MONOVARIANTS

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1. Introduction

Most problems in this note[1] are of the following nature. Suppose we are given a task and we do not know how to accomplish it. We start experimenting and try simple ideas. After a while we notice that one idea seems to work, but we are not sure why. All examples we can do 'by hand' seem to confirm the observation. Still it is not clear why the method should work in general case. Then we can program computer and apply the method to much larger examples, and it works again and again! Still, we do not understand why the method works. A tool which may help sometimes in such situation is a monovariant. To introduce it, we start with examples and postpone the general discussion for later.

2. Examples from Discrete Mathematics and Geometry

Example 1. *(A. Schwarz) Consider a rectangular array with m rows and n columns whose entries are real numbers. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations we can make the sum of numbers along each line (row or column) nonnegative.*

Solution 1. Consider the following algorithm of solving this problem. If there is no line with negative sum we are done. If there is a line with negative sum, change signs of all numbers in this line.

Does this algorithm ever top? Changing the signs of all numbers in one row may affect the sum of the numbers in some columns, and then changing the numbers in those columns may affect the sums in many rows, and so on. It is conceivable that this process will run in circles, leading to no better situation than that with which we started.

Here is a trick: Let M be the sum of all mn numbers from the table. Suppose the sum of numbers in some line is $x < 0$. Then reversing signs of all numbers in this line, we get another array with the sum of all its entries being $M - 2x$ which is greater than M since x is negative. Therefore each step of our algorithm leads to a new table with strictly larger sum.

And now we can see why our algorithm cannot run forever. Our operation can produce only finitely many arrays, because each of the mn entries can take on only 2 values (differing in their sign), and so all in all there could be only *finitely* many different tables obtained this way. Therefore there are only finitely many different values of M. if the algorithm does not stop, it will produce infinitely many ever-increasing, and so, distinct values of M: contradiction. □

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[1]The original version of this note was revised by the second author.

Example 2. *Given $2n$ points on the plane such that no three points lie on one line. Prove that it is possible to draw n segments such that each segment connects a pair of these points and no two segments intersect.*

Solution 2. Consider the following algorithm of solving this problem. Place n segments at random. If no two intersect, then we are done. If they do, pick a pair of intersecting segments, say $\overline{AB}$ and $\overline{CD}$, and apply the following 'move': erase them, and draw two new segments $\overline{AC}$ and $\overline{BD}$. See Figure 1(a). Repeat until no crossings remain.

Does this algorithm ever stop? Clearly, at each move we eliminate one point of intersection, but several other intersections can be formed after this.

First we observe that the number of different configurations (placements of n segments on $2n$ given points) is finite. For every configuration we can compute number M which is the sum of lengths of all n segments. Therefore M can take only finitely many values. Let us show that the value of M strictly decreases with every move.

We do it just by showing that $AC + BD < AB + CD$, since other segments are not affected by the move. Let $\{O\} = \overline{AB} \cap \overline{CD}$. See Figure 1(b). Then $AB + CD = (AO + OB) + (CO + OD) = (AO + CO) + (BO + DO) > AC + BD$. This last part follows from the triangle inequality

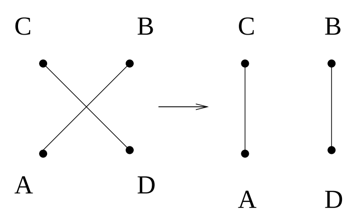

Figure 1(a)

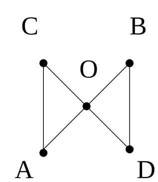

Figure 1(b)

This proves that the algorithm must terminate: otherwise it would produce infinitely many different values of M: contradiction. □

Example 3. *(Friendly Groups.) Given a set of n people some of whom are friends and some are enemies. Prove that they can always be divided into two groups such that every person has at least as many friends in the other group than in his group. (We assume that A is a friend of B if and only if B is a friend of A.)*

Solution 3. This problem can be easily rephrased in terms of coloring of vertices of a graph:

Given a graph on n vertices. Prove that all its vertices can be colored with two colors, say red and blue, such that for every point at most $1/2$ of its neighbors have the same color as the point itself.

(Vertices of the graph correspond to people. Two vertices are joint by an edge if and only if the corresponding people are friends. Vertices of the same color correspond to people belonging to the same group.)

There is a surprisingly simple algorithm which finds such a coloring: First assign colors to the vertices at random. If the condition is satisfied, then we are done. If not, pick a point which has more than half of its neighbors the same color as itself and reverse the color of this point. See Figure 2. Repeat until no such points exist. We then have the desired coloring!

Does this algorithm ever stop?

First we observe that the number of different colorings of the vertices is finite. For every coloring, count the number of edges in the graph whose endpoints are colored in different colors. Call this number M. Therefore M can take only finitely many values. Let us show that after applying our move, the value of M strictly increases.

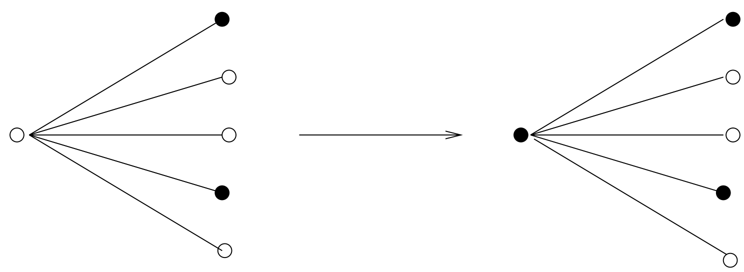

Figure 2

Suppose a chosen point had k neighbors and $s > k/2$ are colored the same. Changing the color of our point we add s to M and subtract $k-s$ from M. Therefore for the new coloring, there are $M + s - (k - s) = M + 2s - k > M$ edges whose endpoints are colored in different colors. (In Figure 2, $k = 5, s = 3 > 5/2$, and M is increased by $2 \cdot 3 - 5 = 1$.) This proves that the algorithm must terminate: otherwise it would produce infinitely many different values of M: contradiction. □

Before proceeding further, let us analyze the solutions of Examples 1-3. In each case we interpreted the problem as a sequence of allowable "moves" between elements of a certain finite set S: the set of "states" or "configurations" associated with the problem.

In Example 1, the finite set S can be defined as the set of all $m \times n$ arrays of real numbers obtained from a given $m \times n$ array by arbitrary changes of signs of its entries.

In Example 2, S is the finite set of all figures obtained by joining n pairs of given $2n$ points by segments.

In Example 3, S is the finite set of all different separations on the set of n people in two groups, or the set of all colorings of the vertices of the graph in colors red and blue.

Sometimes, as in Example 1 both S and the allowable moves were given in the problem. But in Examples 2 and 3 it was up to us to create them.

The only requirements on S are that it has to be *finite* and it has to contain all states which can be produced from the original one by allowable moves.

We next introduced, on S, a certain numerical function M which had the useful property of being strictly monotone with respect to the moves. More precisely, for every move and every pair of states $s,\ s' \in S$, where s' is obtained from s by a move, $M(s) > M(s')$ or for every move and every states $s, s' \in S$, where s' is obtained from s by the move, $M(s) < M(s')$.

We call such a function a *monovariant*, and denote it by M.

In Example 1, $M(s)$ was the sum of all entries of the array s, and it was an increasing function.

In Example 2, $M(s)$ was the sum of lengths of all segments in figure s, and it was a decreasing function.

In Example 3, $M(s)$ was the number of edges in a graph s whose endpoints were colored in different colors, and it was an increasing function.

In each example, the termination of the algorithm followed from the same basic argument:

If you move forever between your states, but have only finitely many of them, then at some point you must revisit some state. But then the alleged monovariant will repeat its value, something a strictly monotone function cannot do.

Let us continue with more examples.

Example 4. *A nonconvex polygon is subjected to the following operation: if it lies on one side of a line AB through nonadjacent vertices A and B, then one of the parts into which the perimeter of the polygon is divided by A and B is reflected about the midpoint of the segment $\overline{AB}$–that is, it's rotated $180°$ about the midpoint (see Figure 3). Prove that after a finite number of such operations the polygon becomes convex.*

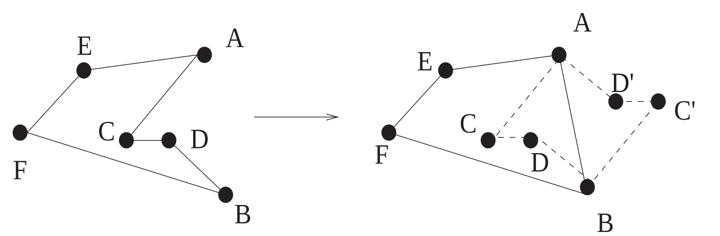

Figure 3

Solution 4. The strictly increasing monovariant here is the area of the polygon, which clearly increases after each move. It is a bit trickier to prove that the set of possible states is finite. This is accomplished by noting that, if you think of each side of the polygon as a segment of given length and given direction, then each move simply rearranges this set of segments, preserving their lengths and directions. Since there is a finite number of ways in which these segments can be arranged to form a polygon, there are only a finite number of possible states. □

Example 5. *Finitely many squares of an infinite square grid drawn on white paper are painted black. At each moment in time $t = 1, 2, \ldots$ each square takes the color of the majority of the following three squares: the square itself, and its top and right-hand neighbors. Prove that some time later there will be no black squares at all.*

Solution 5. Draw a rectangle bounded by grid lines which contains all the black squares. Introduce $x - y$ axes so that the origin is at the lower left corner of the rectangle, and the axes are directed as shown.

A little thought will reveal that at no moment will there be a black square outside of the rectangle. Thus the set of states can be taken to be all colorings of the squares within the rectangle using black or white, and so is finite. To define a monovariant M, compute the sum of the x- and y-coordinates for the upper-right corner of each black square, and let M be the greatest of these sums. It is not hard to see that this sum must decrease after each move.

Note: There is another method called "Going to Extremes" closely related to the method of monovariants. Briefly described, it reduces to the following: Consider the value of a function (essentially our monovariant) on the finite set of all possible states, and then consider some state s which minimizes (or maximizes) this function. Then show that if this state did not have the desired property, then a state with a smaller (or larger) function value can be constructed, contradicting our choice of s.

For example, another solution in Example 1 could go as follows: Among all achievable states, consider that one which maximizes the sum of all entries in the array. It exists since the number of states is finite. If the corresponding array had a line with negative sum, then switching the signs of the elements along that line yields an array with a larger sum (of all entries), contradiction.

More examples and useful discussions can be found in the article by D. Fomin and L. Kurlyandchik [1], on which these notes are based.

EXERCISES

By * we mark more difficult problems, and by ** – the hardest (in our opinion, of course).

(1) Each member of a parliament has no more than three enemies among other members. Prove that the parliament can be split into two houses such that each member has no more than one enemy in same house. (We assume that if A is an enemy of B, then B is an enemy of A.)

(2) N red and N blue points lie in the plane, and no three of them are collinear. Prove that one can draw N non-intersecting segments joining red points to blue points.

(3) Given two positive integers $a > b$, one can find their greatest common divisor gcd (a, b) using the *Euclidean Algorithm* as follows: divide a by b to get a quotient q_1 and a remainder r_1. Thus

$$a = q_1 b + r_1, \text{ with } 0 \leq r_1 < b.$$

if $r_1 = 0$, then gcd $(a, b) = b$. Otherwise, divide b by r_1 to get another quotient q_2 and remainder r_2. Thus

$$b = q_2 r_1 + r_2, \text{ with } 0 \leq r_2 < r_1.$$

If $r_2 = 0$ then gcd $(a, b) = r_2$. Otherwise divide r_1 by r_2 to get new quotient q_3 and remainder r_3, and so on, until you get a remainder r_n equal to 0. Then Then gcd $(a, b) = r_{n-1}$.

Prove that this algorithm terminates. (The proof that the algorithm really gives the greatest common divisor is a bit trickier, and is left as an additional exercise, not related to monovariants.)

(4) Given are n points, no three of which are collinear, and n lines, no two of which are parallel, in the plane. Prove that we can drop a perpendicular from each point to one of the lines, one perpendicular per line, such that no two perpendiculars intersect.

(5) * (*V. Alexeev*) Several numbers are arranged around a circle. If four consecutive numbers a, b, c, and d satisfy the inequality $(a-d)(b-c) > 0$, then we can exchange b and c. Prove that we can perform this operation only a finite number of times.

(6) Let r be an integer and $r \geq 2$. A graph is called *r-partite* if its vertex set V can be partitioned into r (nonempty) subsets $V_1, \ldots, V_r$ such that every edge of the graph joins two points in distinct subsets. The sets V_i are referred to as *partition classes*. Prove the following.

  (a) Every graph with e edges contains a bipartite (i.e., 2-partite) subgraph with at least $e/2$ edges.
  (b) Every graph with e edges contains a 3-partite subgraph with at least $\frac{2}{3}e$ edges.
  (c) For every $r \geq 2$, every graph with e edges contains an r-partite subgraph with at least $\frac{r-1}{r}e$ edges.
  (d) * Every graph on rn vertices and e edges, $2 \leq r \leq n$, contains an r-partite subgraph with at least $\frac{r-1}{r}e$ edges, and with each partition having n vertices.

  Clearly, part (6c) of this problem is a generalization of parts (6a) and (6b). Part (6d) is a stronger version of part (6c). We do not know a proof of part (6d) which uses a monovariant, even for $r = 2$; a solution we have is based on another approach.

(7) ** King Arthur summoned $2N$ knights to his court. Each knight has no more than $N-1$ enemies among the knights present. Prove that Merlin can seat the knights at the Round Table in such a way that no two enemies will sit next to each other.

(8) ** Consider a simple graph G with the vertex set $\{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for all $i = 1, \ldots, n$, and $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that either vertex v_1 is joined to vertices $v_2, v_3, \ldots, v_{d_1+1}$, or one can rearrange some edges of G in such a way that in the obtained graph G', $d_{G'}(v_i) = d_G(v_i) = d_i$ for all $i = 1, \ldots, n$, and vertex v_1 is joined in G' to vertices $v_2, v_3, \ldots, v_{d_1+1}$.

HINTS TO EXERCISES

(1) Compare with an example in the text.

(2) Compare with an example in the text.

(3) The set of states is the set of pairs (a, b), (b, r_1), (r_1, r_2), ... appearing during the algorithm. The moves and the monovariant here are obvious.

(4) Compare with an example in the text.

(5) Show that the sum $ab + bc + cd$ decreases with each move, and try to construct a monovariant based on this observation.

(6)
   (a) Compare with an example in the text. Use the fact that the sum of degrees of all vertices in a graph is twice the number of its edges.
   (b) Compare with an example in the text. Similar to the previous part.
   (c) Similar to the previous two parts.
   (d) Let G be a simple graph on rn vertices and e edges, $2 \leq r \leq n$. Consider all r-partite subgraphs of G with each partitions having n vertices. Show that the average number of edges in all these r-partite subgraphs is $\frac{r-1}{r}e$. Hence, at least one such a subgraph has at least as many edges.

(7) Look for a proof of Dirac's Theorem about Hamilton cycles. It is presented in many graph theory book or discrete mathematics books. Relate your problem to the theorem.
Alternately, you might notice that sometimes, when you take an arcs-worth of knights and reflect their positions along that arc, you decrease the number of pairs of enemies which are seated next to one another.

(8) Show that if v_1 is not joint to a vertex of degree d_i with $1 < i \leq d_1 + 1$, then it is joint to a vertex v_j, with $j > d_1 + 1$, and that $d_i > d_j$. Then show that there exists a vertex t distinct from v_1, v_i, v_j such that tv_i is an edge, and tv_j is not an edge. Delete edges v_1v_j and tv_i, and draw edges v_1v_i, tv_j.

Comment. This statement is the key idea in a proof of a theorem by Havel (1955) and by Hakimi (1962), which gives necessary and sufficient condition for a a sequence of n nonnegative integers to be graphic, i.e., to be a degree sequence of a simple graph:

For $n > 1$, a sequence

$$d_1 \geq d_2 \geq \cdots \geq d_n$$

is graphic if and only if the sequence

$$d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$$

is graphic.

References

[1] D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles (Russian Experience)*, Mathematical World, Vol. **7**, American Mathematical Society, 1996.

[2] D. Fomin and L. Kurlyandchik *Light at the end of the tunnel*, Quantum March/April 1994.