x_2, \ldots, x_{n-1}, x_n = \frac{x_1+\ldots+x_{n-1}}{n-1}$. We have

$$f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \frac{\sum_{i=1}^n f(x_i)}{n} \Leftrightarrow f\left(\frac{\frac{n}{n-1}\sum_{i=1}^{n-1} x_i}{n}\right) \le \frac{\sum_{i=1}^{n-1} f(x_i) + f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)}{n} \Leftrightarrow$$

$$f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right) \le \frac{\sum_{i=1}^{n-1} f(x_i) + f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)}{n}.$$

Solving the last inequality for $f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)$, we obtain

$$f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right) \le \frac{\sum_{i=1}^{n-1} f(x_i)}{n-1}.$$

This completes the proof. The assertion about the equality sign should be a part of the inductive hypothesis, and it follows immediately. $\square$

Several famous inequalities can be obtained as simple corollaries of Jensen's inequality:

- $f(x) = x^2$ gives us that variance of a data set is positive, or that the **quadratic** mean is greater or equal than the **arithmetic** mean:

$$\left(\frac{\sum_{i=1}^n x_i^2}{n}\right)^{1/2} \geq \frac{\sum_{i=1}^n x_i}{n}.$$

- $f(x) = \ln(x),\ x > 0$ gives that the **arithmetic** mean is greater or equal than the **geometric** mean:

$$\frac{\sum_{i=1}^n x_i}{n} \geq \left(\prod_{i=1}^n x_i\right)^{1/n}.$$

- $f(x) = \frac{1}{x},\ x > 0$ gives

$$\frac{\sum_{i=1}^n x_i}{n} \geq \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}.$$

  This is an inequality between the **arithmetic** and **harmonic** means. If we rewrite it as

$$\left(\sum_{i=1}^n x_i\right)^{\left(\frac{1}{2}\right)} \left(\sum_{i=1}^n \frac{1}{x_i}\right)^{\left(\frac{1}{2}\right)} \geq n,$$

  it can be viewed as a special case of the Cauchy–Schwartz inequality.

In order to apply Jensen's inequality, one has to check that the function is concave up (down). This can be done by proving it for $n = 2$ or by using the second derivative test when it is applicable.