ON QUADRATIC POLYNOMIALS.

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This handout is intended to help the students to review some mathematics related to quadratic polynomials and quadratic functions. I assume that most of it has been seen in high school Algebra (or Precalculus) courses. Otherwise the discussions would be at much slower pace.

Our goal is to emphasize the proofs of the facts and logical relations among them.

I do not assume that the reader knows Calculus. Moreover, if you do know Calculus, you may use it to check some of your answers, but you must be able to solve all problems without using Calculus.

1. Completing the square.

Theorem 1.1. *If $a \neq 0$, then*

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$$

Proof. Though the simplest proof will proceed by simplifying the expression on the right hand side, we will prove it by transforming the left hand side (l.h.s.) to the r.h.s..

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a\big(x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}\big) =$$

$$a\big((x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}\big) = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a} \qquad \blacksquare$$

The process of rewriting the quadratic trinomial $ax^2 + bx + c$ in the form

$$a(x + b/2a)^2 + (4ac - b^2)/(4a),$$

is called the **completion of the square** in $ax^2 + bx + c$.

2. Minimum and maximum values of quadratic functions.

We assume that the domain of all quadratic functions, i.e., the ones given by $f(x) = ax^2+bx+c$, a, b, c are fixed real numbers, $a \neq 0$, is $\mathbf{R}$ - the set of all real numbers. A very important application of the completion of the square transformation is that it allows to find the smallest (minimum) or the largest (maximum) value of a quadratic function and the value of x where this value is attained.

For example, consider $f(x) = 2x^2 - 7 = 2x^2 + (-7)$. Since $2x^2 \geq 0$ for all real numbers x, we conclude that $f(x) \geq -7$ for all real x. Therefore the minimum value of $f(x)$ is -7 and it is achieved if and only if $x = 0$. We can write this fact as

$$\min f(x) = f(0) = -7.$$

Since $f(x)$ can take arbitrarily large values (just think x being $10, 100, 1000, \ldots$), we conclude that the maximum value of $f(x)$ does not exist.

Suppose now that $f(x) = -2(x - 3)^2 + 1$. Similarly we can conclude that

$$\max f(x) = f(3) = 1,$$

and that the minimum value of $f(x)$ does not exist.

In the examples above, the quadratic function was given in the "completed square form", and this made the finding of its minimum or maximum easy. In case when the quadratic function is given in the form $f(x) = ax^2 + bx + c$, in order to determine its minimum or maximum value we first complete the square, and then the answer becomes obvious. For example, let $f(x) = 3x^2 + 12x + 4$. Completing the square we get $f(x) = 3x^2 + 12x + 4 = 3(x + 2)^2 + (-8)$. Therefore $\min f(x) = f(-2) = -8$, and $\max f(x)$ does not exist.

3. Quadratic equations.

Another most useful applications of the completion of the square transformation is that it allows to derive a formula for solving quadratic equations or show that no solutions (among real numbers) exist. This makes obsolete the factoring trial-and-error technique, which is often emphasized in high schools, and which can be used only for very special values of the coefficients a, b, c.

Theorem 3.1. *If $a \neq 0$ and $b^2 - 4ac \geq 0$, then*

$$ax^2 + bx + c = 0 \iff x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

If $a \neq 0$ and $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions among real numbers.

Proof. Indeed,

$$ax^2 + bx + c = 0 \iff a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a} = 0 \iff$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \iff x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \iff$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

■

Those values of x which being substituted into $ax^2 + bx + c$ result zero, are often called the **roots** of the quadratic trinomial $ax^2 + bx + c$, or the roots (or solutions) of the quadratic equation $ax^2 + bx + c = 0$. Hence the two numbers $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ are the roots of $ax^2 + bx + c$. They are often denoted by using indices as x_1 and x_2. If $x_1 = x_2$, we will say that the equation still has two solutions (which are equal).

4. Viète's Theorem.

It turns out that the sum and the product of the two roots of a quadratic equation can be computed without actual evaluation of the roots themselves.

Theorem 4.1. *Let x_1 and x_2 be two roots a the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. Then $x_1 + x_2 = -b/a$ and $x_1 x_2 = c/a$.*

Proof. Substituting for x_1 and x_2 their values from (1), we get:

$$x_1 + x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

Similarly,

$$x_1 x_2 = \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{(2a)^2} =$$

$$\frac{(-b)^2-(b^2-4ac)}{(2a)^2} = \frac{c}{a}.$$

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This statement is often referred to as the Viète's Theorem, in honor to a French mathematician F. Viète (1540-1603).

Example. Let x_1 and x_2 be the roots of the quadratic polynomial $-3x^2+19x-10$. Then, according to Viète's Theorem, $x_1+x_2 = -\frac{19}{2\cdot(-3)} = 19/6$, and $x_1 \cdot x_2 = \frac{-10}{(-3)} = 10/3$.

Viète's Theorem can be used to compute values of some other expressions depending on x_1, x_2 without finding the values of x_1, x_2.

Example. Let x_1 and x_2 be the roots of the quadratic polynomial $f(x) = 2x^2-7x-10$. Compute $x_1x_2^2+x_1^2x_2+x_1x_2$.

Solution. By Viète's Theorem, $x_1+x_2 = -(-7/2) = 7/2$ and $x_1x_2 = (-10)/2 = -5$. Now we transform the given expression and substitute the values of x_1+x_2 and x_1x_2. We get

$$x_1x_2^2+x_1^2x_2+x_1x_2 = x_1x_2(x_2+x_1+1) = (-5)(7/2+1) = -45/2 = -22.5$$

5. Factoring quadratic polynomials.

Formula (1) makes the factoring of quadratic polynomials an easy problem.

Theorem 5.1. *Let x_1 and x_2 be two roots a the quadratic equation $ax^2+bx+c=0$, $a \neq 0$. Then $ax^2+bx+c = a(x-x_1)(x-x_2)$.*

Proof. Expanding the right hand side we get:

$$a(x-x_1)(x-x_2) = a(x^2-(x_1+x_2)x+x_1x_2).$$

Using Viète's Theorem we rewrite the latter as

$$a(x^2-(x_1+x_2)x+x_1x_2) = a(x^2-(-b/a)x+c/a)) = ax^2+bx+c. \quad ■$$

Example. Finding roots of $3x^2+x-1$, we get $x_1 = \frac{-1-\sqrt{13}}{6}$ and $x_2 = \frac{-1+\sqrt{13}}{6}$. Therefore $3x^2+x-1 = 3(x-\frac{-1-\sqrt{13}}{6})(x-\frac{-1+\sqrt{13}}{6})$.

The following theorem simply says that if we know two linear factors of ax^2+bx+c, then we know its roots.

Theorem 5.2. *If for some numbers s and t, $ax^2+bx+c = a(x-s)(x-t)$, then s and t are roots of the quadratic equation $ax^2+bx+c=0$.*

Proof. Substituting $x=s$ in both sides of the equality $ax^2+bx+c = a(x-s)(x-t)$, we get $as^2+bs+c = a(s-s)(s-t) = a\cdot 0\cdot(s-t) = 0$. Therefore $as^2+bs+c=0$, which means that s is a root of ax^2+bx+c. Similarly we show that t is also a root. ■

Example. Since $2x^2 - 4x - 6 = 2(x-3)(x+1)$, then $x = 3$ and $x = -1$ are the roots of $2x^2 - 4x - 6$.

6. Parabola.

The curve which is the graph of a quadratic function $y = f(x) = ax^2 + bx + c$ $(a \neq 0)$ in a Cartesian coordinate system XOY is called a *parabola*.

Problems.

In problems asking for numerical or symbolic answer, you have to prove (same as 'explain', same as 'justify') your method. Sometimes we remind about this in the problem's statement, sometimes we do not. Often you can just refer to the main facts proved in this handout. The statement of the form 'If A, then B' means the same as 'Given A, prove B'. Proof of the statement of the form 'A if and only if B' usually amounts to two proofs: 'If A, then B' and 'If B, then A'.

1. For each of the following quadratic polynomials (a) complete the square; (b) find the minimum or the maximum values of the corresponding functions and the value of x for which it is attained; (c) find the roots (i.e. solve the corresponding quadratic equation); (d) sketch the corresponding parabola.

(i) x^2

(ii) $x^2 + 6x + 9$

(iii) $x^2 + 6x$

(iv) $-2x^2 - 5$

(v) $-2x^2 + 5$

(vi) $x^2 - 2x + 3$

(vii) $3x^2 + 4x - 1$

(viii) $-2x^2 + 4x - 1$

2. Prove that a quadratic function $f(x) = ax^2 + bx + c$, with $a > 0$ $(a < 0)$, (with the domain $\mathbf{R}$ - the set of all real numbers) always attains the minimum (maximum) value exactly for one value of x, and does not attain maximum (minimum) value.

What is this value of x?

3. (i) Among all pairs of two real numbers whose sum is 10, find the pair with the greatest product. Justify. What is the maximum value of the product? What can be said about the minimum value of the product?

Generalize, by replacing 10 with an arbitrary positive number s.

(ii) Among all rectangles with perimeter 20 find the one of the greatest area. What is the maximum area?

Generalize, by replacing 20 with an arbitrary positive number p.

4. Find all pairs of real numbers (x, y) such that $x^2 + y^2 + 6x = 10y - 34$.

5. Let x_1 and x_2 be the roots of the quadratic polynomial $f(x) = 5x^2 - 20x - 1$. Compute

(i) $2(x_1 + x_2)$

(ii) $(x_1 x_2)^3$

(iii) $x_1^2 x_2^2 + 5 - x_1 - x_2$

(iv) $1/x_1 + 1/x_2$

(v) $x_1^2 + x_2^2$

(vi) $1/x_1^3 + 1/x_2^3$.

6. Factor the following quadratic polynomials into the product of the first degree polynomials with real coefficients and with the coefficients at x equal to 1, or show that no such factorization exists.

$x^2 - 2x + 3$, $3x^2 + 4x - 1$, $-2x^2 + 4x - 1$, $x^2 + 6x$, $x^2 + 6x + 9$, x^2, $-2x^2 + 5$, $-2x^2 - 5$.

7. Find the coordinates of the points of intersections (or show that they do not exist) of the line $y = 2x + 1$ and the following parabolas. You do not have to draw graphs to solve this problem, but it is instructive to draw them after the solution is obtained in order to see what happens "geometrically".

(i) $y = x^2 + 4x - 3$

(ii) $y = x^2 + 4x + 10$

(iii) $y = x^2 + 4x + 2$

8. Prove that a parabola and a line can intersect only at 0, or 1, or 2 points.

(Often, when a parabola and a nonvertical line intersect at 1 point, we say that they intersect at two equal points, or that the line is *tangent* to the parabola.)

9. Find an equation of a parabola (or show that no parabola exist) which passes through the points

(i) $(1, 1), (-2, 0)$ and $(3, 5)$

(ii) $(1, 1), (1, -10)$ and $(3, 5)$

(iii) $(0, 3)$, and has its vertex at $(-2, -7)$;

(iv) $(2, 5)$, $(-1, 7)$, and has its vertex at $(1, 4)$.

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10. For which values of r, the equation $x^2 + 4rx + (5 + 4r) = 0$ has

(i) two distinct real roots? (ii) two equal real roots?

11. For which values of r, the equation $x^2 + (4 + 2r)x + (5 + 4r) = 0$ has

(i) equal roots? (ii) opposite roots? (i.e., with equal absolute values, but different signs.)

12. Prove that for every three non-collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3), (i.e., the points are not on a line) and with pairwise distinct x-coordinates, there exists exactly one parabola passing through them.

Will the conclusion hold if we allow the three points to be collinear? Will the conclusion hold if we allow two of the points lie on a vertical line?

13. For which values of a, the line $y = 3x + a$ and the parabola $y = x^2 + x$ have (i) two common points? (ii) one common point? (iii) no common points?

14. Prove that the graph of the parabola $y = f(x) = ax^2 + bx + c$ is symmetric with respect to the vertical line $x = -b/2a$.

The line $x = -b/2a$ is called the *axis* of the parabola.

(*Hint:* First realize that it amounts showing that the points of the graph whose x-coordinates are symmetric with respect to the number $-b/2a$ have equal y-coordinates. Symbolically this means that for every number h, $f(-b/2a - h) = f(-b/2a + h)$. Therefore we have to check the latter.)

15. Prove that for all real numbers x, y, the polynomial $x^2 + xy + y^2 \geq 0$ and that it takes value zero if and only if $x = y = 0$.

16. (i) If $2a + 3b = 12$, what is the greatest value of ab? Justify.

(ii) If $ab = 12$, what is the smallest value of $a^2 + b^2$? Justify.

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17. Find all value of a for which the polynomials $x^2 + ax + 1$ and $x^2 + x + a$ have at least one common (real) root.

18. Prove that for all real numbers x, y, z, $x^2 + y^2 + z^2 \geq xy + yz + zx$ and the equality sign is attained if and only if $x = y = z$.

19. Prove that for all real numbers x, y, z, $x + y + z = 1$ implies that $x^2 + y^2 + z^2 \geq 1/3$, and the equality sign is attained if and only if $x = y = z$.

20. Prove that for all real numbers x, y, z, $x + y + z \geq 0$ implies $x^3 + y^3 + z^3 \geq 3xyz$. Then show that $x^3 + y^3 + z^3 = 3xyz$ if and only if $x = y = z$ or $x + y + z = 0$.

21. Among all rectangles with area $a > 0$, find the one of the smallest perimeter (i.e., determine the lengths of its sides).

22. A stone moves in such a way that its height after t seconds of its motion is given by the formula $h(t) = -16t^2 + 96t + 256$ feet. How high will the stone go? At what moment(s) of time will it be 144 feet high? When does it hit the ground?

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Optional Problems.

In all problems below, the term "parabola" is understood in a more general way. In Problems **OP 1** and **OP 2**, "parabola" means a curve which is the graph of an equation $y = ax^2 + bx + c$, $a \neq 0$, in SOME Cartesian coordinate system OXY. The coordinate-free definition of a parabola is given in Problem **OP 3**.

OP 1. Given two parabolas with perpendicular axis which intersect at 4 points. Prove the four points lie on a circle.

OP 2. Prove that one cannot cover the whole plane with finitely many parabolas with their interiors.

OP 3. In this handout the term 'parabola' is used for the graph of a quadratic function $y = f(x) = ax^2 + bx + c$, where a, b, c are real numbers, $a \neq 0$, in some Cartesian coordinate system. Refer to this definition as Definition 1.

A more traditional definition of parabola is the following. Let l and F be a line and a point in a plane, F is not on l. Consider the set S all points in the plane which are at the same distance from l and F. This set is called *parabola* with focus F and directrix l. Refer to this definition as Definition 2.

The two definitions are equivalent and one can show this in the following two steps.

(i) Let parabola P be defined by using Definition 2. Prove that one can choose a Cartesian coordinate system in the plane such that the equation of P in this coordinate system is $y = ax^2 + bx + c$, where a, b, c are some (fixed) real numbers, $a \neq 0$.

(*Hint:* choose y-axis to be the line passing through F and perpendicular to l.)

(ii) Let parabola P be defined by using Definition 1, and $y = ax^2 + bx + c$, where a, b, c are real numbers, $a \neq 0$, be the equation. Prove that one can find a point F and a line l in the plane, such that P is the set of all points in the plane equidistant from F and l.

Hints and/or Answers.

1. To check the answers, one may use calculators or any software.

(a) this can be checked by simplifying the obtained expression back to the polynomial you started with.

(b) follows from part (a);

(c) one can always use formula (1), especially when it is not too easy.

(d) this part should be done after (a),(b), (c). Your graphs should reflect the answers obtained there.

2. Complete the square. The minimum (maximum) is attained at $x = -b/(2a)$.

3. (i) *Hint:* let x be the first number...

The maximum value of the product is 25. It is attained when both numbers are equal to 5. No minimum value exists.

When 10 is replaced with an arbitrary positive number s, the maximum is $s^2/4$. It is attained when both numbers are equal to $s/2$. No minimum value exists.

(ii) Similar to (i). The maximum area is $p^2/16$.

4. *Hint:* Move all terms to the left hand side. Then complete the square with respect to both x and y.

Answer: $(-3, 5)$.

5. (i) 8; (ii) $-1/125$; (iii) $26/25$; (iv) -20; (v) 16.4; (vi) -8300. *Hint:* Use the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and part (v). (Do you know how to prove this formula?)

6. You can check that your factoring is correct by expanding the obtained product into polynomial form. If $b^2 - 4ac < 0$, no real roots exist, and hence no such factoring exists.

7. *Hint:* Just solve the system of two equations.

(i) $(-1 - \sqrt{5}, -1 - 2\sqrt{5})$ and $(-1 + \sqrt{5}, -1 + 2\sqrt{5})$. (ii) The line and the parabola do not intersect. (iii) $(-1, -1)$. The line is tangent to the parabola.

8. *Hint:* Just repeat what you did in the previous problem for a general parabola $y = ax^2 + bx + c$ and a general nonvertical line $y = mx + n$. Then consider a vertical line $x = d$.

9. *Hint:* Let the equation of the parabola be $y = ax^2 + bx + c$. Substitute for x and y the values of the x and y coordinates of each point. We get a system of three equation with respect to a, b, c. Solve the system by any method you wish.

Answers: (i) $y = (1/3)x^2 + (2/3)x$; (ii) does not exist; (iii) $(5/2)x^2 + 10x + 3$; (iv) does not exist.

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10. (i) $r < (1 - \sqrt{6})/2$ or $r > (1 + \sqrt{6})/2$; (ii) $r = (1 - \sqrt{6})/2$ or $r = (1 + \sqrt{6})/2$.

11. *Hint:* Find x_1, x_2 in terms of r. Solve the equations $x_1 = x_2$ and $x_1 = -x_2$ with respect to r.

Answers: (i) $r = -1$ or $r = 1$; (ii) $r = -2$.

12. *Hint:* Repeat the argument suggested for problem 9, using the general coordinates of the three points. The answers to the both questions are ‘No’.

13. (i) $a > -1$; (ii) $a = -1$; (iii) $a < -1$;

14. Follow the Hint.

15. *Hint:* Complete the square with respect to x. Another approach is to begin by using the quadratic formula with respect to x: $a = 1, b = y, c = y^2$.

16. (i) 6; (ii) 24. *Hint:* one may start with $(a-b)^2 \geq 0$. Or one may rewrite $a^2+b^2 = a^2+144/a^2 = (a-12/a)^2+24$.

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17. $a = -2$. If we also considered complex roots, then $a = 1$ would be another such value of a..

18. *Hint:* Think about using the inequality $(a-b)^2 \geq 0$ three times.

19. *Hint:* One can use Problem 18. Another approach: $x + y + z = 1$ implies that the are three numbers a, b, c such that $x = 1/3 + a$, $y = 1/3 + b$, and $z = 1/3 + c$, where $a + b + c = 0$ (just take $a = x - 1/3, b = y - 1/3, c = z - 1/3$). Continue from here.

20. *Hint:* first try to factor $x^3 + y^3 + z^3 - 3xyz$ by using $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and then use Problem 18.

21. Use the hint for Problem 16 (ii).

22. *Answers:* 400 feet; after 7 seconds; after 8 seconds.