NOTES ON INEQUALITIES

FELIX LAZEBNIK

Order and inequalities are fundamental notions of modern mathematics. Calculus and Analysis depend heavily on them, and properties of inequalities provide the main tool for developing these subjects. Often students are not sure what they are allowed to take for granted when they argue about inequalities. These brief notes are intended to help them to review the basics, improve their skills in working with inequalities, and present two inequalities which have many applications. Here is the list of sections.

Content:

1. Order on $\mathbb{R}$ and Basic Properties of Inequalities

We assume that the reader is well familiar with real numbers (or just reals), with the algebraic properties of operations on them, and with basic properties of their ordering. We denote the set of all real numbers by $\mathbb{R}$, the set of all negative reals by $\mathbb{R}^-$, and the set of positive reals by $\mathbb{R}^+$. Then $\mathbb{R}^-$, $\{0\}$, and $\mathbb{R}^+$ partition $\mathbb{R}$, which is another way of saying that every real number is either negative, or zero, or positive, and no real number has two of these properties. We take for granted that the following properties hold.

- *the sum of any two positive reals is positive*
- *the product of two positive reals is positive*
- *the sum of any two negative reals is negative*
- *the product of any two negative reals is positive*
- *the product of a positive real and a negative real is negative*
- *number* 1 *is positive*
- x *positive (negative) if and only if* x^{-1} *is positive (negative)*
- *the ratio of a positive and a negative reals is negative*

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- *for any two reals x, y, $xy = 0$ if and only if $x = 0$ or $y = 0$*
- *for any two reals x, y, $x < y$ if and only if $y - x$ is positive.*
- for any positive (negative) real x, $0 < x$ ($x < 0$).

For convenience, we introduce another symbol $>$, called **greater**, and we write

$$x > y \quad \text{if} \quad y < x.$$

The abbreviation for $(x < y) \vee (x = y)$ is $x \leq y$. If $x \leq y$, we say that x is **at most** y, or, equivalently, y is **at least** x. Similarly, for $x \geq y$. Hence, $3 \leq 3$ and $3 \leq 5$ are true. If $0 \leq x$ or, equivalently, $x \geq 0$, we say that x is **non-negative**, and if $x \leq 0$ or, equivalently, $0 \geq x$, we say that a is **non-positive**.

The following property is used often, and we wish to state it separately.

- *for every x, $x^2 \geq 0$, and $x^2 = 0$ if and only if $x = 0$.*
  Moreover, the sum of $n \geq 2$ non-negative (non-positive) numbers is non-negative (non-positive), and it is equal to 0 if and only if all addends are equal to 0.

Now we want to (finally) prove some more subtle properties which relate the inequalities on $\mathbb{R}$ and operations on $\mathbb{R}$. (So, please forget that you are familiar with them!)

Theorem 1.1.

(1) *For all real x, y, z, if $x < y$ and $y < z$ then $x < z$.*
(2) *For all real x, y, z, $x < y$ if and only if $x + z < y + z$.*
(3) *For all real x, y and positive z, $x < y$ if and only if $xz < yz$.*
(4) *For all real x, y and negative z, $x < y$ if and only if $xz > yz$.*
(5) *For all real a, b, x, y, if $a < b$ and $x < y$, then $a + x < b + y$.*
Similar statement holds for the sum of any $n \geq 2$ inequalities.
(6) *For all positive a, b, x, y, if $a < b$ and $x < y$, then $ax < by$.*
Similar statement holds for the product of any $n \geq 2$ inequalities of positive numbers.
(7) *For all integer $n \geq 1$, and any positive a, b, $a < b$ if and only if $a^n < b^n$.*

Proof. (1) By the definition of $<$: if $x < y$, then $y - x$ is positive, and if $y < z$, then $z - y$ is positive. The sum of positive numbers is positive (a part of the definition of positive reals). So $(y - x) + (z - y) = z - x$ is positive. By the definition of $<$, $x < z$.

(2) $(y + z) - (x + z) = y - x$. Therefore the difference on the left is positive if and only if the difference on the right is positive. This means that the inequalities imply each other.

(3) $yz - xz = (y - x)z$. By the definition of $<$: if $x < y$, then $y - x$ is positive. Since z is positive (given), and the product of positive numbers is positive (part of the definition of positive reals), $(y - x)z$ is positive. Hence $yz - xz$ is positive, and so $xz < yz$.

(4) Left for the reader.

(5) *Proof 1.* By (2), $a < b$ implies $a + x < b + x$, and $x + b < y + b$. As addition of reals is commutative, the last inequality can be written as $b + x < b + y$. Then,

by transitivity of inequalities (Property (1)), $a+x < b+x$ and $b+x < b+y$ imply $a+x < b+y$.

Proof 2. We wish to show that $a+x < b+y$. By the definition of $<$, this amounts to showing that $(b+y)-(a+x) > 0$.

Note that $(b+y)-(a+x) = (b-a)+(y-x)$. As $b-a > 0$ (since $a < b$) and $y-x > 0$ (since $x < y$), and the sum of positive reals is positive, we have $(b-a)+(y-x) > 0$. Hence, $(b+y)-(a+x) > 0$, and the proof is finished.

Let us generalize (5):

For all $n \geq 2$, the inequalities $a_i < b_i$, $i = 1, \ldots, n$, imply
$a_1 + \ldots + a_n < b_1 + \ldots + b_n$.

We prove this by the method of mathematical induction (on n). The base case, $n = 2$, has been established in (5): $a_1 = a, b_1 = b$, $a_2 = x, b_2 = y$. Suppose the statement holds for any $n = k \geq 2$ inequalities. We have to show that it holds for any $n = k+1$ inequalities.

Let $x_i < y_i$ for all $i = 1, \ldots, k+1$. We have to show that

$$x_1 + \ldots + x_k + x_{k+1} < y_1 + \ldots + y_k + y_{k+1}.$$

Note that $x_1+\ldots+x_k < y_1+\ldots+y_k$ by the induction hypothesis, and $x_{k+1} < y_{k+1}$ as given. These two inequalities can be added, as was shown in the base case ($n = 2$). By doing this, we obtain

$$(x_1 + \ldots + x_k) + x_{k+1} < (y_1 + \ldots + y_k) + y_{k+1}.$$

When several real numbers are added, the parentheses can be placed in arbitrary way due to the associative property of addition of reals. Hence, the statement holds for any for $n = k+1$ inequalities, and the proof is finished.

By now the reader should understand the logic of such proofs, and we proceed by **omitting similar simple reasons** in our explanations.

(6) *Proof 1.* By (3), $a < b$ and $x > 0$ imply $ax < bx$. Similarly, $x < y$ and $b > 0$ imply $xb < yb$. As multiplication or reals is commutative, the last inequality is equivalent to $bx < by$. By transitivity of $<$, $ax < bx$ and $bx < by$ imply $ax < by$.

Proof 2. $by - ax = by - bx + bx - ax = b(y-x) + (b-a)x$. As all $b, y-x, b-a, x$ are positive, $by - ax$ is positive. Hence, $ax < by$.

We leave the proof of the generalization of this property (by the method of mathematical induction on n) to the reader.

(7) If we have established generalization of (6) for any $n \geq 2$ inequalities with positive sides, then the result follows from multiplying n same inequalities $a < b$, where $a, b > 0$. If the generalization of (6) has not been established, the result can be proven by induction on n.

□

Example 1. Let $0 < a < b$. Then for any integer $n \geq 2$, $\sqrt[n]{a} < \sqrt[n]{b}$.

Proof. We prove it by assuming the contrary and obtaining a contradiction. Suppose $\sqrt[n]{a} > \sqrt[n]{b}$. Since both sides are positive, $(\sqrt[n]{a})^n > (\sqrt[n]{b})^n$ by (7). As $(\sqrt[n]{a})^n = a$ and $(\sqrt[n]{a})^n = b$, this is equivalent to $a > b$, a contradiction with the assumption that $a < b$. Hence, $\sqrt[n]{a} < \sqrt[n]{b}$. □

The **absolute value** of a real number x, denoted by $|x|$, is defined as x for $x \geq 0$, and as $-x$ for $x < 0$. $|x|$ can be thought as the distance from the point on the real line corresponding to number x to the origin. More generally, for two points $A(a)$ and $B(b)$ on the real line, the distance $AB = |a - b|$. Important properties of absolute value are the following: for all x, y

$$|xy| = |x|\,|y|, \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ (y \neq 0), \quad |x + y| \leq |x| + |y|.$$

These properties of the absolute value function follow from the definition of operations with reals and their geometric interpretation as points of a line.

2. Solving inequalities: Case analysis

In the examples below we use symbols $\vee$ and $\wedge$ to denote the disjunction and conjunction of predicates (i.e., open sentences, equations, inequalities, ...). If the reader is unfamiliar with them, the usage of connectives "or" and "and", respectively, does the same.

Example 2. Find all real solutions of the following inequalities.

$$\text{(i)}\ x^2 - 3x + 2 > 0 \qquad \text{(ii)}\ \frac{3x+2}{x} \leq 1 \qquad \text{(iii)}\ \frac{x^2 - x - 1}{x + 2} < 3.$$

Solution. (i)

$$x^2 - 3x + 2 > 0 \Leftrightarrow$$
$$(x-1)(x-2) > 0 \Leftrightarrow$$
$$\Big[(x - 1 > 0) \wedge (x - 2 > 0)\Big] \bigvee \Big[(x - 1 < 0) \wedge (x - 2 < 0)\Big] \Leftrightarrow$$
$$\Big[(x > 1) \wedge (x > 2)\Big] \bigvee \Big[(x < 1) \wedge (x < 2)\Big] \Leftrightarrow$$
$$(x > 2) \vee (x < 1).$$

The solution set is $\boxed{(-\infty, 1) \cup (2, \infty).}$

(ii)

$$\frac{3x+2}{x} \leq 1 \Leftrightarrow$$
$$\Big[(x > 0) \wedge (3x + 2 \leq x)\Big] \bigvee \Big[(x < 0) \wedge (3x + 2 \geq x)\Big] \Leftrightarrow$$
$$[(x > 0) \wedge (x \leq -1] \bigvee [(x < 0) \wedge (x \geq -1)] \Leftrightarrow$$
$$(x \in \emptyset) \vee (-1 \leq x < 0).$$

The solution set is $\boxed{[-1, 0).}$

(iii)

$$\frac{x^2-x-1}{x+2} < 3 \Leftrightarrow$$

$$\Big[(x+2>0)\wedge(x^2-x-1<3(x+2))\Big] \bigvee \Big[(x+2<0)\wedge(x^2-x-1>3(x+2))\Big] \Leftrightarrow$$

$$\Big[(x>-2)\wedge(x^2-4x-7<0)\Big] \bigvee \Big[(x<-2)\wedge(x^2-4x-7>0)\Big] \Leftrightarrow$$

$$\Big[(x>-2)\wedge(2-\sqrt{11}<x<2+\sqrt{11})\Big] \bigvee \Big[(x<-2)\wedge[(x<2-\sqrt{11})\vee(x>2+\sqrt{11})]\Big] \Leftrightarrow$$

$$(2-\sqrt{11}<x<2+\sqrt{11}) \bigvee \Big[[(x<-2)\wedge(x<2-\sqrt{11})]\vee[(x<-2)\wedge(x>2+\sqrt{11})]\Big] \Leftrightarrow$$

$$(2-\sqrt{11}<x<2+\sqrt{11}) \bigvee [(x<-2)\vee(x\in\emptyset)] \Leftrightarrow$$

$$(2-\sqrt{11}<x<2+\sqrt{11}) \vee (x<-2).$$

The solution set is $\boxed{(2-\sqrt{11}, 2-\sqrt{11}) \cup (-\infty,-2)}$.

The method we used above is often referred as **case analysis**. It boils to partitioning the set of all possible values of x into disjoint subsets, and considering each of them separately. The following examples provide additional illustrations of the method.

Example 3. Find all real solutions of the inequality

$$\left|\frac{x-2}{x+1}\right| \le 2.$$

Solution. As $|a| \le b \;\Leftrightarrow\; -b \le a \le b \;\Leftrightarrow\; (-b \le a) \wedge (a \le b)$, we have:

$$\left|\frac{x-2}{x+1}\right| \le 2 \Leftrightarrow$$

$$-2 \le \frac{x-2}{x+1} \le 2 \Leftrightarrow$$

$$[(x+1>0)\wedge(-2x-2 \le x-2 \le 2x+2)] \bigvee [(x+1<0)\wedge(-2x-2 \ge x-2 \ge 2x+2)] \Leftrightarrow$$

$$[(x>-1)\wedge(x\ge 0)\wedge(x\ge -4)] \bigvee [(x<-1)\wedge(x\le 0)\wedge(x\le -4)] \Leftrightarrow$$

$$(x \ge 0) \vee (x \le -4).$$

The solution set is $\boxed{(-\infty,-4] \cup [0,\infty)}$.

Example 4. Find all real solutions of the inequality

$$|x+1|+|x+3|+|x-5| \le 8.$$

Solution. In order to rewrite the inequality without absolute value signs, we divide reals into four intervals defined by the points where the expressions under the absolute value signs change their signs, namely by $x=-3$, $x=-1$, and $x=5$. Then we have:

$$|x+1|+|x+3|+|x-5| \le 8 \Leftrightarrow$$
$$[(x \le -3) \wedge (-(x+1)+(-(x+3))+(-(x-5)) \le 8] \bigvee$$
$$[(-3 < x \le -1) \wedge (-(x+1)+(x+3)+(-(x-5)) \le 8] \bigvee$$
$$[(-1 < x \le 5) \wedge ((x+1)+(x+3)+(-(x-5)) \le 8] \bigvee$$
$$[(5 < x) \wedge ((x+1)+(x+3)+(x-5) \le 8] \quad \Leftrightarrow$$

$$[(x \le -3) \wedge (-3x+1 \le 8] \bigvee [(-3 < x \le -1) \wedge (-x+7 \le 8] \bigvee$$
$$[(-1 < x \le 5) \wedge (x+9 \le 8)] \bigvee [(5 < x) \wedge (3x-1 \le 8] \quad \Leftrightarrow$$

$$[(x \le -3) \wedge (x \ge -7/3] \bigvee [(-3 < x \le -1) \wedge (x \ge -1] \bigvee$$
$$[(-1 < x \le 5) \wedge (x \le -1)] \bigvee [(5 < x) \wedge (x \le 3] \quad \Leftrightarrow$$

$$(x \in \emptyset) \vee (x=-1) \vee (x \in \emptyset) \vee (x \in \emptyset) \quad \Leftrightarrow \quad x=-1.$$

The solution set is $\boxed{\{-1\}.}$

The answer suggests the existence of a simpler solution. Interpreting $|a-b|$ in geometric terms as the distance on the real line between points $A(a)$ and $B(b)$, we are looking at points $X(x)$ such that the sum of distances from X to points corresponding to -3, -1 and 5 is at least 8. Note that 8 is the distance between points corresponding -3 and 5. Therefore $|x+3|+|x-5|=8$ for all x in $[-3,5]$, and $|x+3|+|x-5|>8$ for all x not in $[-3,5]$. If $x \ne -1$, then $|x+1|>0$, and $|x+1|+|x+3|+|x-5|>8$. Hence, $x=-1$, is the *only* value of x such that and $|x+1|+|x+3|+|x-5| \le 8$. □

Example 5. Solve: $x^8 - x^5 + x^2 - x + 1 \le 0$.

Solution. Trying to factor the polynomial in the right hand side seems like a hard problem. Experimenting with different values of x, we fail to find any solutions. Therefore we will try to prove that no solutions exists, or in other words, that $f(x) = x^8 - x^5 + x^2 - x + 1 > 0$ for all x. Trying partitioning all reals into several subsets such that the statement can be easily verified in each of them, one may finally arrive to the following three cases: $x \le 0$, $0 < x < 1$, and $x \ge 1$.

Case 1: $x \le 0$. For such x, $f(x)$ is the sum of four nonnegative terms $x^8, -x^5, x^2, -x$, and 1. Hence $f(x) > 0$ in this case.

Case 2: $0 < x \le 1$. We rewrite $f(x)$ in the following way: $f(x) = x^8 + x^2(1 - x^3) + (1-x)$. As $0 < x \le 1$, $1 - x^3 \ge 0$ and $1 - x > 0$. Hence $f(x)$ is the sum of two positive numbers $x^8, 1-x$, and one nonnegative number $x^2(1-x^3)$. Hence $f(x) > 0$ in this case.

Case 3: $x \geq 1$. We rewrite $f(x)$ in the following way: $f(x) = (x^8 - x^5) + (x^2 - x) + 1 = x^5(x^3-1) + x(x-1) + 1$. As $x \geq 1$, both $x^3 - 1$ and $x - 1$ are nonnegative. Hence $f(x)$ is the sum of two nonnegative numbers $x^5(x^3-1)$ and $+\, x(x-1)$, and a positive number 1. Hence $f(x) > 0$ in this case.

Therefore the solution set of our inequality is the empty set $\boxed{\emptyset.}$ □

3. Solving Inequalities: Method of Intervals

There exists another way of solving inequalities, which uses continuity of functions and the Intermediate Value Theorem (IVT). Though the justification of this approach is based on calculus, the Method of Intervals, which we state further below, can be easily understood and used without any knowledge of calculus.

We remind the reader that a function f is **continuous on** $[a, b]$ if $\lim_{x\to c} f(x) = f(c)$ for all $c \in (a, b)$, $\lim_{x\to a+} f(x) = f(a)$, and $\lim_{x\to b-} f(x) = f(b)$. Intuitively, f is continuous if its graph can be drawn without taking the pen off the paper. It can be shown that all polynomials are continuous functions on $\mathbb{R}$, and all rational functions are continuous at the points they are defined. More generally, all elementary functions are continuous in their domain. Elementary functions include polynomial, rational functions, power functions, exponential functions, trigonometric functions, and inverses of all these functions where they exist. Absolute value function is another example of a continuous function. It can be proven that the sum, difference, product and ratio of continuous functions is continuous (in all points of its domain). Similarly, the compositions of continuous functions are also continuous.

We also remind the reader the IVT.

Theorem 3.1. (Intermediate Value Theorem)
A continuous function $f : [a, b] \to \mathbb{R}$ has the property that if $f(a)$ and $f(b)$ are distinct, then for every number M between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = M$.

In particular, when one of $f(a), f(b)$ is positive and another negative, and $M = 0$, there exists $c \in (a, b)$ such that $f(c) = 0$. An immediate corollary from the IVP is the following statement:

Corollary 3.2. *Let $f : [a, b] \to \mathbb{R}$ be continuous function and let $z_1, \ldots, z_n$, $z_1 < z_2 < \ldots < z_n$, be all distinct solutions of the equation $f(z) = 0$ in (a, b). Then on each of the $n + 1$ intervals (a, z_1), (z_i, z_{i+1}), and (z_n, b), all values of f are of the same sign, i.e., are all positive or all negative.*

The validity of this corollary should be clear: if at least one of these intervals contains two points such that f is positive at one of them and negative at another, by the IVT there would be a point between them (and so inside the interval) where f would take value zero. But there are no such points in the interval because z_i are the only points on (a, b) where f is zero. Hence, the sign of all values of f on (z_i, z_{i+1}) can be determined by its sign at one point of the interval.

A method of solving inequalities based on this corollary is often referred to as the **Method of Intervals**. Suppose we have to solve an inequality

$$f(x) > 0,$$

where f is a function continuous on an interval (a, b), except, maybe, of a finite number of (discontinuity) points. We allow $a = -\infty$ or $b = \infty$.

Consider the equation

$$f(x) = 0.$$

Suppose it has finitely many solutions on (a, b). Let $x_1, x_2, \ldots, x_k$ represent **all** points of (a, b) where f is not continuous, together with **all** solutions of the equation $f(x) = 0$. Assume

$$x_1 < x_2 < \ldots < x_k.$$

These k points divide the interval into $k + 1$ subintervals:

$$(a, x_1),\ (x_1, x_2),\ (x_2, x_3),\ \ldots, (x_k, b).$$

On each of these intervals, f is continuous and nonzero. Therefore, in order to determine the sign of f on each interval, we just chose a point from it and determine the sign of f at this point. By the Corollary 3.2, for all points of the interval, this sign will be the same.

Example 6. Solve the inequality

$$\frac{x^2 - x - 6}{x^2 - 1} > 0.$$

Solution. Let $f(x) = \frac{x^2-x-6}{x^2-1}$. Being a rational function, f is continuous at all points where it is defined, i.e., for all real numbers, except values of x which make $x^2 - 1$ equal to zero. As $x^2 - 1 = (x - 1)(x + 1)$, f is not continuous at $x = -1$ and $x = 1$.

The equation $f(x) = 0$ is equivalent to $x^2 - x - 6 = 0 \ \wedge\ (x \neq \pm 1)$. As $x^2 - x - 6 = (x - 3)(x + 2)$, $f(x) = 0$ for $x = -2$ and $x = 3$.

Therefore we consider the following five intervals:

$$(-\infty, -2),\ (-2, -1),\ (-1, 1),\ (1, 3), (3, \infty).$$

Choosing a point in each of them, and deciding whether f is positive or negative at this point, we obtain

$$f(-100) > 0,\ f(-1.5) < 0,\ f(0) > 0,\ f(2) > 0,\ f(100) > 0.$$

Therefore, $f(x) > 0$ precisely on $\boxed{(-\infty, -2),\ (-1, 1), \quad \text{and} \quad (3, \infty).}$ □

Before proceeding to other examples, we wish to make several simple comments.

- It is clear that the inequality $f(x) < 0$ can be solved in a similar way.
- An inequality of the form $f(x) > g(x)$ ($f(x) < g(x)$) is equivalent to $h(x) > 0$ ($h(x) < 0$), where $h(x) = f(x) - g(x)$. Instead of introducing h, one can just solve the equation $f(x) = g(x)$, and proceed as before by testing a number from each obtained interval.

- In case we have $f(x) \geq 0$, or $f(x) \leq 0$, or $f(x) \geq g(x)$, or $f(x) \leq g(x)$, we have to add the solutions of the corresponding equation to the points of some intervals.

  For example, the solution of $\frac{x^2-x-6}{x^2-1} \geq 0$ would consist of all points of the intervals
$$(-\infty, -2],\ (-1, 1), \quad \text{and} \quad [3, \infty).$$
- In many cases, in order to determine the sign of a function at a point, one does not have to compute the exact value of the function.

Example 7. Find all real solutions of the inequality
$$|x^2 - 7x + 1| < 5.$$

Solution. First we consider the corresponding equation.
$$|x^2 - 7x + 1| = 5 \ \Leftrightarrow\ [(x^2 - 7x + 1 = 5) \vee (x^2 - 7x + 1 = -5)] \ \Leftrightarrow$$
$$(x^2 - 7x - 4 = 0) \vee (x^2 - 7x + 6 = 0)] \ \Leftrightarrow$$
$$[(x = (7 - \sqrt{65})/2) \vee (x = (7 + \sqrt{65})/2)] \vee [(x = 1) \vee (x = 6)].$$

We consider five intervals:
$(-\infty, (7 - \sqrt{65})/2),\ ((7 - \sqrt{65})/2, 1),\ (1, 6),\ (6, (7 + \sqrt{65})/2),\ ((7 + \sqrt{65})/2, \infty).$

Let $f(x) = |x^2 - 7x + 1| - 5$. As
$$f(-100) > 0,\ f(0) < 0,\ f(2) > 0,\ f(7) < 0, \text{ and } f(100) > 0,$$
we conclude that the solution set of the inequality is $\boxed{\left(\frac{7-\sqrt{65}}{2}, 1\right) \bigcup \left(6, \frac{7+\sqrt{65}}{2}\right).}$

4. Proving Inequalities by Induction.

The method of mathematical induction is often very effective for proving inequalities. We remind ourselves that there are two most often used versions of the method. We refer to the first one as to "mathematical induction", and the second – as to "strong mathematical induction". The adjective "mathematical" is used to stress the fact that we mean a particular mathematical statement, since the word "induction" is applied to a much broader notion.[1]

Let n_0 be a fixed integer, and suppose we wish to prove that for all integers $n \geq n_0$, and $P(n)$ be a statement referring to n. We remind the reader[2] that according to the method of **Mathematical Induction** , we prove the statement
$$\text{“for all } n,\ n \geq n_0,\ P(n)\text{”}$$

[1]See, e.g. `http://en.wikipedia.org/wiki/Inductive_reasoning`

[2]We assume that the reader has had some experience with mathematical induction.

if we establish the following two facts:

(i) $P(n_0)$ is correct (the base case);
(ii) if $P(k)$ is true, then $P(k+1)$ is true for each $k \geq n_0$.

The method of **Strong Mathematical Induction** asserts that we prove the statement

$$\text{"for all } n,\ n \geq n_0,\ P(n)\text{"}$$

if we establish the following two facts:
(i) $P(n_0)$ is correct (the base case);
(ii) if $P(k)$ is true for all $n_0 \leq k < n$, then $P(n)$ is true.

At the first glance, the second method seems to be more convenient to use than the first, since we are allowed to assume more. This is true. Nevertheless, one can show that the methods are equivalent.

Let us present several examples of proofs of inequalities by mathematical induction. In all our examples we will apply the following logic. Suppose we have $A > B$, and we wish to show that $A > C$. Then, if we show that $B \geq C$, we are done. If $B \geq C$ is false, or if we are unable to show this, then A still can be greater than C, but it has to be established in some other way.

Example 8. Prove that for all integers n, $n \geq 2$,

$$1 + \frac{1}{2} + \ldots + \frac{1}{n} > \frac{2n}{n+1}$$

Solution. We use the method of Mathematical Induction.

Let $n = 2$. We have to show that $1 + \frac{1}{2} > \frac{4}{3}$. As $1 + 1/2 = 3/2$, and $3/2 > 4/3$ ($3^2 = 9 > 2 \cdot 4 = 8$), the base case is established.

Suppose $k \geq 2$ and the statement is true for $n = k$, i.e.,

$$1 + \frac{1}{2} + \ldots + \frac{1}{k} > \frac{2k}{k+1}. \tag{1}$$

We wish to show that the statement is true for $n = k+1$, i.e.,

$$1 + \frac{1}{2} + \ldots + \frac{1}{k+1} > \frac{2(k+1)}{(k+1)+1} = \frac{2(k+1)}{k+2}. \tag{2}$$

From (1), we have:

$$\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) + \frac{1}{k+1} > \frac{2k}{k+1} + \frac{1}{k+1}. \tag{3}$$

As the expressions in the left hand sides of (2)) and (3) are equal, it is sufficient to show that

$$\frac{2k}{k+1} + \frac{1}{k+1} \geq \frac{2(k+1)}{k+2} \tag{4}$$

The last inequality is equivalent to

$$\frac{2k+1}{k+1} \geq \frac{2k+2}{k+2} \;\Leftrightarrow\; (2k+1)(k+2) \geq (k+1)(2k+2) \;\Leftrightarrow$$
$$2k^2 + 5k + 2 \geq 2k^2 + 4k + 2 \;\Leftrightarrow\; k \geq 0.$$

As $k \geq 2$, (4) holds, and the proof is finished. □

To see how this solution reflects on our general scheme of inductive proofs discussed above, take $n_0 = 2$, $P(n)$ to be the statement $1 + \frac{1}{2} + \ldots + \frac{1}{n} > \frac{2n}{n+1}$, $P(k)$ to be the statement (2), $P(k+1)$ to be the statement (3), $A = 1 + \frac{1}{2} + \ldots + \frac{1}{k+1}$, $B = \frac{2k}{k+1} + \frac{1}{k+1}$, and $C = \frac{2(k+1)}{k+2}$.

Example 9. Prove that for all integers n, $n \geq 1$,

$$2^{n-1} \leq n!$$

Solution. We use the method of Mathematical Induction.

Let $n = 1$. We have to show that $2^0 \leq 1!$. As $2^0 = 1! = 1$, the base case is established.

Suppose $k \geq 2$ and the statement is true for $n = k$, i.e.,

$$2^{k-1} \leq k! \tag{5}$$

We wish to show that the statement is true for $n = k + 1$, i.e.,

$$2^{(k+1)-1} \leq (k+1)! \quad \Leftrightarrow \quad 2^k \leq (k+1)! \tag{6}$$

From (5), we have:

$$2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k! \tag{7}$$

As the expressions in the left hand sides of (6)) and (7) are equal, it is sufficient to show that

$$2 \cdot k! \leq (k+1)! \tag{8}$$

The last inequality is equivalent to

$$2 \leq k + 1$$

As $k \geq 1$, (8) holds, and the proof is finished. □

Note that the inequality in Example 9 could be established faster, by multiplying obvious n inequalities:

$$1 \leq 1, \quad 2 \leq 2, \quad 2 \leq 3, \quad 2 \leq 4, \quad \ldots \quad 2 \leq n.$$

Solution of many problems suggested at the end follow the patterns of our solutions to Examples 8 and 9.

The following example illustrates how one can use the Strong Mathematical Induction for proving inequalities.

Example 10. Let the sequence $\{a_n\}_{n\geq 1}$ be defined as follows: $a_1 = 1$, $a_2 = 2$, and for all $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. Here are several first successive terms of the sequence:

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \ldots$$

Prove that for all n, $n \geq 1$, $a_n < 1.7^n$.

Solution. We use the method of Strong Mathematical Induction.

Let us check the statement for the first two values of n, $n = 1$ and $n = 2$. We have to show that $a_1 < 1.7^1$, and $a_2 < 1.7^2$. This is obvious, as $1 < 1.7$, and

$$2 \; < \; 1.7^2 = 2.89.$$

It establishes the base case. Why we need to check the statement for $n = 2$ will become clear later.

Suppose the statement is true for all k, $1 \le k < n$, i.e.,

$$a_k \le 1.7^k. \tag{9}$$

We wish to show that

$$a_n < 1.7^n. \tag{10}$$

From the definition of the sequence and the induction hypothesis, we have:

$$a_n = a_{n-1} + a_{n-2} < 1.7^{n-1} + 1.7^{n-2}. \tag{11}$$

Comparing (10)) and (11), we conclude that in order to prove (10)), it is sufficient to show that

$$1.7^{n-1} + 1.7^{n-2} < 1.7^n. \tag{12}$$

Dividing both sides of the last inequality by 1.7^{n-2} (which is positive), we obtain an equivalent inequality

$$1 + 1.7 < 1.7^2$$

As $2.7 < 2.89$, (10) holds for all $n \ge 3$, and the proof is finished. □

Question: Let a_n be as in the last example. Can $a_n < \alpha^n$ still holds for all $n \ge 1$ for some $\alpha < 1.7$? What is the least α such that $a_n < \alpha^n$ holds for all $n \ge 1$?

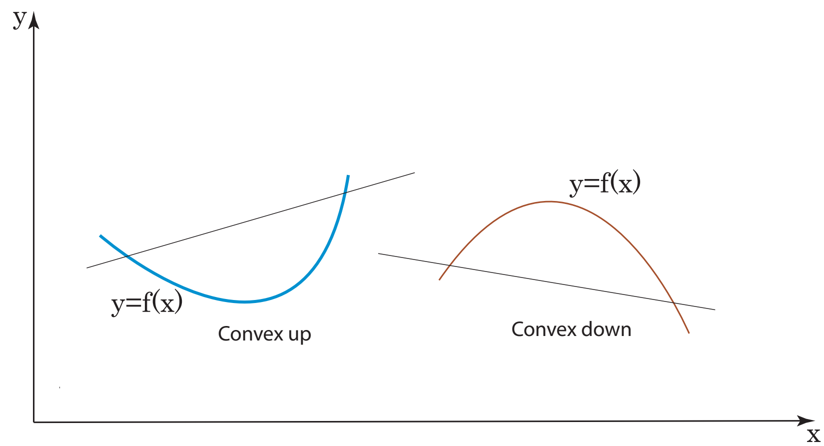
5. Jensen's Inequality

A function $f : (a, b) \to \mathbb{R}$ is called **convex up** (same as concave up) on (a, b), if for all $x_1, x_2 \in (a, b)$,

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}. \tag{13}$$

Changing the sign in the inequality above to $\ge$, we obtain the definition of a function **convex down** (same as concave down) on (a, b).[3] If f is convex up (down) on (a, b), then its graph lies below (above) the line passing through any of its two points, as illustrated below.

[3]To remember the shape of a convex up function, think about the shape of letter "U" in "Up".



It turns out that for a convex up (down) function, an analog of the inequality (13) also holds for any $n \geq 2$ numbers from the interval. The following inequality was published by J.L. Jensen (1859 - 1925) in 1906.

Theorem 5.1. (Jensen's Inequality) *Let f be a function convex up on (a, b). Then for any $n \geq 2$ numbers $x_i \in (a, b)$,*

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{\sum_{i=1}^{n} f(x_i)}{n},$$

and that the equality is attained if and only if f is linear or all x_i are equal.

For a convex down function, the sign of the inequality changes to $\geq$.

Proof. The proof we present uses the idea of 'up' and 'down' induction (having nothing to do with 'up' and 'down' convexity) which is credited to A.L. Cauchy (1789 - 1857).

The idea of 'up' and 'down' induction is the following. After establishing the base case (for $n = n_0$), one then proves the statement for a particular infinite increasing sequence of integers. The sequence can be, e.g., $n = 2^k$, $k \geq 1$, and a proof that the statement holds for these n can use an independent inductive (with respect to k) argument. Once it is done, one shows that if a statement is true for $n = k > n_0$, then it must me true for $n = k - 1$. After it is done, one concludes that the statement is true for all the integers n, $n \geq n_0$.

Meditating a minute over this approach, we conclude that it is as valid as one of the more traditional versions of mathematical induction.

In this proof we first use induction on k to show that the inequality is satisfied for all $n = 2^k$ (the 'up' part). The statement is correct for $k = 1$, as it states the given fact f is convex up on (a, b). To illustrate the main step (the passage from $n = 2^k$ to $n = 2^{k+1}$), we first consider next two values of n, namely $n = 2^2 = 4$ and $n = 2^3 = 8$.

Suppose $n = 2^2 = 4$. Then

$$f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) = f\left(\frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2}\right) \leq$$

$$\frac{f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)}{2} \le$$
$$\frac{\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_3)+f(x_4)}{2}}{2} =$$
$$\frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}.$$

Hence, the statement is proven for $n = 2^2$. Similarly, for $n = 2^3 = 8$, we have

$$f\left(\frac{x_1+\ldots+x_8}{8}\right) = f\left(\frac{\frac{x_1+\ldots+x_4}{4}+\frac{x_5+\cdots+x_8}{4}}{2}\right) \le$$
$$\frac{f\left(\frac{x_1+\ldots+x_4}{4}\right)+f\left(\frac{x_5+\ldots+x_8}{4}\right)}{2} \le$$
$$\frac{\frac{f(x_1)+\ldots+f(x_4)}{4}+\frac{f(x_5)+\ldots+f(x_8)}{4}}{2} =$$
$$\frac{f(x_1)+\ldots+f(x_8)}{8}.$$

The transition from $n = 2^k$ to $n = 2^{k+1}$ follows similarly to the preceding particular cases. As

$$\frac{x_1+\ldots+x_{2^{k+1}}}{2^{k+1}} = \frac{\frac{x_1+\ldots+x_{2^k}}{2^k}+\frac{x_{2^k+1}+\ldots+x_{2^{k+1}}}{2^k}}{2}$$

for all $k \ge 2$, we have:

$$f\left(\frac{x_1+\ldots+x_{2^{k+1}}}{2^{k+1}}\right) = f\left(\frac{\frac{x_1+\ldots+x_{2^k}}{2^k}+\frac{x_{2^k+1}+\ldots+x_{2^{k+1}}}{2^k}}{2}\right) \le$$
$$\frac{f\left(\frac{x_1+\ldots+x_{2^k}}{2^k}\right)+f\left(\frac{x_{2^k+1}+\ldots+x_{2^{k+1}}}{2^k}\right)}{2} \le$$
$$\frac{\frac{f(x_1)+\ldots+f(x_{2^k})}{2^k}+\frac{f(x_{2^k+1})+\ldots+f(x_{2^{k+1}})}{2^k}}{2} =$$
$$\frac{f(x_1)+\ldots+f(x_{2^{k+1}})}{2^{k+1}}.$$

Now we suppose that the statement is proven for $n \ge 3$ values of x_i's and show that it implies the statement for $n-1$ values of x (the 'down' part). Let $x_i \in (a,b)$, $i \in [n-1]$, be arbitrary $n-1$ numbers on (a,b). Apply the inequality to the following n numbers:

$$x_1, x_2, \ldots, x_{n-1}, \text{ and } x_n = \frac{x_1+\ldots+x_{n-1}}{n-1} = \frac{\sum_{i=1}^{n-1} x_i}{n-1}.$$

We have

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \le \frac{\sum_{i=1}^{n} f(x_i)}{n} \Leftrightarrow$$
$$f\left(\frac{\frac{n}{n-1}\sum_{i=1}^{n-1} x_i}{n}\right) \le \frac{\sum_{i=1}^{n-1} f(x_i)+f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)}{n} \Leftrightarrow$$
$$f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right) \le \frac{\sum_{i=1}^{n-1} f(x_i)+f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)}{n}.$$

Solving the last inequality for $f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right)$, we obtain

$$f\left(\frac{\sum_{i=1}^{n-1} x_i}{n-1}\right) \leq \frac{\sum_{i=1}^{n-1} f(x_i)}{n-1}.$$

This completes the proof. The assertion about the equality sign should be a part of the inductive hypothesis, and it follows immediately. □

Several famous inequalities can be obtained as simple corollaries of Jensen's inequality:

- $f(x) = x^2$ gives us that the quadratic mean is greater or equal than the arithmetic mean:

$$\frac{\sum_{i=1}^{n} x_i}{n} \leq \left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)^{1/2}.$$

  The **quadratic mean** of numbers $x_1, x_2, \ldots, x_n$ is the name of the number on the right. This inequality is called the **arithmetic-quadratic mean inequality**. This inequality is useful in Probability theory and Statistics.

- $f(x) = \ln x,\ x > 0$ gives that the arithmetic mean of n positive real numbers is greater or equal than the geometric mean:

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

  This inequality is called the **arithmetic-geometric mean inequality (AGM)**. It allows to minimize the sum of numbers whose product is fixed, or to maximize the product of numbers whose sum is fixed.

- $f(x) = 1/x,\ x > 0$ gives

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

  The expression on the right is called the **harmonic mean** of n positive numbers, and the inequality is called the **arithmetic-harmonic mean inequality**. An equivalent way to write it is:

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \geq n^2.$$

In order to apply Jensen's inequality, one has to check that the function is concave up (down). This can be done by using the definition ($n = 2$), or by using the second derivative test when it is applicable. Let's do it for the three function above.

1. $f(x) = x^2$. Using the definition, we have:

$$f\Big(\frac{x_1 + x_2}{2}\Big) = \Big(\frac{x_1 + x_2}{2}\Big)^2 \leq \frac{x_1^2 + x_2^2}{2} = \frac{f(x_1) + f(x_2)}{2},$$

and the equality is attained only if it is attained in

$$\left(\frac{x_1+x_2}{2}\right)^2 \le \frac{x_1^2+x_2^2}{2} \quad \Leftrightarrow \quad (x_1-x_2)^2 \ge 0,$$

i.e., for $x_1 = x_2$.

Using the second derivative, we have: $f''(x) = 2 > 0$ for all real x. Each argument proves that f is convex up on $\mathbb{R}$.

2. $f(x) = \ln x$, $x > 0$. Using the definition, we have:

$$f\Big(\frac{x_1+x_2}{2}\Big) = \ln\Big(\frac{x_1+x_2}{2}\Big),$$

$$\frac{f(x_1)+f(x_2)}{2} = \frac{\ln x_1 + \ln x_2}{2} = \frac{1}{2}\ln(x_1x_2) = \ln(\sqrt{x_1x_2}),$$

$$\ln\Big(\frac{x_1+x_2}{2}\Big) \ge \ln(\sqrt{x_1x_2}) \quad \Leftrightarrow \quad (\sqrt{x_1}-\sqrt{x_2})^2 \ge 0.$$

Hence,

$$f\Big(\frac{x_1+x_2}{2}\Big) \ge \frac{f(x_1)+f(x_2)}{2},$$

and the function is convex down. The equality is attained only if $x_1 = x_2 > 0$.

Using the second derivative, we have: $(\ln x)'' = (1/x)' = -1/x^2 < 0$ for all $x > 0$. Hence, f is convex down on $(0, \infty)$.

3. $f(x) = 1/x$, $x > 0$. We leave it to the reader to show that f is convex up on $(0, \infty)$.

6. **The Arithmetic-Geometric Mean Inequality (AGM)**

Here we wish to present several more examples of applications of the AGM inequality:

For all non-negative real numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1x_2\cdots x_n} \le \frac{x_1+x_2+\ldots+x_n}{n},$$

with equality taking place if and only if all x_i *are equal.*

Example 11. What is the maximum product of four positive numbers which add to 20? Which numbers maximize the product?

Solution. Let $a, b, c, d \ge 0$ be the numbers. Then $a+b+c+d = 20$. By the AGM inequality,

$$\sqrt[4]{abcd} \le \frac{a+b+c+d}{4} = 5.$$

Hence $abcd \le 5^4 = 625$, with equality if and only if $a = b = c = d\,(= 5)$. Hence, the greatest value of the product is 625, and it is attained if and only if $a = b = c = d = 5$. □

Those who are familiar with calculus of several variables may try to apply it to the Example 11. It will not work as fast.

Example 12. Out of all rectangular parallelepipeds (boxes) of unit volume, which one has the smallest surface area?

Solution. Let $x, y, z > 0$ denote the dimensions of a parallelepiped. Then its volume is $xyz = 1$, and its surface area is $A = 2(xy + yz + zx)$. As numbers xy, yz, zx are positive, the AGM inequality gives:

$$\frac{xy + yz + zx}{3} \geq \sqrt[3]{xy \cdot yz \cdot zx} = \sqrt[3]{(xyz)^2} = 1.$$

Hence, $xy + yz + zx \geq 3$, and $A \geq 6$. The equality is achieved if and only if $xy = yz = zx$. As $x, y, z > 0$, we obtain $x = y = z$ $(= 1)$. Therefore he parallelepiped of minimum surface area is the unit cube. $\square$

Example 13. Show that $2x^3 + 5x + \frac{1}{x^4} > 6$ for all positive x.

Solution. Applying the AGM inequality for three positive numbers $2x^3, 5x$, and $\frac{1}{x^4}$, we obtain:

$$\sqrt[3]{10} = \sqrt[3]{2x^3 \cdot 5x \cdot \frac{1}{x^4}} \leq \frac{2x^3 + 5x + \frac{1}{x^4}}{3}.$$

Therefore $3\sqrt[3]{10} \leq 2x^3 + 5x + \frac{1}{x^4}$. Now observe that $6 < 3\sqrt[3]{10}$ as $6^3 = 216 < (3\sqrt[3]{10})^3 = 270$. $\square$

Is it easy to obtain the result of Example 13 by using Calculus?

Example 14. Prove that for all integer $n \geq 2$, $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$.

Proof. Consider $n + 1$ positive numbers: $x_1 = x_2 = \ldots = x_n = 1 + 1/n$, and $x_{n+1} = 1$. Applying the AGM inequality to these numbers, we obtain

$$\begin{aligned} \left((1 + \frac{1}{n})^n \cdot 1\right)^{\frac{1}{n+1}} &< \frac{n \cdot (1 + \frac{1}{n}) + 1}{n+1} \\ \left(1 + \frac{1}{n}\right)^n &< \left(\frac{n+2}{n+1}\right)^{n+1} \\ \left(1 + \frac{1}{n}\right)^n &< \left(1 + \frac{1}{n+1}\right)^{n+1}. \ \square \end{aligned}$$

7. Problems

All letters in these exercises represent REAL numbers, and we will not be repeating this. Many of the problems below can be solved without references to Calculus, but you are welcome to use Calculus as much as you wish. Problems which require use of Calculus (or greatly benefit from it) are marked by red symbol ⓒ. Most problems can be approached by the methods discussed in this paper, such as using the elementary properties of inequalities, case analysis, using the the method of intervals, the method of mathematical induction, Jensen's inequality and its corollaries (especially AGM or AQM inequalities). Horizontal lines separate problems of various difficulties (in the opinion of the author): easier, medium, harder.

1. Prove each correct statement. Give a counterexample to each false statement. If you see a short way to modify false statements such that they become correct ones, please do it.
   (a) For all a, b, c, d, $a > b$ and $c > d$ imply $a - c > b - d$.
   (b) For all a, b, c, d, $a > b$ and $c > d$ imply $ac > bd$.
   (c) For all a, $a^2 > a^3$.
   (d) For all $a > 1$, $1/a < 1$.
   (e) For all nonzero a, b, if $a > b$ then $1/a < 1/b$.
   (f) For all positive a, b, c, d, $a > b$ and $c > d$ implies $a/c > b/d$.
   (g) For all a, b, $a > b$ implies $a^2 > b^2$
   (h) For all a, b, $a^2 > b^2$ implies $a > b$.
   (i) For all a, b, $a^3 > b^3$ if and only if $a > b$.
   (j) For all positive integer n, and all a, b, $a^{2n+1} > b^{2n+1}$ if and only if $a > b$.
   (k) For all $a \geq 0$ and all b, $\sqrt{a} > b$ implies $a > b^2$.
   (l) For all a, b, $a > b^2$ implies $\sqrt{a} > b$.
   (m) For all $a \geq 0$ and all b, $\sqrt{a} < b$ implies $a < b^2$.
   (n) For all $a \geq 0$ and all b, $a < b^2$ implies $\sqrt{a} < b$.

2. Justify the following statements.
   (a) If $x > 1$, then $x^5 > x^2$.
   (b) If $0 < x < 1$, then $x^{10} < x^2 < x^{10} + 1$.
   (c) If $f(x) = x^2 - 5x - 1$, then $f(1234567) > 0$, and $f(0.1234567) < 0$.

3. Without using calculator, or computers, decide which number is greater and prove your answer. (But you are welcome to use anything you wish in order to *check* whether your answer is correct.)
   (a) $\sqrt[2]{2}$ or $\sqrt[3]{3}$; $\sqrt[4]{4}$ or $\sqrt[5]{5}$
   (b) $\sqrt{10} + \sqrt{30}$ or $\sqrt{11} + \sqrt{29}$
   (c) $1/4$ or $\frac{\sqrt{5}+1}{5\sqrt{10-2\sqrt{5}}}$
   (d) $\sqrt[3]{10} + \sqrt{8}$ or 5

4. (i) Without using a calculator or computer, explain which of the two numbers is larger:
$$2^{1,000,000} \quad \text{or} \quad 1,000,000^{1,000}.$$
(ii) ⓒ Do graphs of the curves $y = 2^x$ and $y = x^{1000}$ intersect over $[3, \infty)$?

5. A boat travels in the river from A to B and then back to A. The speed of a boat in still water is v miles per hour, and the speed of the current is u miles per hour ($u < v$). Suppose t_1 is the time of the round trip between A and B if there were no current, and t_2 is the time of the round trip between A and B with the current. Compare t_1 and t_2.

6. Solve the following inequalities.
   (a) (i) $x^2 - 3x - 5 \geq 0$ (ii) $x^2 - 3x + 5 \geq 0$ (iii) $2x^2 - x + 1 < 0$
   (b) (i) $\frac{x^2+3x-10}{x^2-3x-28} < 0$ (ii) $\frac{x^2-25}{(x-2)(x^2-x+6)} \geq 0$
   (c) (i) $|2x - 4| \geq 5$ (ii) $|3x - 4| + x \leq 6$ (iii) $|x^2 - 4x| \geq 6$
   (d) (i) $|x + 5| + |x - 11| > 20$ (ii) $|x + 5| + |x - 11| < 15$
   (e) (i) $\frac{x-1}{x+2} > \frac{x+1}{x+5}$. (ii) $\left|\frac{x+2}{x-1}\right| > 3$.
   (f) $5 - 3x < \sqrt{(x-1)(2-x)}$
   (g) $\sqrt{2-x} \leq x + 1$.

7. Use the method of mathematical induction to show that for $n \geq 2$, inequalities $a_1 < b_1$, $a_1 < b_1$, $\ldots$, $a_n < b_n$ with all a_i positive, imply $a_1 a_2 \ldots a_n < b_1 b_2 \ldots b_n$. Conclude from here that $0 < a < b$ implies $a^n < b^n$ for all integer $n \geq 2$.

8. Prove the following inequalities.
   (a) $2x^3 > x + 1$ if $x > 1$, and $2x^3 < x + 1$ if $x < 1$.
   (b) For all x, $x^{12} - x^9 + x^4 - x + 1 > 0$.
   (c) For all x, $2x^4 + 1 \geq 2x^3 + x^2$.

9. Let a, b be the lengths of two legs of a right triangle, and let c be the length of its hypothenuse. Prove that $a^3 + b^3 < c^3$.

10. Find all solutions of the equation $\cos^{2008} x + \sin^{2008} x = 1$ on $[0, 2\pi]$.

11. Let $x_1, , x_2, \ldots, x_n$ be real numbers, let m be the smallest of them, and M be the largest of them. Prove that
$$m \leq \frac{x_1 + x_2 + \ldots + x_n}{n} \leq M,$$
with equality in each part if and only if all n numbers are equal.

12. Prove the following inequalities.
   (a) If $ab > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$. Describe completely when the equality takes place.
   (b) If $a, b, c > 0$, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$, and the equality occurs if and only if $a = b = c$.
   (c) For all x, y, z, $x^2 + y^2 + z^2 \geq xy + yz + zx$, and the equality occurs if and only if $x = y = z$.

(d) If $x + y + z = 1$, then $x^2 + y^2 + z^2 \geq 1/3$. Describe completely when the equality takes place.
(e) If not all a, b, c are equal, then $a^2 + b^2 + c^2 + 3 > 2(a + b + c)$.
(f) Suppose $a_1 a_2 \dots a_n = 1$ and all $a_i > 0$. Prove that $(1 + a_1)(1 + a_2) \dots (1 + a_n) \geq 2^n$. Describe completely when the equality takes place.

13. Out of all rectangular parallelepipeds
 (i) of a given perimeter, which one has the largest volume?
 (ii) of a given volume, which one has the smallest perimeter?
 (iii) of a given surface area, which one has the largest volume?

14. Let a, b, c be the lengths of sides of a triangle ABC, and let $p = (a+b+c)/2$ be its semiperimeter. Then the area of the triangle can be found by the famous Archimedes-Heron formula:
$$\text{Area} \triangle ABC = \sqrt{p(p-a)(p-b)(p-c)}.$$
Use the AGM inequality to prove that
 (i) out of all triangles with a given perimeter ($= 2p$), the equilateral triangle has the greatest possible area.
 (ii) out of all triangles with a given area, the equilateral triangle has the smallest perimeter.

15. Prove that for $n \geq 5$, $2^n > n^2$

16. Prove that for $n \geq 0$, $3^n > (n-1)^3$

17. Investigate for which positive integers n, $n! > n2^n$, and prove your result.

18. Investigate for which positive integers n, $3^n > 10n^2$, and then prove your result

19. Prove that for $n \geq 2$, $(1 + 1/3)^n > 1 + n/3$

20. (Bernulli's Inequality) Prove that for any fixed real number x, $-1 < x \neq 0$, and every integer $n \geq 2$,
$$(1 + x)^n > 1 + nx.$$
 This inequality is useful to provide a simple rough lower bound on the values of exponential functions. For example, what does it imply about the values of 1.0002^{100} or $.9995^{1000}$?

21. Let $x_1, x_2, \dots, x_n$ be n real numbers, $n \geq 2$. Prove that
$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

22. Let $x_1, x_2, \dots, x_n$ be n real numbers, $n \geq 2$. Prove that
$$|\sin(x_1 + x_2 + \cdots + x_n)| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_n|.$$

23. Determine whether the following functions are convex up, or down, or neither on the given intervals. In all parts, using the second derivative test provides a faster solution.
(a) $f(x) = \cos x$ on $[0, \pi/2]$.
(b) $f(x) = \cos x$ on $[0, \pi]$.
(c) © $f(x) = \tan x$ on $[0, \pi/2)$.
(d) $f(x) = \frac{x}{1-x}$ on $(1, \infty)$.

(e) ⓒ $f(x) = \sum_{k=1}^{100} \binom{n}{k} x^k$ on $(-1, \infty)$.
(f) ⓒ $f(x) = x \ln x$ on $(0, \infty)$.

24. Verify that the function $f(x) = x^3$ is convex up on $(0, \infty)$. Apply Jensen's inequality to function f and its values at three positive real numbers a, b, c. What inequality do you obtain?

25. Verify that the function $f(x) = \sin x$ is convex up on $[0, \pi]$. Apply Jensen's inequality to function f and its values at α, β, γ from $[0, \pi]$. What inequality do you obtain?

26. Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be two functions convex up on (a, b).
(i) Is $f + g$ be convex up on (a, b)?
(ii) ⓒ Is $f \cdot g$ convex up on (a, b)?

27. Let $f : (a, b) \to (0, \infty)$ be a convex up function on (a, b).
(i) Is $1/f$ convex up on (a, b)?
(ii) Is $1/f$ convex down on (a, b)?

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28. Prove that for $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > 1/2.$$

This inequality leads to an easy proof of the famous fact that the sum $1/1 + 1/2 + 1/3 + \ldots + 1/n$ can exceed any fixed number provided n being sufficiently large.

29. Prove that for every positive integer n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

This inequality implies that the sum $1/1^2 + 1/2^2 + 1/3^2 + \cdots + 1/n^2$ cannot exceed 2 no matter how large n is. It was proven by L. Euler that as n becomes larger, this sum becomes closer and closer to $\pi^2/6$.

30. Prove that for every positive integer n,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

31. Prove that for $n \geq 2$,

$$\frac{1}{2\sqrt{n}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}.$$

32. Prove that for every two non-negative real numbers a and b, and every integer $n \geq 2$,

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2},$$

where the equality is attained if and only if $a = b$.

33. Use the AGM inequality. Let the sum of positive numbers x, y, z be 60. Use the AGM inequality to determine the maximum value of
 (i) $(x-3)(y+1)(z+5)$
 (ii) $(x-3)(2y+1)(3z+5)$.

34. Consider a sequence $\{a_n\}_{n\geq 1}$, where $a_1 = 2$ and $a_{n+1} = \sqrt{2a_n+5}$ for $n \geq 1$.
 (i) Prove that $a_{n+1} > a_n$ for all $n \geq 1$.
 (ii) Prove that $a_n < 4$ for all $n \geq 1$.

35. Consider a sequence $\{b_n\}_{n\geq 1}$, where $b_1 = 2$ and $b_{n+1} = \sqrt{b_n + 2}$ for $n \geq 1$.
 (i) Prove that $b_{n+1} < b_n$ for all $n \geq 1$.
 (ii) Prove that $b_n > 2$ for all $n \geq 1$.

36. Prove that
 (i) $\sin t < t$ for all $t > 0$;
 (ii) $\tan t > t$ for $t \in (0, \pi/2)$.

37. ⓒ (i) Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function such that $f(a) \geq 0$ and $f'(x) > 0$ for all $x > 0$. Prove that $f(x) > 0$ for all $x > 0$.

 (ii) Let $f, g : [a, \infty) \to \mathbb{R}$ be continuous functions such that $f(a) \geq g(a)$ and $f'(x) > g'(x)$ for all $x > 0$. Prove that $f(x) > g(x)$ for all $x > 0$.

38. ⓒ Prove that
 (a) $\cos x > 1 - x^2/2$ for all $x \in \mathbb{R}$.
 (b) $\sin x > x - x^3/6$, for all $x > 0$.
 (c) $\cos x \geq 1 - \frac{2x}{\pi}$ on $[0, \pi/2]$.
 (d) $e^x > 1 + x + x^2$ for all $x \in \mathbb{R}$.
 (e) $x - x^2/2 < \ln(1 + x) < x$ for all $x > 0$.

39. ⓒ Prove that for for all $0 < x < y < \pi/2$,
 (a) $\frac{x}{\sin x} < \frac{y}{\sin y}$, $\quad x - \tan x > y - \tan y$.

40. ⓒ The Mean Value Theorem states that for any function continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$, such that $f(b) - f(a) = f'(c)(b - a)$. Use this theorem to show that
 (i) $|\sin x - \sin y| \leq |x - y|$ for any $x, y \in \mathbb{R}$.
 (ii) $|\tan x - \tan y| \leq |x - y|$ for any $x, y \in (0, \pi/2)$.

41. What is the greatest value of $\sin A + \sin B + \sin C$, where A, B, C are measures of three angles of a $\triangle ABC$?

42. ⓒ Use Jensen's inequality to prove that for any a, b, c on $(0, \infty)$,

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

43. Use Jensen's inequality to prove that for any $n \geq 2$ real numbers $x_1, x_2, \ldots, x_n$ on $(1, \infty)$,

$$n \cdot \frac{\sum_{i=1}^{n} x_i}{n - \sum_{i=1}^{n} x_i} \geq \sum_{i=1}^{n} \frac{x_i}{1 - x_i}.$$

44. ⓒ Show that if $f''(x) > 0$ ($f''(x) < 0$) for all $x \in (a, b)$, then f is convex up (down) on (a, b).

45. ⓒ What is greater:
(i) $\sqrt[78]{78}$ or $\sqrt[79]{79}$? (ii) e^{π} or π^{e}?

46. ⓒ The goal of this problem is to show that the function $y = f(x) = (1+\frac{1}{x})^x$ is increasing on $I = [1, \infty)$. This result can be used to justify the fact that when the interest in the bank is compounded more times per year, the better this is for a customer.

(a) Find $y'(x)$. Is it obvious that $y'(x) > 0$ for all x in I?
(b) Prove that $\ln(1 + t) > t - \frac{t^2}{2}$ for any $t > 0$.
(c) Prove that $\frac{1}{x} - \frac{1}{2x^2} > \frac{1}{x+1}$ for $x > 1$.
(d) Prove that $\ln(1 + \frac{1}{x}) > \frac{1}{x+1}$ for $x > 1$.
(e) Conclude that f is increasing on I.

Compare this problem with Example 14, where a similar statement had to be proved for values of the sequence $f(n)$, $n = 1, 2, 3, \ldots$.

47. Let $a_1 < a_2 < \ldots < a_n$ be fixed numbers, and

$$f(x) = |x - a_1| + |x - a_2| + \ldots + |x - a_n|.$$

Find $\min_{x\in\mathbb{R}} f$ (the minimum value of f over all reals), and all value(s) of x for which this minimum is attained. What about $\max_{x\in\mathbb{R}} f$?

48. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a(a - 1) + b(b - 1) + c(c - 1) \geq 0.$$

49. Prove that out of all n-gons, $n \geq 3$, inscribed in a circle, the regular one has the greatest area. You can assume without proof that the center of the circle is inside the n-gon.

50. Prove that for all $n \geq 2$, $2 < (1 + \frac{1}{n})^n < 3$.

51. Represent 1000 as a sum of several positive integers which product is the greatest. (The number of integers is not given.)

8. Hints and Answers to Problems

These are NOT complete solutions. Just some answers and hints.

1. In order to show that the statement "for all $a, b, c, d, \ldots$ $A \to B$" is false, one can show that there exist values of $a, b, c, d, \ldots$ such that A is true and B is false.
   (a) False.
   (b) False. True if all parts of all inequalities are positive.
   (c) False.True if $\ldots$ $0 \neq a < 1$
   (d) True.
   (e) False.
   (f) False. A counterexample:
   (g) False. True if $\ldots$
   (h) False. True if $\ldots$
   (i) True.
   (j) True.
   (k) False: take $a = 1$ and $b = -2$. True if $\ldots$
   (l) True.
   (m) True.
   (n) True.
2. (a) Hint: $x^5 - x^2 = x^2(x^3 - 1)$.
   (b) Hint: consider each part of the inequality separately.
   (c) Hint: use idea of part (a).
3. (a) Hint: Raise both numbers we compare to the nth power for some special values of n.
   (b) Hint: Begin by squaring both numbers.
   (c) Hint: Begin by squaring both numbers.
   (d) Hint: First subtract $\sqrt{8}$ from both numbers.
4. (i) Hint: use laws of exponents.
   (ii) Answer: yes.
5. Answer: $t_1 < t_2$.
6. (a) Answers: (i) $(-\infty, \frac{3-\sqrt{29}}{2}) \cup (\frac{3+\sqrt{29}}{2}, \infty)$; (ii) $\mathbb{R}$; (iii) $\emptyset$.
   (b) Answers: (i) $(-5, -4) \cup (2, 7)$; (ii) $[-5, 2) \cup [5, \infty)$.
   (c) Answers: (i) $(-\infty, -0.5] \cup [4.5, \infty)$; (ii) $[-1, 2.5]$; (iii) $(-\infty, 2 - \sqrt{10}] \ \cup \ [2 + \sqrt{10}, \infty)$.
   (d) Answers: (i) $(-\infty, -7) \cup (13, \infty)$; (ii) $\emptyset$.
   (e) Answers: (i) $(-5, -2) \cup (7, \infty)$; (ii) $(0.25, 1) \cup (1, 2.5)$.
   (f) Answer: (1.5, 2]
   (g) Answer: $((-3 + \sqrt{13})/2, 2]$
7. Straightforward.
8. (a) Hint: $2x^3 = x^3 + x^3$
   (b) hint: Consider three cases: $x \leq 0$, $0 < x < 1$, and $x \geq 1$.

(c) Factor the polynomial $f(x) = 2x^4 - 2x^3 - x^2 + 1 \geq 0$. Or use Calculus to find $\min_{x \in \mathbb{R}} f(x)$.

9. Hint: use the Pythagoras theorem in the form $(a/c)^2 + (b/c)^2 = 1$.
10. Answer: $\{0, \pi/2, \pi, 3\pi/2, 2\pi\}$.
11. Hint: Replace each number with m to get the first inequality.
12. (a) Very simple.
    (b) Use the AGM inequality.
    (c) Hint: complete the square with respect to each variable, or apply the obvious inequality $(a - b)^2 \geq 0$ three times. Another approach is to "symmetries": write $x = 1/3 + \alpha$, $y = 1/3 + \beta$, $z = 1/3 + \gamma$. The third approach is to find the distance from the origin to the plane $x + y + x = 1$.
    (d) Hint: several different ideas will work. For example, one can use the inequality from part (c).
    (e) Easy.
    (f) Hint: $a_i + 1 \geq 2\sqrt{a_i}$. The equality takes place if and only if $a_i = 1$.
13. Use the AGM inequality.
14. Use the AGM inequality.
15. Use the Method of Mathematical Induction.
16. Use the Method of Mathematical Induction. The usual logic of showing $A > C$ given $A > B$ and proving $B \geq C$, will work here beginning with $n \geq 4$, though the inequality holds for $n = 0, 1, 2, 3$ as well. This illustrates the *sufficiency* of showing $B \geq C$, rather than its *necessity*.
17. Substitute several first values of n: $n = 1, 2, \ldots$. Use the Method of Mathematical Induction.
18. Substitute several first values of n: $n = 1, 2, \ldots$. Use the Method of Mathematical Induction.
19. Use the Binomial formula, or use the Method of Mathematical Induction.
20. Use the Method of Mathematical Induction.
21. Use the Method of Mathematical Induction. The most important is the base case: $n = 2$.
22. Use the formula for the sine of the sum of two angles, and the the Method of Mathematical Induction. The most important is the base case: $n = 2$.
23. (a) Answer: convex down. Use trigonometric formuli.
    (b) Answer: not convex.
    (c) Answer: convex up. Use Calculus.
    (d) Answer: convex down.
    (e) Answer: convex up. Use the Binomial Theorem.
    (f) Answer: convex up. Use Calculus.
24. Hint: Easy.
25. Hint: similar to Problem 23 (a).

26. (i) Answer: yes. (ii) Answer: no. Find a counterexample.
27. Let $f : (a, b) \to (0, \infty)$ be a convex up function on (a, b). (i) Answer: no. Find a counterexample. (ii) Answer: no. Find a counterexample.

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28. Compare each term with 1/2. Or use the Method of Mathematical Induction.
29. Hint: Check that $\frac{1}{k^2} < \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}$, and use this fact. Or use the Method of Mathematical Induction.
30. Use the Method of Mathematical Induction.
31. Use the Method of Mathematical Induction.
32. Use the Method of Mathematical Induction. Consider two cases: $a \le b$ and $a > b$. Another approach is to use the second derivative test for the function $y = x^n$ on $[0, \infty)$.
33. Use the AGM inequality. Hints: (i) Use the AGM inequality. (ii) $2y + 1 = 2(y + 1/2)$, $3z + 5 = 3(z + 5/3)$.
34. Hints: (i), (ii) Use the Method of Mathematical Induction.
35. Hints: (i), (ii) Use the Method of Mathematical Induction.
36. (i) Let $0 < t \le 1$, and let $P_t : (\cos t, \sin t)$ be the point on the unit circle $x^2 + y^2 = 1$ corresponding to the angle t radians. Let O be the origin. Compare the area of $\triangle OP_tP_0$ with the area of the sector OP_tP_0. For $t > 1$, the statement is obvious.

    (ii) Let $0 < t < \pi/2$, and let $P_t : (\cos t, \sin t)$ be the point on the unit circle $x^2 + y^2 = 1$ corresponding to the angle t radians. Let O be the origin, and let M be the point of intersection of the line OP_t with the line $x = 1$. Compare the area of the sector OP_tP_0 with the area of the triangle OMP_0.

37. (i) Hint: Use the Mean Value Theorem.

    (ii) Hint: let $h(x) = f(x) - g(x)$. Apply part (i) to the function h.
38. Hint: use the statement from Problem 37.
39. No other hints.
40. No other hints.
41. Use Jensen's inequality.
42. Hint: consider the function $y = x \ln x$ on $(0, \infty)$.
43. Hint: consider the function $y = \frac{x}{1-x}$ on $(1, \infty)$.

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44. Hint: a proof can be found in most Calculus books.
45. Hint: Consider the logarithms of the numbers. Use Calculus.
46. Use the statement from Problem 37.
47. Hint: Think about $|a - b|$ geometrically. Experiment with $n = 2, 3, 4$.

48. Use AGM and Jensen's inequalities.
49. Hint: Use Jensen's inequality.
50. Hint: The first inequality is easy. For the second one, use the Binomial formula, and bound from above the sum of all terms starting from the third by $1/2 + 1/2^2 + \ldots + 1/2^n$. Then show that this sum is less than 1.
51. Hint: first experiment with a similar question, when 1000 is replaced by 9, or 10, or 11, or 12. Use the fact that for an integer $a \geq 5$, $2 \cdot (a-2) > a$, and that $3^2 > 2^3$.

9. Acknowledgement

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These notes represent a very brief introduction to the vast subject of inequalities. Much more can be found in the references below.

References

[1] E. Beckenbach, R. Bellman, An Introduction to Inequalities, Random House, 1961.

(A great introduction to inequalities. Covers more than these notes. Can be easily understood by good high school students. No use of Calculus.)

[2] E. Beckenbach, R. Bellman, Inequalities, Springer-Verlag, 1961.

(An advanced version of [1] *for college students. Begins with twelve proofs of the AGM.)*

[3] G. H. Hardy, J. E. Littlewood, and G. Plya, Inequalities, Cambridge University Press; 2 edition 1988.

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A nice collection of various examples and techniques. More advanced than these notes. Based on notes for Math Olympiad Program (MOP), 1999.

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[6] D. O. Shklarsky, N. N. Chentzov, and I. M. Yaglom , Geometric Inquality and Problems on Maximum and Minimum, Nauka, 1970. (In Russian).

(A great discussion of inequalities related to geometry. Can be easily understood by good high school students. No use of Calculus.)

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(An outstanding collection of fifteen "stories" related to maxima and minima problems. A very useful reading for everyone who likes mathematics and who knows Calculus and a little beyond it.)