On a Regular Simplex in $\mathbb{R}^n$

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We begin with stating and proving the following theorem.

Theorem 1. *For $n \geq 1$, there exist $n+1$ points in $\mathbb{R}^n$ with all pairwise distances 1.*

Proof. Induction on n. For $n = 1, 2, 3$ – the endpoints of a unit segment, a regular triangle and a regular tetrahedron provide the desired set of points.

Suppose we know that the theorem is correct in $\mathbb{R}^{n-1}$, $n \geq 2$, and we want to prove it in $\mathbb{R}^n$.

Consider a hyperplane H in $\mathbb{R}^n$, and identify it with $\mathbb{R}^{n-1}$. By induction hypothesis, there is a set $S = \{a_1, \ldots, a_n\}$ of points (vectors) in H with all pairwise distances 1. Therefore $\|a_i - a_j\| = 1$. Then $c = \frac{1}{n}\sum_{i=1}^n a_i$ is the centroid of S. Let $b_i = a_i - c$. Then

$$\sum_{i=1}^n b_i = 0, \text{ and } \|b_i - b_j\| = \|a_i - a_j\| = 1 \text{ for all } i \neq j.$$

This implies that $b_i^2 - 2b_ib_j + b_j^2 = 1^2 = 1$. Fix i in this equality, then sum the equalities over all $n-1$ values of j, and, finally, substitute $-b_i$ for $\sum_{j\neq i} b_j$. We get $(n-1)b_i^2 + 2b_i^2 + \sum_{j\neq i} b_j^2 = n-1$, hence

$$b_i^2 = \frac{n-1}{n} - \frac{1}{n}\sum_{i=1}^n b_i^2.$$

Therefore all b_i^2 are equal, and the common value of $\|b_i\|$ is $\sqrt{\frac{n-1}{2n}}$.

Let u be a unit vector orthogonal to H. Then there exists a value of $\lambda \in \mathbb{R}$ such that all $\|\lambda u - b_i\| = 1$. Indeed, $\lambda^2 - 2\lambda b_i u + b_i^2 = 1$. Using $b_i u = 0$ and $b_i^2 = \frac{n-1}{2n}$, and assuming $\lambda > 0$, we get $\lambda = \sqrt{\frac{n+1}{2n}}$. For this value of λ, let $a_{n+1} = c + \lambda u$. Then $(a_{n+1} - a_i)^2 = (b_i - \lambda u)^2 = 1$ for all $i \in [n+1]$. Hence $\{a_1, \ldots, a_n, a_{n+1}\}$ is the desired set of $n+1$ points in $\mathbb{R}^n$. $\square$

Let $\{a_1, \ldots, a_n, a_{n+1}\}$ be a set of points in $\mathbb{R}^n$ with all pairwise distances 1. It is easy to show that all such sets in $\mathbb{R}^n$ are congruent, and it has been essentially done in the proof above. It could be just added to the statement of the theorem. When a new point is added in the next dimension, it is defined uniquely up to the sign of λ.

The convex hull of such a set in $\mathbb{R}^n$, i.e. the set of all linear combinations $\sum_{i=1}^{n+1} \lambda_i a_i$, where all $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$ is called the **standard regular**

simplex in $\mathbb{R}^n$. If instead of distance 1 we assume all pairwise distances $d > 0$, we obtain a **regular simplex** .

Here are some question related to the notion of regular simplex. I enjoyed thinking about them. Hopefully you will too.

Problems.

1. Compute the volume of the standard regular unit simplex in $\mathbb{R}^n$ and its surface area.
2. Compute the smallest radius of a sphere in $\mathbb{R}^n$ which contains the standard regular simplex, and the largest radius of a sphere which is contained in it. What is the the limit of each of them when $n \to \infty$?
3. Prove that the greatest number of points in $\mathbb{R}^n$ with all pairwise distances d is $n + 1$ and they must be the vertices of a regular simplex.
4. Choose an orthonormal basis in $\mathbb{R}^n$ and find the coordinates of the vertices of the standard regular simplex in with respect to this basis.
5. Let $ABCD$ be an arbitrary tetrahedron in $\mathbb{R}^3$. Prove that any plane passing through the midpoints of the edges AB and CD

   (i) divides two other its edges in the same ratio;

   (ii) divides its volume in half.