SURPRISES

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Perhaps the most surprising thing about mathematics is that it is so surprising.
– E.C. Titchmarsh.

In great mathematics there is a very high degree of unexpectedness, ...
– G.H. Hardy.

My momma always said, "Life is like a box of chocolates. You never know what you're gonna get."
– Forrest Gump.[1]

In my study and teaching of mathematics, surprises have always played an important part. Here I want to share some of these experiences with you.[2] I feel a bit uneasy talking about them. Surprises are subjective, and others may not feel the same. It is like a famous joke about two people standing at the edge of a cliff. One looks around and says: "God... What a beauty!". Another looks, and looks, and looks, and then asks: "Where?". The first does not know what to say or do ... He just pushes the second off the cliff. Gently.

So, just in case, do not be too close to me while reading this.

My surprises are of various natures. Here I tried to organize them in four categories, marked by A, B, C, and D.

A. "How could I not know this for so long?!"

A good thing about these ones is that we can blame them on our teachers.

B. "Surprising, but not that much, after I look at them closer... Maybe, I could discover this myself ... "

These are the most numerous for me. Their frequency depends on how cool one is. It depends on how patient one is in getting to the "bottom of things", on one's knowledge of related subjects, and on one's self-confidence.

C. "Even after I see a proof, the fact is still mind boggling. I could not discover this... "

These depend on how comfortable one is leaving with mysteries, and on one being secure and honest with oneself.

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[1]From the movie *Forrest Gump*, 1994.

[2]This article developed from the notes of a lecture given by the author in April, 2007 to mathematics graduate students at the University of Delaware.

D. "How is it possible that this problem is not solved yet?"

These are numerous. Listing few of them, I tried to avoid famous unsolved problems (with few exceptions), and those problems I never thought about myself. I also picked ones which can be easily understood by most readers.

Let me now illustrate each category with examples and comments. Some of the examples are very elementary. I believe that the reader may enjoy spending a few minutes thinking about the problems. Solutions to most of the problems are omitted. Some of them can be easily found, and for others we provide references.

Here are several examples of A: "How could I not know this for so long?!"

1. *All parabolas are similar.*

   Comments. A figure A in a plane is called similar to a figure B in the plane, if there exists a positive number k and a bijection $f : A \to B$ such that $\text{dist}(f(x), f(y)) = k{\cdot}\text{dist}(x, y)$ for any $x, y \in A$. Every two segments are similar, every two circles are similar, every two equilateral triangles are similar. However, not every two ellipses are similar, and not every two hyperbolas are similar.

   So why are all parabolas are similar? Is it not true that the parabola with equation $y = x^2$ is wider than the one with equation $y = 2x^2$ and narrower than the one having equation $y = \frac{1}{2}x^2$? No! The second is just smaller, and the third is larger, but all three are of the same shape, i.e., are similar. Though stretching or shrinking along the y-axis alone does not transform a curve to a similar one, it does for parabolas.

   The property is obvious from the definition of a parabola as a locus. The coefficient k is just the ratio of two distances from the foci to the corresponding directrix. If one knows that any parabola can be represented in a certain Cartesian coordinate system as a graph of $y = ax^2$, then one can check this property by considering the transformation $(x, y) \to (kx, ky)$, which is clearly a similarity transformation. Then $y = ax^2 \Leftrightarrow (ky) = a(kx)^2$.

   This trivial discovery was made in 2007, when O. Byer, D.L. Smeltzer and the author were working on their book [12]. Later we learned that, of course, the property was known. It turns out that Johannes Kepler mentions it with great excitement in 1604! See [19].

2. *Suppose we have perfectly spherical earth, density is distributed spherically symmetrically, and a cannonball is moving without drag under the influence of the gravitational field. Then the trajectory is a conic section. What is it?*

   Comments. It is ... an ellipse, not a parabola, as we are often taught.

   I learned this from an article by L.M. Burko and R.H. Price [11]. It immediately made perfect sense to me. I had known for a long time that parabolic trajectory of a projectile is rare. However, for some strange reason, many texts stated that every thrown stone moved along a parabola... This also made a perfect sense given the fact that the shape of a very narrow ellipse is close to a parabola: the latter can be considered as an ellipse with one focus removed to infinity. It was also surprising to see in the article a quotation from Newton's "*Principia*", where he points out that Galileo's model, which assumes flat-earth-uniform gravitation, leads to a parabolic trajectory. But the central force model, which is used in astronomy, leads to an ellipse. I do not remember seeing this discussion in calculus texts...

3. *For a fixed position of Earth and the Moon, where do we have high tide, and where do we have low tide?*

Comments. Answer: Let the line passing through the centers of Earth and the Moon (both are balls) intersect Earth surface in points A and B. Then the high tide is close to these two points, and the low tide is close to points C and D, where segment CD is the diameter perpendicular to the diameter AB.

For some reason, I always thought that if A is farther than B from the center of the Moon, the tide at A is lower. This all can be proved by simple computations. See, A.I. Kitaigorodsky and L.D. Landau [21].

4. *When we apply the Principle of Mathematical Induction, we use a deductive reasoning.*

Comments. The deductive method is a passage from a general statement to a particular one. For example, applying the Pythagorean Theorem to a right triangle with legs 5 and 12, we get (deduce) that the hypothenuse is of length 13. Therefore, application of any theorem (including the Mathematical Induction Theorem (Principle)) to a specific statement of the form *"for all $n \geq n_0$ $P(n)$"* is a deduction.

5. *Characterize the set of all functions f which have continuous nth derivatives on an open interval $I \subset \mathbb{R}$ and satisfy the differential equation*

$$f^{(n)} + p_1 f^{(n-1)} + \dots p_{n-1} f' + p_n f = 0,$$

for $n \geq 1$ and some continuous functions $p_1, \dots, p_n$ on I.

Comments. I asked this question when I taught an undergraduate course on Differential Equations. There was an exercise in the book by E. Boyce and R.C. DiPrima [10] which asked to show that $f(x) = \sin(x^2)$ cannot be a solution of such an equation for $n = 2$. This can be easily seen from the theorem about the uniqueness of the solution of the initial value problem in this case. Indeed, the function which is identically zero on the interval is, obviously, a solution of this equation. As $f(0) = f'(0) = 0$, the function f would represent another solution with the same initial values. Contradiction.

Asking the question above was a natural thing for a person with background in algebra. It was surprising to me that this question was new to people who work with differential equations. And that it was not interesting to them.

I found the answer, but had difficulty proving it. It was my colleague David Bellamy who provided the first proof. The question appeared as a Problem in *Monthly* [5]. Several people submitted much simpler solutions than ours, see [6].

6. *Let $\mathbb{Z}$ denote the ring of integers. Find all solutions $x \in \mathbb{Z}^n$ of a system of linear equations $Ax = b$, where A is a given $m \times n$ matrix over $\mathbb{Z}$, and $b \in \mathbb{Z}^m$ is a given vector.*

Comments. The question appeared when I taught a graduate topics course on Asymptotic Design Theory. It was very surprising to me that having taught Algebra and Linear Algebra many times, and knowing about the structure of finitely generated modules over PIDs, I had not recognized the question. It turned out that my experience was not so unique: the related paper got accepted quickly in the *Mathematics Magazine*. It was exactly this question which led H.J.S. Smith to his normal form for matrices [26]. For more on this, see [22].

7. *Find all real values of a such that the sequence $\{a_n\}_{n\geq 0}$ defined by $a_0 = a$, and $a_{n+1} = a_n^2 - 2$ for $n \geq 0$, converges.*

Comments. I can think of at least three surprises related to this problem. The first surprise was the unusual story behind the question. I assigned it as one of many other homework problems on limits in a high school where I was working at the time (1977). I had no idea that it was a challenge. I just thought that it was a nice extention to a simple question: what is the limit of $\{a_n\}_{n\geq 0}$ if it converges? After my students could not solve the problem, I tried it for two days, with no success. There was something very unusual about the sequence... I began asking my colleagues, and, soon, Yurii Pilipenko saw the light, and we finished a proof quickly. Several years later, we submitted it as a problem to *Monthly*. See [23].

A surprising thing about this sequence is that if it converges, then it must stabilize: all terms, starting at an arbitrary term, must be equal to its limit, which, obviously, can take one of two values: -1 or 2 (depending on a). This allows one to find a, and the set of all such a's allows a simple description. I had not seen anything like this before I asked the question. As I understood later, this was my first exposure to an interesting dynamical system and repulsors. In 1985, Emil Grosswald pointed out to me that the set of values of a for which the sequence converges is uniformly dense on $[-1, 2]$.

The third surprise was when I saw solutions and extensive comments sent by readers of *Monthly*, see [24]. Eighty two people from twenty three countries! It turned out that this type of sequence had been studied long ago, since 1918 at least, and by many mathematicians. The corresponding area (which used to be called just analysis) is now called topological dynamics. Many references and generalizations were mentioned. Some people pointed out that it was a good example of how one can get a wrong answer by experimenting with computer.

8. *Given two polygonal regions in a plane of the same area, one can be dissected by straight lines into finitely many smaller polygons such that the other can be assembled from them.*

   Comments. Sometimes this statement is called the Wallace-Bolyai-Gerwien Theorem, and proofs were found independently by W. Wallace, F. Bolyai, and P. Gerwien, who published them in 1831, 1833, and 1835, respectively. The polygonal regions do not have to be convex, or have equal number of sides. I found it very surprising that many people I mentioned this result to did not know it. It is a little better known that in space (3-dimensional), the analog of the theorem does not hold: for example, one cannot dissect a cube by planes and assemble a regular tetrahedron from them. This follows from M. Dehn's solution of the Hilbert's third problem in 1900. For details, extensions, and a proof of the Wallace-Bolyai-Gerwien Theorem, see V.G. Boltyanskii [8], or [12]. The proof of Dehn's theorem was simplified many times, and the version in [8] is one of the most beautiful proofs I have ever understood. Still, when it comes to surprises, the affirmative result in the plane is far ahead (for me) of the negative one in space.

Let us now discuss several examples of B: " Surprising, but not that much, if I look closer... Maybe, I could discover this ... "

9. *Watermelon is 99% water. 1 ton of watermelons was shipped, and during the shipment some water evaporated. The watermelons arrived made up of 98% water. What was the weight of the shipment when it arrived?*

   Comments. Well, solve it, as I did. The answer is $\frac{1}{2}$ ton. After this problem, my belief that I had good intuition about percents was shattered.

10. *It takes three days for a motor boat to travel from A to B down the river, and it takes it four days to come back. How long will it take a wooden log to be carried from A to B by the current?*

    Comments. This problem was one among twenty that my mathematics school teacher, L. I. Bogomolny, assigned for the summer after the eighth grade. I spent a lot of time on it, unable to solve it. I was sure that some data was missing. My older brother Lazar solved it for me. I was amazed with the power of algebra, when he introduced more unknowns than he could find, and the answer appeared as the ratio of two of them. By the way, the answer is 24 days, and it is impossible to find the speed of the current, the speed of the boat, or the distance AB.

11. *Consider any positive integer N whose (decimal) digits read from left to right are in non-decreasing order, but the last two digits (tens and ones) are in increasing order. Prove that the sum of the digits of $9N$ is always 9.*

Comments. It was hard to believe, since N could be really large. For example, if $a = 1778$, $b = 2344459$, and $c = 12225557779$, then

$$9a = 16002,\ 9b = 21100131,\ 9c = 110030020011,$$

and the sum of digits in each case is 9. The proof is easy. If you find it for for 3- or 4-digit numbers, the generalization is trivial. This problem was mentioned to me about ten years ago by V.A. Kanevsky.

12. *Take a 4-digit number (in base 10) with not all digits equal. Rearranging its digits in decreasing order we get a number M. Rearranging its digits in increasing order we get a number m. Consider $M - m$, and repeat the procedure. Do it again, and again... After several iterations we get to the number 6174.*

Comments. If this is not surprising, then I give up... I found a proof, but it was not illuminating (case analysis). A similar question can be asked for numbers with an arbitrary number of digits, and when bases different than 10 are used. Answers become more interesting. Play with a computer. See an article on Wikipedia (`http://en.wikipedia.org/wiki/6174_number`) for more information.

13. $e^{i\pi} + 1 = 0$*, or, more general,* $e^{ix} = \cos x + i \sin x$.

Comments. Please do not become angry with me for placing this result in this section ("... Maybe, I could discover this"). Yes, the equality $e^{i\pi}+1 = 0$ is considered to be one of the highest standards of mathematical beauty: it ties five of the most celebrated and independent mathematical constants: 0, 1, π, e and i! But is it surprising?

Well, it needs a definition of $e^{i\pi}$, and it clearly follows from $e^{ix} = \cos x + i \sin x$ when $x = \pi$. How mysterious is the latter? In order to answer this question, one has to assign meaning to powers with complex exponents. How can this be done? Similarly to reals, where $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$? For complex z, we can try to define e^z using same power series $e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Its convergence for every complex number z, and the property $e^{z_1} e^{z_2} = e^{z_1+z_2}$ for any complex z_1, z_2, is easy to establish (try the latter). Substituting $z = ix$ for real x, and combining real and imaginary parts (even without justification), gives $e^{ix} = \cos x + i \sin x$.

It is interesting that the formula $e^{ix} = \cos x + i \sin x$ was known before Euler. Roger Cotes published it 34 years before, describing the equivalent logarithmic relation in words.

14. *100 women board an airplane with 100 seats. Each of them has a seat assigned. For some reason, the first woman who gets in takes her seat at random. Then the second passenger takes her seat if it is not occupied (by the first), and picks a seat at random if her seat is occupied. Then the third passenger takes her seat if it is not occupied (by the first or second), and picks a seat at random if her seat is occupied. And so on. What is the probability that the last person will sit in her seat?*

Comments. The answer is 1/2 (!), and it is not hard to prove it. I did it by constructing a recurrence for the sequence $\{p_n\}$, $n = 1, \ldots, 100$, where p_i is the probability that the ith woman sits in her seat. A solution without any computations can be found in P. Winkler [27].

15. *There exists a number of the form* $111\ldots111$ *which is divisible by* 2013.

Comments. Very surprising! It is also surprising that if the digit 1 in the desired number is replaced by any sequence of digits (e.g., 1776), and 2013 is replaced by any odd integer not divisible by 5, the result will still hold.

A solution below is an impressive application of the Pigeonhole Principle. Here it is. Let $a_1 = 1$, $a_2 = 11$, $a_3 = 111$, and so on, $a_{2013} = 111\ldots111$ (2013 ones). Divide each a_n by 2013. If one number is divisible (the remainder is 0), then we are done. If not, two of the remainders must necessarily repeat, as there are at most 2012 distinct nonzero remainders. Subtracting the corresponding numbers, we obtain a number M which is divisible by 2013, and is of the form $111\ldots1111000\ldots000$. Hence $M = N \cdot 10^a$, where the digits of N are all 1's, and a is the number of zeros in M. Since M is divisible by 2013, and $\gcd(2013, 10^a) = 1$, then 2013 divides N.

Here are several examples of C: "Even after I see a proof, the fact is still a mystery." "I could not discover this...

16. *Consider a continuous curve $y = f(x)$ on $[0, 1]$ such that $f(0) = f(1)$. A segment joining two points on the graph of the curve is called a chord. Consider only horizontal chords, i.e., those which are parallel to the x-axis. What lengths can they have?*

Comments. The answer is very striking. It turns out that for any positive integer n, the curve will have a horizontal chord of length $\frac{1}{n}$, and that no other horizontal chord length is guaranteed! The last statement can also be phrased this way: for every α which is not a reciprocal of a positive integer, there exists a curve $y = f(x)$ which satisfies the conditions of the statement and which has no horizontal chord of length α.

A solution can be found in R.P. Boas [7], or in A.M. Yaglom and I.M. Yaglom [28]. See also comments in [7], concerning the history and applications of this problem.

17. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Comments. As we know, finding the closed form $\frac{\pi^2}{6}$ for the sum of the series on the left (Basel problem), made young Euler a superstar. Here is a sketch of Euler's proof as presented in W. Dunham [13].

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \quad \Rightarrow \\ \frac{\sin x}{x} &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n} \quad \Rightarrow \\ \frac{\sin \sqrt{x}}{\sqrt{x}} &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{n}. \end{aligned}$$

Given a polynomial $a_0 + a_1 x + \ldots + a_n x^n$ with nonzero roots $x_1, \ldots, x_n$, we have, by Viète's theorem,

$$\sum_{i=1}^{n} \frac{1}{x_i} = \frac{\sum_{i=1}^{n} x_1 \cdot \ldots \cdot x_{i-1} x_{i+1} \cdot \ldots \cdot x_n}{x_1 x_2 \cdots x_n} = -\frac{a_1/a_n}{a_0/a_n} = -\frac{a_1}{a_0}.$$

Looking at $(\sin \sqrt{x})/\sqrt{x}$ as a polynomial of infinite degree (so what?), realizing that its roots are $x_n = \pi^2 n^2$, $n \geq 1$, and applying Viète's theorem the same way (still holds, of course ...), we get

$$\sum_{i=1}^{\infty} \frac{1}{\pi^2 n^2} = -\frac{a_1}{a_0} = -\frac{-1/3!}{1} = \frac{1}{6}.$$

I would never be able to think of this!

18. *Let $0 < r \leq R$, and $S(r, R) = \{x \in \mathbb{R}^3 : r \leq \|x\| \leq R\}$ be a uniform density spherical layer. Let A be any point inside it or on it. Then the gravity at A is zero.*

Comments. I think that the result is impossible "to feel". If the Law of Gravity had 2 ± 10^{-100} as the exponent in the denominator, this would not be true. Still, Newton had an intuitive geometric argument with infinitesimals. It is described, e.g., in V.I. Arnold [2], or (an electrostatic version) in [14]. I did not find the argument convincing. The way I convinced myself that the fact was true was by using spherical coordinates, triple integrals, and Maple. I did not see this problem in Calculus texts. Nor did I see problems asking to demonstrate that solid balls can be replaced by point masses at their centers, when we study motions of planets. I think these are great classical applications of triple integrals, and they should find a place in our courses.

The question was mentioned to me by Yves Crama, while we were driving on I-295 to the University of Delaware in 1988. We tried to find a simple explanation for it for several days, but could not. Do similar statements hold for annuli in 1- and 2-dimensions?

19. *A person writes two distinct integers on two cards, one per card, and puts them on the table face down. You can pick either of the two, look at it, and then you have to guess whether the other number is larger or smaller. Suppose you have a good random number generator. Prove that you have a strategy to make a correct guess with probability strictly greater than* $1/2$.

    Comments. The first time I heard this question, and its solution, was from Peter Winkler, at a dinner which followed his talk at Penn many years ago. Though the proof was short and convincing, I have difficulties believing the statement. So does everyone who I tell this problem.

    For a discussion and a solution, see D. Gale [16], where the problem is attributed to David Blackwell's modification of a related question.

20. *Alice and Bob have one of two consecutive positive integers* n *and* $n+1$ *written on their foreheads. Alice sees Bob's number, and Bob sees Alice's number. Each of them can ask another the question: "Do you know your number?" Suppose Alice and Bob are infinitely intelligent: if there is a way to find out the number on their own forehead, then they will do it. Each of them can answer only "Yes", or "No". Prove that after finitely many question and answers, one of them will know their number.*

    Comments. The first time I heard this question was from Peter Winkler, at the same dinner I mentioned above. Can you prove it by using mathematical induction? See D. Gale [16].

21. *An automorphism* f *of a field is a bijection on it such that* $f(x+y) = f(x) + f(y)$ *and* $f(xy) = f(x)f(y)$. *There are infinitely many automorphisms of the field of complex numbers* $\mathbb{C}$.

    Comments. The statement contrasts with the widely known facts that the only automorphism of the field of rational numbers and of the field of real numbers is the identity map.

    I remember this property of $\mathbb{C}$ was mentioned by L.A. Kaluźnin, in one of his lectures in early 1970's. Though the result was established at the beginning of the 20th century, (see an interesting article by H. Kestelman [20]), it is still not known by many. It is easy to argue that if we ask for only continuous automorphisms of $\mathbb{C}$, then it can be either identity or conjugation. Two other surprising results about isomorphisms of algebraic structures are the following. The additive groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic, as are the multiplicative groups $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : |z| = 1\}$. These two isomorphisms can be derived from the structure theorem of divisible abelian groups, see, e.g., L. Fuchs [15]. Maybe all these results become a bit less surprising if we note that the axiom of choice is used to establish them.

Finally we offer several examples of D: "How is it possible that this problem is not solved yet?"

22. *What is the smallest number of people in a group such that there must be 5 of them who know one another or 5 who do not know one another?*

    Comments. This is a famous problem. The best known result is that this number is between 43 and 49. If 5 in the statement of the problem is replaced by 2, the answer is 2. If 5 is replaced by 3, the answer is 6. If it is replaced by 4, the answer is 18. For details and related questions, see S. Radziszowski [25].

23. *Are there infinitely many positive integers n such that* $\tan n > n$?

    Comments. The question was asked by David Bellamy. It is instructive to experiment with Maple, and see that the positive integer solutions of this inequality are very rare. It can be shown that each of the inequalities $\tan n < -n$, and $\tan n > \frac{1}{4}n$ have infinitely many solutions in positive integers, but the original problem is still open. See D.L. Bellamy, J.C. Lagarias, F. Lazebnik [4].

24. *How many distinct points of intersection can n lines in a plane have? How many regions can they form?*

    Comments. The second question was asked by the author in 1998. It is easy to find the minimum and the maximum of these numbers. For the number of intersection points, it is 0 and $\binom{n}{2}$, respectively. For the number of regions, it is $n+1$ and $\binom{n}{2}+\binom{n}{1}+\binom{n}{0}$, respectively. On the other hand, it is not clear which numbers can appear in between. For details and related results, see B. Grünbaum [17], O.A. Ivanov [18].

25. *Let p be a prime, and $p \geq 5$. Take an arbitrary invertible $n \times n$ matrix A with entries in $\mathbb{Z}_p$ (field of p elements), $n \geq 3$. It is conjectured that there always exists a vector $x = (x_1, x_2, \ldots, x_n)$ with all $x_i \in \mathbb{Z}_p$ such that no x_i is zero, and no component of xA is zero.*

    Comments. The statement is trivial over infinite fields. N. Alon and M. Tarsi [1] proved that the conjecture is true if $\mathbb{Z}_p$ is replaced by any finite field with *nonprime* number of elements, more precisely, by $GF(q)$, where $q = p^e \geq 4$, p is prime and $e > 1$. For prime $q = p \geq 5$, and n much larger than p, the conjecture is still open. For some related results, see R.D. Baker, J. Bonin, F. Lazebnik, E. Shustin [3], and Y.Yu [29].

I will stop here. Dear reader, please share with me your surprises.

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