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Characterization of the American Put Option Using Convexity

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ABSTRACT *Understanding the behaviour of the American put option is one of the classic problems in mathematical finance. Considerable efforts have been made to understand the asymptotic expansion of the optimal early exercise boundary for small time near expiry. Here we focus on the large-time expansion of the boundary. Based on a recent development of the convexity property, we are able to establish two integral identities pertaining to the boundary, from which the upper bound of its large-time expansion is derived. The bound includes parameter dependence in the exponential decay to its limiting value. In addition, these time explicit identities provide very efficient numerical approximations to the true solution to the problem.*

KEY WORDS: asymptotic analysis, free boundary-value problem, American put option

1. Introduction

When holding a put option, the contract holder has the right to sell, at a preset strike price K, an underlying asset. Using the assumptions of standard mathematical finance theory (cf. Etheridge, 2004 or Willmott, 1999), the value of the asset follows geometric Brownian motion. In an American put option, the contract holder may exercise the option at any time during the contract. Then the value of the option contract v satisfies the Kolmogorov equation when it is not optimal for early exercise.

If the holder does decide to exercise, he receives a payoff of

$$P(s) = (K - s)^+, \tag{1}$$

where s is the stock price at the time of exercise. Hence at any time $\tilde{t}$ (measured backwards from the expiration date), there must be an optimal stock price $s = g(\tilde{t})$ such that if $s \leq g(\tilde{t})$, it is advantageous to exercise the option and receive the payoff.

To determine $g(\tilde{t})$, we must solve the following free boundary value problem (cf. Ekström, 2004) for v and g:

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$$\begin{cases} \dfrac{\partial v}{\partial \tilde{t}} - \dfrac{\sigma^2}{2} s^2 \dfrac{\partial^2 v}{\partial s^2} - rs\dfrac{\partial v}{\partial s} + rv = 0, & \text{for} \quad s > g(\tilde{t}), \tilde{t} > 0 \\ v = P(s) = (K - s)^+, & \text{for} \quad s \le g(\tilde{t}), \tilde{t} > 0 \\ \dfrac{\partial v}{\partial s}(g(\tilde{t}), \tilde{t}) = -1, & \\ v(s, 0) = P(s) = (K - s)^+ \text{ for all } s, & \end{cases} . \tag{2}$$

where $r > 0$ denotes the risk-free interest rate and $\sigma > 0$ denotes the volatility.

We introduce the following changes of variables:

$$t = \frac{\sigma^2}{2}\tilde{t}, \quad x = \ln\left(\frac{s}{K}\right) \quad V(x, t) = \frac{v(s, \tilde{t})}{K}. \tag{3}$$

Hence we are normalizing both the option and asset prices by the strike price. The time scale we are choosing is associated with the volatility. Hence, compared with a low-volatility environment, the same amount of "real time" $\tilde{t}$ will correspond to a larger value of t in a high-volatility environment. This is because diffusion of the option price occurs more rapidly in a high-volatility environment.

Substituting Equation (3) into Equation (2), we obtain the following:

$$\begin{cases} \dfrac{\partial V}{\partial t} - \dfrac{\partial^2 V}{\partial x^2} - (k-1)\dfrac{\partial V}{\partial x} + kV = 0, & \text{for} \quad x > h(t), t > 0 \\ V = 1 - e^x, & \text{for} \quad x \le h(t), t > 0 \\ \dfrac{\partial V}{\partial x}(h(t), t) = -e^x, & \\ V(x, 0) = (1 - e^x)^+, & \text{for} \quad x \in \mathbb{R} \end{cases} \tag{4}$$

where $h(t)$ is the free boundary in the new variables and by introducing the scaling, we have reduced the parameter set to the single dimensionless parameter

$$k = \frac{2r}{\sigma^2}.$$

Note that k measures the relative importance of the risk-free and volatile aspects of the market. High values of k correspond to low-volatility situations or those with a high risk-free rate; low values correspond to the opposite situations.

Because of the importance of the American put option in theory and practice, there exists a considerable literature on the problem, both analytical and numerical, for example, Barles *et al.* (1995), Kuske and Keller (1998) and Stamicar *et al.* (1999). A recent survey is provided by Barone-Adesi (2005). The well-posedness of Equation (4) is proved by Chen and Chadam (2007). In addition, Ekström (2004) and Chen *et al.* (2008) proved that the free boundary $h(t)$ is strictly decreasing and convex.

Previous analytical results have been largely focused on the asymptotic expansion of the free boundary $h(t)$ for t small, that is, near the expiry date. In this article, we are interested in the behaviour of the early exercise boundary for t large, that is, near the beginning of the contract. Hence, our results would be of interest to traders who are holding long-term contracts and wish to know the circumstances under which it would be advantageous to exercise shortly after the contract is purchased.

With our scalings in Equation (3), we note that large values of $\widetilde{t}$ may translate only into moderate values of t. Hence even though our techniques formally hold only in the limit of large t, they must hold for moderate t to be useful in the marketplace. We will demonstrate that they are.

Starting with integral equations that hold along the free boundary, we first derive several useful relations for the key integral operator appearing in the integral representation of the solution. These integral relations, together with the convexity of the free boundary, allow us to derive two explicit inequalities for the upper bound of the free boundary. In particular, the sharper of the two bounds establishes that $h(t)$ decays to its infinite horizon h_* in the following manner:

$$h_* \leq h(t) \leq h_* + H_2(t), \quad t \to \infty; \qquad h_* = \log\left(\frac{k}{k+1}\right)$$

$$H_2(t) = \int_t^\infty \frac{k+1}{2\sqrt{\pi\tau}} e^{-k\tau - \left[\frac{h_*+(k-1)\tau}{2\sqrt{\tau}}\right]^2} \left[k - \frac{1}{\sqrt{\pi\tau}}\left(1 + \frac{h_*}{(k+1)\tau}\right)^{-1} e^{-k\tau - \left[\frac{h_*+(k-1)\tau}{2\sqrt{\tau}}\right]^2}\right]^{-1} d\tau, \tag{5}$$

as described more fully in Section 5.

We remark that Equation (5) can be used as an analytical approximation to the true solution. The full integral equation defining $h(t)$ is highly singular near expiry ($t = 0$), hence causing technical difficulty for any standard numerical scheme used to compute h at any time t. (A discussion of this problem can be found in Peskir (2005) or Rogers and Talay (2007), for instance.) Because of its analytical nature, we believe our approach provides an effective alternative way for handling such problems. From a mathematical perspective, the techniques we use may also provide useful hints for similar problems involving non-linear integral equations.

The derived upper bound on $h(t)$ will also lead to the following bound on $V(h(t), t)$:

$$1 - \frac{k}{k+1} e^{H_2} \leq V(h(t), t) \leq \frac{1}{k+1},$$

and hence the value of the option contract when it is written. This will provide useful data to customers looking to purchase such contracts at reasonable prices.

2. Integral Equation Formulation for $h'(t)$

To solve our problem, we must first determine the solution V of Equation (4), and then use the conditions at $x = h(t)$ to determine an expression for the free boundary. Following Chen and Chadam (2007), one can rewrite the differential equation in Equation (4) as

$$\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2} - (k-1)\frac{\partial V}{\partial x} + kV = kH(h(t) - x), \tag{6}$$

valid for all $x \in \mathbb{R}$, $t > 0$, where $H(\cdot)$ is the Heaviside function. Given that the operator in Equation (6) has a Green's function

$$G(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-kt - \frac{(x+(k-1)t)^2}{4t}\right), \tag{7}$$

we have the following integral representation of the transformed contract value from Green's identity (Friedman, 1982):

$$V(x,t) = \int_{-\infty}^{0} (1-e^y) G(x-y,t)\, dy + k \int_0^t \int_{-\infty}^{h(\tau)} G(x-y,t-\tau)\, dy\, d\tau, \tag{8}$$

where the first integral solves the homogeneous equation with initial conditions (4), and the second integral solves the inhomogeneous Equation (6) with zero initial conditions.

To simplify our notation, we rewrite Equation (8) as

$$V(x,t) = I_1(x,t) - I_2(x,t) + k \int_0^t I_1(x-h(t-\tau),\tau)\, d\tau, \tag{9}$$

where

$$I_1(x-z,t) := \int_{-\infty}^{z} G(x-y,t)\, dy = \frac{1}{2} e^{-kt} \operatorname{Erfc}\left(\frac{x-z+(k-1)t}{2\sqrt{t}}\right), \tag{10}$$

$$I_2(x,t) := \int_{-\infty}^{0} e^y G(x-y,t)\, dy = \frac{1}{2} e^{x} \operatorname{Erfc}\left(\frac{x+(k+1)t}{2\sqrt{t}}\right), \tag{11}$$

where in calculating the final integral in V we have reversed the roles of τ and $t-\tau$. For further algebraic simplicity, we note that

$$\frac{\partial I_1}{\partial x} - \frac{\partial I_2}{\partial x} = -I_2(x,t), \tag{12}$$

$$\frac{\partial I_1}{\partial t} - \frac{\partial I_2}{\partial t} = -kI_1(x,t) + \frac{1}{2\sqrt{\pi t}} e^{-kt-[\frac{x+(k-1)t}{2\sqrt{t}}]^2}. \tag{13}$$

Because the second equation in Equation (4) holds at $x = h(t)$, we see that

$$V(h(t),t) = 1 - e^{h(t)}, \;\; kQ = 1 - e^{h(t)} - I_1(h(t),t) + I_2(h(t),t), \tag{14}$$

where we define

$$Q(t) := \int_0^t I_1(h(t) - h(t-\tau),\tau)\, d\tau. \tag{15}$$

Given an integral identity such as Equation (15), the classical numerical approach is to implement certain iterative methods (chosen based on the identity's form) to find $h(t)$ numerically. However, the integral terms appearing in Equation (15) are highly singular; thus traditional methods such as Gaussian quadratures usually provide low accuracy and slow convergence. For this reason, in part, we make further manipulations so that integral-free identities may be established.

In particular, we have the following theorem:

Theorem 1 The free boundary $h(t)$ obeys the following identity:

$$\frac{1}{2}+\frac{k+1}{2}\mathcal{J}[1]=\mathcal{J}[h'(\tau)]-\frac{1}{2}\mathcal{J}\left[\frac{h(t)-h(\tau)}{t-\tau}\right], \tag{16}$$

where

$$\mathcal{J}[u]:=\int_0^t u\frac{\partial I_1}{\partial x}(h(t)-h(t-\tau),\tau)\,d\tau, \quad u\in C^1([0,t]). \tag{17}$$

To begin the proof, we consider the derivative condition on $h(t)$:

$$\frac{\partial V}{\partial x}(h(t),t)=-e^{h(t)},$$

$$\frac{\partial I_1}{\partial x}(h(t),t)-\frac{\partial I_2}{\partial x}(h(t),t)+k\int_0^t\frac{\partial \mathrm{I}_1}{\partial \mathrm{x}}(\mathrm{h(t)}-\mathrm{h}(\mathrm{t}-\tau),\tau)\,\mathrm{d}\tau=-\mathrm{e}^{\mathrm{h(t)}}.$$

Rewriting the above using our definition of $\mathcal{J}$ in Equation (17), we obtain

$$k\mathcal{J}[1]=I_2(h(t),t)-e^{h(t)}, \tag{18}$$

where we have used Equation (12). Moreover, we note that Equation (14) can be written as

$$kQ=1-I_1(h(t),t)+k\mathcal{J}[1]. \tag{19}$$

The problem with using Equation (18) or (19) to determine $h(t)$ is that due to the presence of Q and $\mathcal{J}$, they are still integral equations, which is not desirable. We would like to reduce our system to algebraic or differential equations, so we must manipulate Equation (14) to that form. As a first step, we take the derivative of Equation (14) with respect to t and simplify:

$$k\frac{dQ}{dt}=kh'\mathcal{J}[1]+kI_1(h(t),t)-\frac{1}{2\sqrt{\pi t}}e^{-kt-\left[\frac{h(t)+(k-1)t}{2\sqrt{t}}\right]^2}, \tag{20}$$

where we have used Equations (12) and (13). Then we take the derivative of the integral directly:

$$\frac{dQ}{dt} = I_1(h(t), t) + h'(t)\mathcal{J}[1] - \mathcal{J}[h'(\tau)]. \tag{21}$$

Combining Equations (20) and (21), we may obtain an expression for $\mathcal{J}[h'(\tau)]$:

$$\mathcal{J}[h'(\tau)] = \frac{1}{2k\sqrt{\pi t}} e^{-kt - [\frac{h(t)+(k-1)t}{2\sqrt{t}}]^2}. \tag{22}$$

Next we integrate Q by parts:

$$\begin{aligned} Q = & \left[\frac{1}{-2k} e^{-k\tau} \mathrm{Erfc}\left(\frac{h(t) - h(t-\tau) + (k-1)\tau}{2\sqrt{\tau}}\right)\right]_0^t \\ & - \int_0^t \frac{1}{-2k} e^{-k\tau} \left[\frac{h'(t-\tau) + k - 1}{2\sqrt{\tau}} - \frac{h(t) - h(t-\tau) + (k-1)\tau}{4\tau^{3/2}}\right] \times \\ & - \frac{2}{\sqrt{\pi}} \exp\left(-\left[\frac{h(t) - h(t-\tau) + (k-1)\tau}{2\sqrt{\tau}}\right]^2\right) d\tau. \end{aligned} \tag{23}$$

We may rewrite Equation (23) in the $\mathcal{J}$ notation by switching the roles of t and τ in Equation (17) to obtain

$$\begin{aligned} \mathcal{J}[u] &= \int_0^t u \frac{\partial I_1}{\partial x}(h(t) - h(\tau), t - \tau)\, d\tau, \\ &= -\int_0^t \frac{u e^{-k(t-\tau)}}{\sqrt{\pi(t-\tau)}} e^{-\left[\frac{h(t)-h(\tau)+(k-1)(t-\tau)}{\sqrt{t-\tau}}\right]^2} d\tau. \end{aligned} \tag{24}$$

Then substituting Equation (24) into Equation (23), we have the following:

$$kQ = \frac{1}{2} - I_1(h(t), t) + \mathcal{J}[h'(\tau)] + \frac{k-1}{2}\mathcal{J}[1] - \frac{1}{2}\mathcal{J}\left[\frac{h(t) - h(\tau)}{t - \tau}\right]. \tag{25}$$

Continuing to simplify Equation (25) using Equation (19), we obtain

$$\frac{1}{2} + \frac{k+1}{2}\mathcal{J}[1] = \mathcal{J}[h'(\tau)] - \frac{1}{2}\mathcal{J}\left[\frac{h(t) - h(\tau)}{t - \tau}\right],$$

which is exactly Equation (16). Hence our proof is complete.

The secant term $\mathcal{J}[(h(t) - h(\tau))/(t - \tau)]$ in Equation (16) is what makes it an intractable integral equation. By contrast, the other $\mathcal{J}$ terms in Equations (16) are easily analysable once we use Equations (18) and (22). Hence in the next sections, we shall bound the secant term in Equation (16) by more tractable expressions to obtain useful information about the long-term behaviour of $h(t)$.

3. Integral Inequalities for $h'(t)$

Ekström (2004) and Chen *et al.* (2008) have established the convexity of $h(t)$, which implies that

$$h'(t) \geq \frac{h(t) - h(\tau)}{t - \tau} \geq h'(\tau). \tag{26}$$

We may also obtain another bound by noting that

$$h'(t) \geq \frac{h(t) - h(\tau)}{t - \tau} \geq \frac{h(t)}{t},$$

but it can be shown that the resulting bound on the large-t behaviour of $h(t)$ is weaker than that obtained from Equation (26).

Because the kernel in $\mathcal{J}$ is always negative, the inequality in Equation (26) provides the proof of the following theorem:

Theorem 2 The free boundary $h(t)$ obeys the following inequality:

$$\mathcal{J}[h'(t)] = h'(t)\mathcal{J}[1] \leq \mathcal{J}\left[\frac{h(t) - h(\tau)}{t - \tau}\right] \leq \mathcal{J}[h'(\tau)]. \tag{27}$$

We then substitute Equation (27) into the integral identity along the free boundary; we expect to derive explicit inequalities for $h(t)$, which approach the true $h(t)$ very well for large (and even moderate) t.

Equation (27) also provides the value of $h_* = h(\infty)$. Because h asymptotes to a constant h_* as $t \to \infty$, we see that the left-hand bound in Equation (27) goes to zero as $t \to \infty$. Similarly, we see from Equation (22) that the right-hand bound in Equation (27) goes to zero as $t \to \infty$. Thus by the Sandwich Theorem, we have that the middle quantity in Equation (27) goes to 0 as $t \to \infty$. Using that fact in Equation (16), we obtain

$$\frac{1}{2} + \frac{k+1}{2k}[I_2(h_*, \infty) - e^{h_*}] = 0,$$

or equivalently

$$h_* = \log\left(\frac{k}{k+1}\right), \tag{28}$$

a result which is consistent with those obtained with other methods in the existing literature such as Chen and Chadam (2007).

4. First Upper Bound for $h(t)$

We shall find that both sides of the inequality in Equation (27) provide upper bounds on $h(t)$. The first is described in the following theorem:

Theorem 3 The free boundary $h(t)$ obeys the following inequality:

$$h(t) \leq h_* + H_1(t), \qquad t \geq t_1 := \frac{|h_*|}{k+1} \tag{29}$$

$$H_1(t) = \left\{ \frac{(k+1)t - h_*}{2[(k+1)t + h_*]} \right\} \frac{1}{k\sqrt{\pi t}} e^{-kt - \left[\frac{h_* + (k-1)t}{2\sqrt{t}}\right]^2}. \tag{30}$$

We begin the proof by substituting the upper bound in Equation (27) into Equation (16) to obtain

$$\frac{1}{2} + \frac{k+1}{2}\mathcal{J}[1] \geq \mathcal{J}[h'(\tau)] - \frac{1}{2}\mathcal{J}[h'(\tau)],$$

$$\frac{1}{2} + \frac{k+1}{2}\left[\frac{I_2(h(t),t) - e^h}{k}\right] \geq \frac{1}{2}\left\{ \frac{1}{2k\sqrt{\pi t}} e^{-kt - \left[\frac{h(t) + (k-1)t}{2\sqrt{t}}\right]^2} \right\}.$$

This expression can be rewritten as

$$2k - 2(k+1)e^h + (k+1)e^h \operatorname{Erfc}\left(\frac{h + (k+1)t}{2\sqrt{t}}\right) - \frac{1}{\sqrt{\pi t}} e^{-kt - \left[\frac{h(t) + (k-1)t}{2\sqrt{t}}\right]^2} \geq 0. \tag{31}$$

From Abramowitz and Stegun (1972), 7.1.14, we have that

$$\operatorname{Erfc}\left(\frac{h + (k+1)t}{2\sqrt{t}}\right) \leq \frac{1}{\sqrt{\pi}}\left(\frac{h + (k+1)t}{2\sqrt{t}}\right)^{-1} e^{-\left[\frac{h + (k+1)t}{2\sqrt{t}}\right]^2}, \quad t \geq t_1, \tag{32}$$

where t_1 is defined in Equation (29) and the restriction on t comes from the fact that the inequality holds only for positive argument. Substituting Equation (32) into Equation (31) and rewriting, we obtain

$$(k+1)e^h - k \leq \left[\sqrt{t}\left(\frac{k+1}{h + (k+1)t}\right) - \frac{1}{2\sqrt{t}}\right] \frac{1}{\sqrt{\pi}} e^{-kt - \left[\frac{h(t) + (k-1)t}{2\sqrt{t}}\right]^2}.$$

Also, as $t \to \infty$, we let

$$h(t) = h_* + h_0(t), \qquad h_0(t) \ll 1. \tag{33}$$

Substituting Equation (33) into the above and simplifying, we obtain

$$h_0 \leq H_1(t), \tag{34}$$

where $H_1(t)$ is as defined in Equation (30). In making the simplification, we have used the facts that, for a and b positive,

$$e^a \geq 1 + a, \qquad (a+b)^{-1} \leq a^{-1}, \qquad e^{-(a+b)} \leq e^{-a}.$$

Substituting Equation (34) into Equation (33) completes the proof.

We note that $H_1(t)$ decays in proportion to

$$t^{-1/2} \exp\left(-\frac{(k+1)^2 t}{4}\right). \tag{35}$$

Hence the quick exponential decay ensures that $H_1(t)$ becomes quite small, even for moderate values of t. Hence $H_1(t)$ is a tight bound. This is particularly useful in the financial context. Recall that for an actual option, the maximum value of $\tilde{t}$ possible is the length T of the contract. Hence for a bound to be useful, it must have quick convergence for moderate values of t. (Note that this quick convergence also implies that the asymptote h_* is also a good estimate for $h(t)$.)

An important feature of Equation (35) is the inclusion of the parameter value k in the exponent. This distinguishes our bound from others in the literature, for example, Hedenmalm (2006), where the bound is proportional to $t^{-3/2}e^{-t}$. Although the algebraic decay is faster in Hedenmalm (2006), the appearance of the parameter k in our exponent can make our bound tighter under certain conditions. (Although it is tempting to say that our bound is better when $k > 1$, in reality scaling differences make the comparison more subtle.)

The inclusion of k in our bound clarifies the effect of the underlying financial parameters r and σ on our solution. In particular, from our discussion of k above, we see that the bound is tighter in low-volatility situations.

5. Second Upper Bound for *h(t)*

Using the lower bound in Equation (27) yields the main result of this article:

Theorem 4 The free boundary $h(t)$ obeys the following inequality:

$$h(t) \leq h_* + H_2(t),$$

$$H_2(t) = \int_t^\infty \frac{k+1}{2D(\tau)\sqrt{\pi\tau}} e^{-k\tau - \left[\frac{h_* + (k-1)\tau}{2\sqrt{\tau}}\right]^2} d\tau, \quad t > t_2, \tag{36}$$

$$D(\tau) = k - \frac{1}{\sqrt{\pi\tau}}\left(1 + \frac{h_*}{(k+1)\tau}\right)^{-1} e^{-k\tau - \left[\frac{h_* + (k-1)\tau}{2\sqrt{\tau}}\right]^2}, \tag{37}$$

where $t_2 > t_1$ is defined by $D(t_2) = 0$.

We begin the proof by substituting the lower bound in Equation (27) into Equation (16) to obtain

$$\frac{1}{2} + \frac{k+1}{2}\mathcal{J}[1] \leq \frac{1}{2k\sqrt{\pi t}} e^{-kt - \left[\frac{h(t) + (k-1)t}{2\sqrt{t}}\right]^2} - \frac{1}{2}\mathcal{J}[h'(t)],$$

$$h'\left[e^h - \frac{1}{2}e^h \mathrm{Erfc}\left(\frac{h + (k+1)t}{2\sqrt{t}}\right)\right] \geq k$$

$$-(k+1)\left[e^h - \frac{1}{2}e^h \mathrm{Erfc}\left(\frac{h + (k+1)t}{2\sqrt{t}}\right)\right] - \frac{1}{\sqrt{\pi t}} e^{-kt - \left[\frac{h(t) + (k-1)t}{2\sqrt{t}}\right]^2}. \tag{38}$$

Here we have used Equations (11), (14), (21) and (22). Substituting Equations (32) and (33) into Equation (38), we obtain

$$h_0'\left[e^{h_\infty} e^{h_0} - \frac{1}{2\sqrt{\pi}}\left(\frac{h_\infty + (k+1)t}{2\sqrt{t}}\right)^{-1} e^{-kt - [\frac{h_\infty + (k-1)t}{2\sqrt{t}}]^2}\right] \geq -\frac{1}{2\sqrt{\pi t}} e^{-kt - [\frac{h_\infty + (k-1)t}{2\sqrt{t}}]^2}.$$

Note that because we have used Equation (32), the above expression holds only for $t \geq t_1$. Continuing to simplify, we obtain

$$h_0' \geq -\frac{k+1}{2D(t)\sqrt{\pi t}} e^{-kt - \left[\frac{h_* + (k-1)t}{2\sqrt{t}}\right]^2},$$

where D is as defined in Equation (37).

We now consider the behaviour of D. For $t > t_1$, D is strictly increasing, because each of the factors in the second term are strictly decreasing (keeping in mind that $h_* < 0$). However, as $t \to t_1^+$, the term with h_* in it diverges, so $D(t_1) \to -\infty$. Hence there must be a unique $t_2 > t_1$ for which $D(t) > 0$ for all $t > t_2$. To preserve the direction of the inequality in the above, we must have that $D(t) > 0$.

This is yet another upper bound as well, because we have

$$\int_t^\infty h_0'(\tau)\,d\tau \geq \int_t^\infty -\frac{k+1}{2D(\tau)\sqrt{\pi\tau}} e^{-k\tau - \left[\frac{h_* + (k-1)\tau}{2\sqrt{\tau}}\right]^2} d\tau.$$

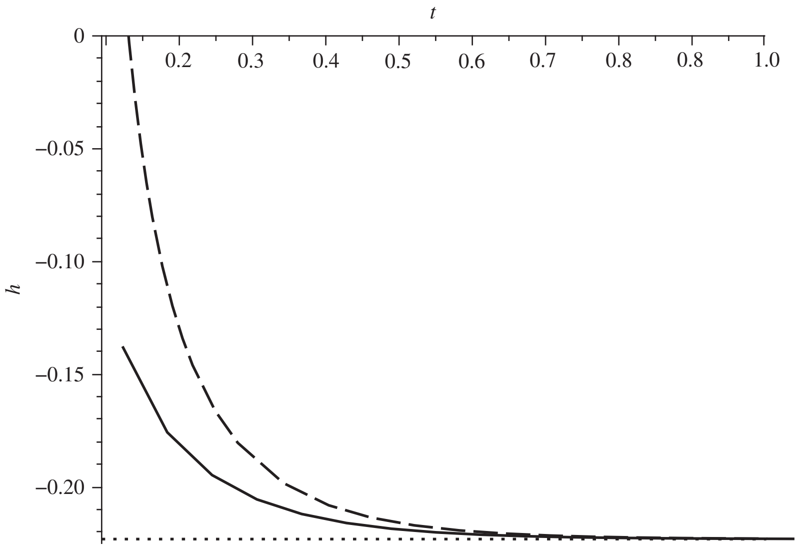

Figure 1. Comparison of bounds on $h(t)$ for $k = 4$. For this case, $t_1 = 0.04$ and $t_2 = 0.07$. Dashed line: expression in Theorem 3. Solid line: expression in Theorem 4. Dotted line: $h = h_*$

Noting that the left-hand side is $-h_0(t)$ and the right-hand side is $-H_2(t)$, the theorem follows.

Because the exponential decay in Equation (36) is the same as that in Equation (30), the same remarks apply regarding the rapid convergence of the bound for even small t, as shown in Figure 1. Hence these bounds are useful even for contracts where $\sigma^2 T/2$ is small.

Note also from Figure 1 that the bound in Equation (36) is better than the bound in Equation (30). This result has held for every value of k we have examined. However, because a proof of this result is beyond the scope of this article, we present both bounds for practitioners. Although weaker, the bound in Equation (30) is much easier to calculate, as the expression does not involve performing an integration.

6. The Option Price

Once a bound on the exercise boundary $h(t)$ has been obtained, it is trivial to obtain a bound on V there. Given that $h_* \le h(t) \le h_* + H_j$, we see that

$$1 - e^{h_* + H_j} \le 1 - e^h \le 1 - e^{h_*}$$

$$1 - \frac{k}{k+1} e^{H_j} \le V(h(t), t) \le V_* := \frac{1}{k+1}, \tag{39}$$

where we have used the second equation in Equation (4) evaluated at $x = h$, as well as the definition of h_* in Equation (28).

Note also that Equation (39) holds for either H_1 or H_2. Because these are computed numerically, it is a straightforward calculation to obtain the lower bound in Equation (39).

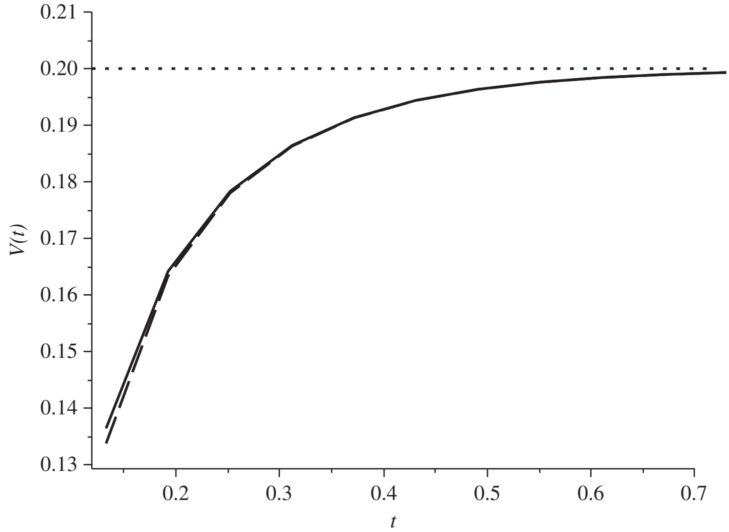

Figure 2. Comparison of bounds on $V(t)$ for $k = 4$ and H_2. For this case, $t_2 = 0.07$. Solid line: Bound on V from (39). Dashed line: Bound on V from (40). Dotted line: $V = V_*$

However, if a simpler expression is needed for analytical purposes, we may exploit the fact that $e^{-H_j} \geq 1 - H_j$ to obtain the following weaker bound:

$$1 - \frac{k}{(k+1)(1-H_j)} \leq V(h(t), t) \leq V_*. \tag{40}$$

The comparison between the bounds is shown in Figure 2. Note that the bounds are nearly identical for this parameter regime. From the t-scale in the figure, it is clear that the rapid decay of H_2 forces rapid decay in V. Hence our bounds will be useful in the financial context, where moderate t can correspond to times near the contract origination date.

7. Conclusions and Further Research

Because the American version of the put option allows early exercise, the resulting mathematical formulation takes the form of a free boundary value problem for the early exercise boundary $h(t)$. The full equation (14) (or equivalently, Equation (16)) governing $h(t)$ is an integral equation with singularities that preclude the use of standard numerical methods. Hence the numerical simulation of these equations, though useful to examine the true accuracy of our bounds, is beyond the scope of this article and will form the basis for future research.

The two upper bounds we constructed for $h(t)$ exploited its recently proved convexity property (Ekström, 2004; Chen *et al.*, 2008). The bounds have several advantages. For instance, both have analytical expressions. The first bound in Equation (30) has a particularly simple expression. The second bound in Equation (36) has a more complicated expression, but one which can be easily computed using symbolic mathematical software. Although our investigations indicate that Equation (36) is a better bound than Equation (30), a formal proof of such a result is beyond the scope of this article and will form the basis of future research.

Our numerical simulations of both bounds verified that these bounds approach the true value of $h(t)$ very quickly due to their exponential decay. Hence, these results are useful for the analysis of real-world financial contracts, where $\tilde{t}$ doesn't go to infinity, but rather is bounded above by T.

By scaling the problem, we were able to reduce the number of parameters to a single dimensionless combination k. Our bounds show direct dependence on k, both in the exponential decay term and in the algebraic decay factors. Hence it is easy to see how the efficacy of our bound depends on the underlying financial parameters r and σ. Once the bounds on $h(t)$ are established, it is a trivial exercise to convert them into bounds on V at the early exercise boundary.

Finally, the analytical techniques we used were not dependent on the particular mathematical system under consideration. Hence we expect these techniques to be useful in other problems where the free boundary is known to be convex.

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