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Simulation Solution to a Two-Dimensional Mortgage Refinancing Problem

Dejun Xie[1] · **Nan Zhang**[2] · **David A. Edwards**[3]

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Abstract This work studies a mortgage borrower's optimal refinancing strategy, which is formulated as the solution to a stochastic minimization problem with contingent conditions. The problem is framed in a business economic environment where the underlying discounting factor and mortgage interest rate are assumed to follow a two-dimensional stochastic process of Vasicek type. A complete Monte Carlo algorithm is developed and implemented. This algorithm generates the optimal refinancing surface as a function of time and the risk-free rate. Numerical examples with financial implications are provided.

Keywords Mortgage refinancing · Stochastic modeling · Monte Carlo simulation · Financial optimization

1 Introduction

Mortgage securities are one of the most heavily traded financial instruments throughout the world's major markets, both in terms of volume and frequency. Valuation of

✉ Dejun Xie
xie.dj@sustc.edu.cn

Nan Zhang
nan.zhang@xjtlu.edu.cn

David A. Edwards
dedwards@udel.edu

1 Department of Finance, South University of Science and Technology of China, Shenzhen, China

2 Department of Computer Science and Software Engineering, Xian Jiaotong-Liverpool University, Suzhou, China

3 Department of Mathematical Sciences, University of Delaware, Newark, DE, USA

mortgage securities has been increasingly crucial to investors, bankers, and financial regulators (for discussion with respect to this aspect, see Saunders and Allen 1999, 2010; Lea 1999; Crotty 2009, for instance) in helping their decision making from various perspectives. The efficacy of using mortgage securities as financial indicators certainly depends on whether such securities are valued properly, and, more importantly, whether the contract holders react rationally to the market movement. In particular, the valuation of mortgage securities needs to take into account the choices available to the mortgage loan borrowers as part of their contracts. In fact, without simultaneously accommodating these choices, the mathematical analysis (via a partial differential equation approach, for instance) is usually incomplete or ill-posed (Xie et al. 2007).

Usually the borrowers are granted the right to prepay, *i.e.*, to settle the remaining loan balance at any time during the contract period of the mortgage loan. The valuation of mortgage securities with the possibility of prepayment has been studied considerably in the literature, especially via the so-called structural approach. In that approach, the problem is mathematically formulated as a free-boundary problem where the free boundary denotes the interest rate at which the mortgage security value equals the face value of the loan (Lo et al. 2009; Xie et al. 2007).

On the other hand, there has not been enough mathematical attention to the valuation of mortgage securities embedded with the possibility of refinancing. As we will treat it, mortgage refinancing refers to replacing the existing loan with a new loan which has

- an initial loan amount equal to the remaining face value of the original loan,
- a term which is the remaining duration of the original loan, and
- an interest rate equal to the market interest rate at the time of refinancing.

The reasons accounting for the lack of scholarly interest can be: (1) historically, refinancing was less commonly claused in industry contracts compared to prepayments; and (2) competition between banks was less severe in earlier times compared to the present. But the contemporary financial market has reversed both these factors, and refinancing has become increasingly common. It is with this observation of market reality that we assume the significance of the current study on the optimal refinancing strategy for mortgage borrowers.

The rest of the paper is organised as follows. Section 2 discusses some of the related work found in the literature. Section 3 lays out key assumptions for the problem under study. Section 4 formulates the problems with postulations on likely solution scenarios. Section 5 outlines the numerical schemes based on Monte Carlo simulation. Section 6 provides example outputs of our algorithm. Section 7 concludes the paper with possible directions of further studies.

2 Literature Review

There has been considerable work done on the topic of modelling mortgage refinance and prepayment behaviours. These works endeavoured to understand the conditions under which a borrower will pay back his/her outstanding debt before the end of the contract period. Most of this earlier work modelled the optimal mortgage prepayment problem out of a different motivation from ours. Their purpose was to determine the fair price of a mortgage contract under the condition that the loan may be prepaid. This mortgage contract pricing problem is closely related to the valuation of residential mortgage backed securities (MBSs): an important problem, as the MBS market has been one of the largest and fastest growing bond markets in the US. One approach to mortgage contract pricing is to view the prepayment opportunity as a built-in option in the mortgage contract that can be exercised by the borrower under favourable conditions. This approach inevitably borrows techniques from option pricing to calculate prices of mortgage contracts.

For example, Dunn and McConnell (1981a, b) applied contingent claim techniques to estimate the present value of the mortgage contract. The resulting partial differential equations were solved using the finite-difference method. Following the option pricing approach, Chen and Ling (1989) applied the binomial tree method to calculate the prices of the prepayment option and the mortgage contract. They assumed that a borrower will prepay the outstanding debt when the contract rate drops deep enough. Their model incorporated the possibility of recursive refinancing. However, the optimal refinancing threshold rate r_* (the rate under which refinancing, if it takes place, will be optimal) cannot be obtained directly from the constructed binomial tree. To approximate r_*, multiple trees have to be constructed with varying initial mortgage rates until the initial rate is high enough for refinancing to be optimal at the present time. The difference in basis point between this rate and the original contract rate is deemed as how much the mortgage rate has to drop in order to make refinancing at the present time optimal.

More recently, Lee and Rosenfield (2005) applied dynamic programming techniques to estimate the overall cost to a borrower if he/she refinances his/her outstanding debt at a particular time with a new mortgage rate. The authors assumed that refinancing will happen if this cost is lower than the overall cost without refinancing. As in Chen and Ling (1989), r_* can be approximated only through multiple tests.

In contrast to these earlier works, our method computes r_* directly without calculating the value of the mortgage contract. Monte Carlo simulation is the fundamental approach for realizing our goal. As a powerful computing tool, Monte Carlo simulation has been used in many applied fields such as system engineering and managerial sciences (cf. Chen et al. 2003; Barat et al. 2006). Longstaff (2004) applied the least-squares Monte Carlo method to compute the prices of prepayment options and mortgage contracts.

A seminal work of refinancing strategy which is close in methodology to our current study is presented in Zheng et al. (2012), Gan et al. (2012), where a Monte Carlo simulation scheme is introduced to find the probability distribution of the optimal refinancing time as a function of t. To find the optimal refinancing strategy according to Zheng et al. (2012), Gan et al. (2012), one can simulate the alternative market interest rate and compare the total discounted value of all installments under the existing payment stream to alternative ones. Then the installment path with minimum net present value of total payment is chosen.

While the seminal work in Zheng et al. (2012), Gan et al. (2012) is interesting, there are a couple of important issues deserving solid theoretical clarification. Among other things, the mathematical assumptions and framework surrounding the refinancing strategy are not explicitly stated. Also the previous work in Gan et al. (2012) does not differentiate between the alternative mortgage interest rate and the discounting factor—they are usually not the same, despite possible strong statistical correlations between these two processes. In addition, the Matlab-based simulation algorithm can be improved in terms of runtime and convergence, as admitted in Gan et al. (2012).

The current work overcomes these weaknesses by generalising the refinancing problems discussed in Gan et al. (2012). We then reformulate the problem rigorously with a discrete stochastic optimisation approach, and provide solutions with enhanced Monte Carlo simulations.

3 Economic and Model Assumptions

3.1 Business Economic Assumptions

As the mortgage market increasingly diversifies, the mortgage contract itself has become rather complicated in actual practice, the documentation of which concerns not only financial and business consultants, but also commercial lawyers and regulatory officers, etc. This said, it is agreeable for us to summarise, with reasonable simplifications, the following key assumptions regarding common contract specifics and the economic environment in which the mortgage deals are cultivated.

1. The payment streams, installments, underlying interest rates and discounting factors, as well as any other financial calculations pertaining to the contract, are all based on a finite discretisation of the duration of the original loan contract.
2. The duration of the original loan contract is divided into subintervals (usually months). Payments and refinancing can be made only at the end of a subinterval.
3. The mortgage loan is fully amortised, which yields equal installment of cash flow per subinterval.
4. Neither the original lender nor the second lender (who are allowed to be the same) charges a fee from the borrower in the event of refinancing.
5. Only one refinancing is granted throughout the whole duration of the original loan.
6. The market is complete, and both the lender and the borrower have equal access to the market information.
7. The borrower does not have enough funds to pay off either the original loan or the second loan after refinancing.

Among these assumptions, 1–5 are contract clauses or interpretations of these clauses, and 6–7 are assumptions about the market and economic environment. In particular, assumption 6 guarantees the method and solutions contained in this work are arbitrage-free. We realize that assumption 4 may not be realistic in practice. However, such an assumption helps to simplify the mathematical formulation of the problem (which is not uncommon in financial mathematics, as demonstrated by the seminal Black–Scholes model). It is our hope that more complicated market conditions can be added in future research after fundamental patterns are well understood using the current study.

3.2 The Mortgage Rate and Risk-Free Rate Processes

Consider the risk-neutral processes for the mortgage rate r_t and risk-free rate f_t, both of which we treat as annual rates with units year^{-1}. We assume they follow Vasicek's instantaneous short rate model (Vasicek 1977); hence we have

$$\begin{aligned} dr_t &= \kappa_1(\theta_1 - r_t)dt + \sigma_1 dW_t^1, \\ df_t &= \kappa_2(\theta_2 - f_t)dt + \sigma_2 dW_t^2, \end{aligned} \tag{1}$$

where the reversion rates κ_1, κ_2, long-term mean levels θ_1, θ_2, and volatilities σ_1, σ_2 are positive constants, and W^1, W^2 are ρ-correlated Brownian processes. Here t is measured in years, and hence the units for the parameters θ, κ, σ, and W are respectively year^{-1}, year^{-1}, year$^{-1.5}$, and year$^{-0.5}$.

The Vasicek stochastic process admits an explicit solution of the form

$$r_t = \theta_1 + (r_0 - \theta_1)e^{-\kappa_1 t} + \sigma \int_0^t e^{-\kappa_1(t-s)} dW_s.$$

This model incorporates mean reversion in that the short rate r (respectively, f) is pulled to the long-term mean level θ_1 (respectively, θ_2) at the speed κ_1 (respectively, κ_2). The second part $\sigma_1 dW^1$ (respectively, $\sigma_2 dW^2$) is a normally distributed stochastic term superimposed upon the mean reversion.

Our Monte Carlo simulations will require a discretized version of (1); in that case, the short rates $r_{t+\Delta t}$, $f_{t+\Delta t}$ at time $t + \Delta t$ are calculated from the rates r_t, f_t at time t as

$$\begin{aligned} r_{t+\Delta t} &= r_t + \kappa_1(\theta_1 - r_t)\Delta t + \sigma_1 \epsilon_t^1 \sqrt{\Delta t}, \\ f_{t+\Delta t} &= f_t + \kappa_2(\theta_2 - f_t)\Delta t + \sigma_2 \epsilon_t^2 \sqrt{\Delta t}, \\ \epsilon_t^1 &= u, \ \epsilon_t^2 = \rho u + \sqrt{1-\rho^2} v. \end{aligned} \tag{2}$$

where Δt is the time interval between changes in the rates (typically taken to be a day). Here u and v are sampled as uncorrelated variables with standard normal distributions, and ρ is the correlation between ϵ_t^1 and ϵ_t^2.

3.3 Model for Mortgage Payment and Refinancing

In this work we consider a fully amortised model, in which a fixed payment is made each month during the whole period of the mortgage contract. Typically, a principal p_0 is borrowed at time $t = 0$ with annual interest rate r_0, and the principal is to be paid back over a period of N months. The first payment is made at month 1, and the last at month N. In each month a fixed payment m_1 is made and this monthly payment m is calculated by

$$m_1 = \frac{p_0 r_0/12}{1 - (1 + r_0/12)^{-N}}, \tag{3}$$

where the extra factor of 12 comes from the fact that r_0 is the annual interest rate.

Let $t = t_k$ correspond to the end of the kth month, $k \in \{1, 2, \ldots, N\}$. After the kth monthly payment m has been made, the outstanding balance owed to the lender is p_{t_k}. Since by assumption 2, the ends of the months are the only times when payments and refinancings can occur, we choose to simplify our notation by writing $p_{t_k} = p_{[\![k]\!]}$, with similar notation for r and f. Calculating the remaining balance, we have

$$p_{[\![k]\!]} = \frac{m_1}{r_0/12}\left[1 - (1 + r_0/12)^{k-N}\right]. \tag{4}$$

We model the refinancing process as follows. Suppose after the kth monthly payment, the mortgage rate for new loans is $r_{[\![k]\!]}$. If $r_{[\![k]\!]} < r_0$, the debtor may decide to refinance by borrowing $p_{[\![k]\!]}$ from a second bank to settle the balance owed to the first lender. The borrower enters into another mortgage contract with the second bank, which is parametrised with principal $p_{[\![k]\!]}$ and repayment months $k+1, k+2, \ldots, N$. (The assumption that both mortgages would end at the same time may be unrealistic in practice, but will not appreciably affect our results.)

In the assumed case, the total payment M made by the borrower under the two contracts is

$$M = km_1 + (N - k)m_2, \tag{5}$$

where

$$m_2 = \frac{p_{[\![k]\!]} r_{[\![k]\!]}/12}{1 - (1 + r_{[\![k]\!]}/12)^{k-N}}, \; r_{[\![k]\!]} < r_0, \tag{6}$$

and m_1 and $p_{[\![k]\!]}$ are given by (3) and (4).

4 Mathematical Setup

4.1 Preliminary Analysis

Suppose at t_k (the end of the kth subinterval), a decision to refinance must be made. Intuitively, one would possibly refinance then only if $r_{[\![k]\!]} < r_0$, though in this case a borrower may keep waiting, betting on a even better deal in the future. To heuristically illustrate, we display, in the following Fig. 1, the comparative level plots of initial mortgage rate r_0 and the interest rate process. For the convenience of explanation, all these plots are based on an analysis with the assumption that $\sigma_1 = 0$ and the discounting factors are constant across time.

1. Case where $r_0 < r_{[\![k]\!]}$ (see Fig. 1a). In this scenario, the borrow cannot optimally refinance today. If he refinances, he immediately pays a higher monthly installment, giving up the existing lower interest rate r_0, and also the possibility of any future refinancing at a lower rate.
2. Cases where $r_{[\![k]\!]} < r_0$ and r_t is set to climb in the future trend (see Fig. 1b,c). These are typical scenarios where the borrower would possibly refinance today. If he does so, he immediately enjoys a lower interest rate and a lower monthly payment. The longer he waits, the higher the new interest rate would be. Also, the longer he waits, the lower the face value of the original loan, which reduces

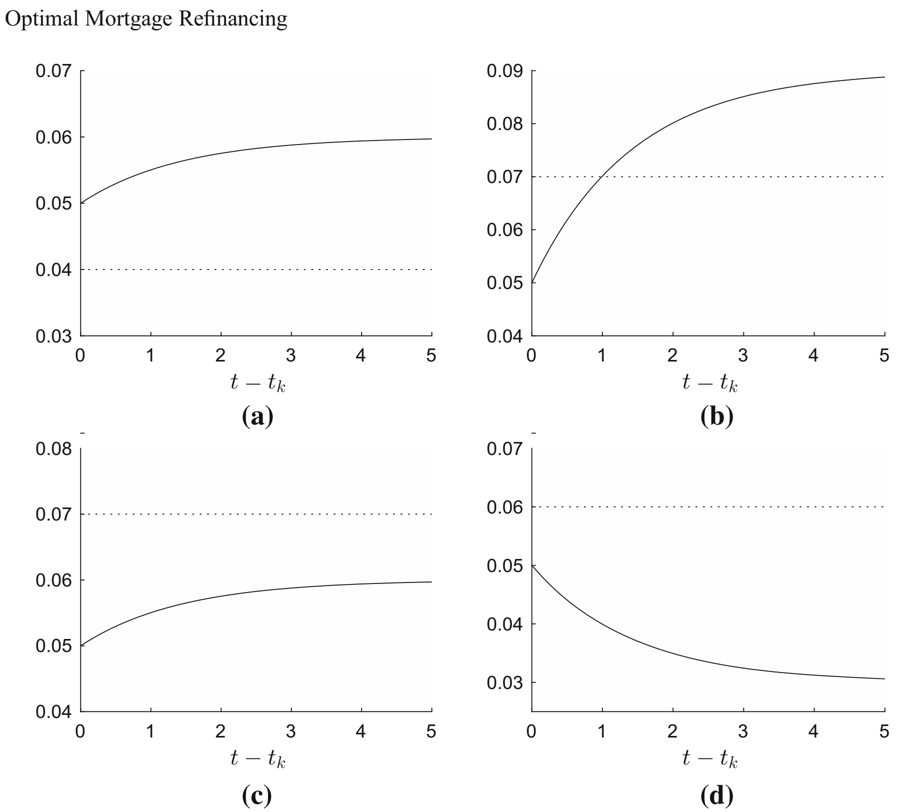

Fig. 1 Comparison between r_0 and interest process for $\sigma_1 = 0$. *Dotted lines*: r_0. *Solid curves*: r_t. **a** $r_0 < r_t$ for all $t > t_k$ (*no refinancing*). **b** $r_0 > r_t$ for some $t > t_k$. **c**, **d** $r_0 > r_t$ for all $t > t_k$ (*refinancing*)

the benefit of refinancing. This observation has been numerically verified in Gan et al. (2012) with plots of the density functions of the optimal refinancing time, showing that the optimal refinancing, if it exists, usually occurs at the early stage of the contract. However, an immediate refinancing may not be optimal due to the uncertainty caused by market volatility.

3. Case where $r_{[\![k]\!]} < r_0$ and r_t is strictly decreasing (see Fig. 1d). Even if $\sigma = 0$ for this case (a higher $\sigma > 0$ is usually the main reason for borrower to take a wait-and-see strategy), chances are the borrower can wait for a while to optimally refinance. How long to wait depends on how fast and how low r_t decreases in the future. This is the most interesting case, and will be the focus of our analysis in subsequent sections.

It is noted, however, that the presence of transaction costs may complicate the issue and change the optimality of the strategies discussed above (except for case 1). In particular, the “earlier the better” strategy may not be the best choice for case 2, especially when the transaction cost is charged in proportion to the remaining loan balance. In addition, for case 3, it may never be worth refinancing if the rates do not drop enough, the likelihood of which will only increase when the transaction cost is imposed.

4.2 Problem Discretization

Let $dt = T/q$ be the unit subinterval length for partitioning the duration of the contract $[0, T]$, where q is a positive integer. Suppose refinancing is made at the time of n^*dt. Viewing the total discounted payment as a function of n^*, with all other parameters prescribed, we are to solve the following minimisation problem. For $n^* = 1, 2, \ldots$, find

$$\min_{n_*} M(n^*) = \sum_{i=0}^{n^*-1} m_1 f(0,i) + \sum_{i=n^*}^{N} m_2 f(0,i) \tag{7}$$

where

$$\begin{cases} m_1 &= \frac{P_0 c/q}{1-(1+c/q)^{-Nq}} \\ m_2 &= \frac{P^* r^*}{1-(1+r^*)^{-(N-n^*)}} \\ P^* &= \frac{m_1}{c/q}\left[1-(1+c/q)^{-(N-n^*)}\right] \\ r^* &= \frac{\theta_1}{q} + \left(\frac{r_0}{q} - \frac{\theta_1}{q}\right)\left(1+\frac{k_1}{q}\right)^{-n^*} + \frac{\sigma_1}{q}\left(1+\frac{k_1}{q}\right)^{-n^*} \sum_{i=1}^{n^*}\left(1+\frac{k_1}{q}\right)^{i}(W_i^1 - W_{i-1}^1) \\ f(0,i) &= \prod_{j=1}^{i}(1+f_j)^{-1} \\ f_j &= \frac{\theta_2}{q} + \left(\frac{f_0}{q} - \frac{\theta_2}{q}\right)\left(1+\frac{k_2}{q}\right)^{-j} + \frac{\sigma_2}{q}\left(1+\frac{k_2}{q}\right)^{-j} \sum_{l=1}^{j}\left(1+\frac{k_2}{q}\right)^{l}(W_l^2 - W_{l-1}^2) \end{cases} \tag{8}$$

and the sequences of $\{W_i^j, j = 1, 2 | W_0^j = 0, j = 1, 2\}$ are determined by

$$\begin{pmatrix} W_i^1 \\ W_i^2 \end{pmatrix} = \begin{pmatrix} W_{i-1}^1 \\ W_{i-1}^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix},$$

with $(u_i, v_i)^T$ being an i.i.d. sample from $N(0, \delta t)$. Here $f(0, i)$ is the accumulated discount rate from $t = 0$ to $t = i$. Once the n^* is found for a particular pair of paths, we can repeat the simulation for a given number of times. Statistically, a period with highest frequency of n^* is the optimal refinancing time to refinance. That is, let n^* be a function in c while all other parameters pertaining to the problem in equation (7) are fixed, we are to find the value of c such that

$$\sup_{c \leq \theta_1} \arg \mathrm{Mode}(n^*) = 0$$

Refer to Fig. 12 in Gan et al. (2012), where the initial borrowing rate is set as $c = 0.051$, with the corresponding parameters $\theta_1 = 0.05$, $\sigma_1 = 0.003$. That example clearly shows that when the initial borrowing borrowing rate is high enough compared to a long-term trend, a minimum total payment cost would occur sometime later than $t = 0$. Thus it is a better deal for the borrower to take the wait-and-see strategy. Once the optimisation problem is solved for a given initial borrowing rate c, one naturally arising question to ask is at what level of interest c is the optimal time of refinance, denoted as t^*, is 0. Equivalently, at what a low level of c the solution of the minimiser n^* in (7) actually $n^* = 0$? In our work, the search for the optimal threshold is done by a bisection scheme. For the special cases when (1) $\sigma_1 = 0$ and the discounting factor is identically 0; or (2) both $\sigma_i = 0$, $i = 1, 2$, it can be shown that a closed-form solution in the form of algebraic equations is attainable. In these cases, it can be numerically verified that t^*, the continuation of n^* when $N \to \infty$, is indeed a decreasing function of c for the interested scenarios (Gan et al. 2012).

5 Numerical Solution Based on Monte-Carlo Simulation

5.1 Aggregated Discounted Monthly Payments

In order to determine the optimal refinancing strategy, we follow the general motivation in Gan et al. (2012). In particular, given a series of mortgage rates r_t and risk-free rates f_t, we wish to find the refinancing month k which minimizes the present value of the payment stream M given by (5).

Taking the time value of money into consideration, the monthly payments m_1, m_2 in (3) and (6) need to be properly discounted. Since payments are made monthly, we discount a payment m made at month k first to month $k-1$, and then from month $k-1$ to month $k-2$, continuing on until we discount from month 1 to time 0. To generate the monthly risk-free discounting rate applied to the period from month k to month $k-1$, we take the arithmetic average of the rates f_t, $t_{k-1} \le t < t_k$, where the f_t are generated using (2). We denote this discounting rate by $\overline{f_{[\![k]\!]}}$.

In particular, let $t_k - t_{k-1} = q_k \Delta t$ (in other words, there are q_k days in month k). Then

$$\overline{f_{[\![k]\!]}} = \frac{1}{q_k} \sum_{i=0}^{q_k - 1} f_{t_{k-1} + i\Delta t}. \tag{9}$$

Therefore, at month k the time 0 discounting factor $F(0, k)$ is

$$F(0, k) = \prod_{i=1}^{k} \left[1 + \overline{f_{[\![i]\!]}}(t_i - t_{i-1}) \right] \tag{10}$$

where $\overline{f_i}$ is the averaged annualised discounting rates applied from month i to month $i-1$. Here $t_i - t_{i-1}$ is the fraction of a year occupied by month i.

If we use $F(0)^{-1} M_k$ to denote the present value at time 0 the monthly payments made under the two contracts when the refinancing takes place at month k, we have

$$F(0)^{-1} M_k = m_1 \sum_{i=1}^{k} \frac{1}{F(0, i)} + m_2 \sum_{i=k+1}^{N} \frac{1}{F(0, i)}, \tag{11}$$

where m_1 and m_2 are defined in (3) and (6).

5.2 Optimal Refinancing

Throughout the following discussion about optimal refinancing, we allow only one single refinancing to take place during the contract period. At month k, the borrower can choose to refinance all of his/her outstanding debt, or wait until a future month to do so. Hence we must compare the present value at time 0 of such monthly payment streams $\{F(0)^{-1}M_k, F(0)^{-1}M_{k+1}, \ldots, F(0)^{-1}M_N\}$. We say that refinancing at month j is *optimal* for the given paths of mortgage rates and risk-free rates if for $r_{[\![j]\!]} < r_0$,

$$F(0)^{-1}M_j = \min\left\{F(0)^{-1}M_k, F(0)^{-1}M_{k+1}, \ldots, F(0)^{-1}M_N\right\}.$$

In fact, in determining optimality we do not need to consider the monthly payments made before month k, as they have already been paid, and we need discount only to month k, when the decision is taking place. Hence as in (10), for a given sequence of risk-free rates we define the month k discounting factor $F(k, j)$ for a future month j with $j \geq k$ to be

$$F(k, j) = \begin{cases} 1, & j = k, \\ \prod_{i=k+1}^{j}\left[1 + \overline{f_{[\![i]\!]}}(t_i - t_{i-1})\right], & j > k. \end{cases} \tag{12}$$

If refinancing happens at month j, $j \geq k$, the present value at month k of the payment stream under the two contracts is given by

$$F(k)^{-1}M_j = m_1 \sum_{i=k}^{j} \frac{1}{F(k, i)} + m_2 \sum_{i=j+1}^{N} \frac{1}{F(k, i)}, \tag{13}$$

where monthly payments m_1 and m_2 are defined in (3) and (6). Therefore, for given paths of mortgage rates and risk-free rates, starting from month k, a refinancing at month j, $k \leq j \leq N$, with new mortgage rate $r_{[\![j]\!]}$, $r_{[\![j]\!]} < r_0$ is optimal if

$$F(k)^{-1}M_j = \min\left\{F(k)^{-1}M_k, F(k)^{-1}M_{k+1}, \ldots, F(k)^{-1}M_N\right\}. \tag{14}$$

5.3 Optimal Refinancing Threshold Rate

Now consider a borrower who wishes to decide whether to refinance at t_k (the end of month k). As discussed in Sect. 4.1, this decision will be based upon a consideration of the size $r_{[\![k]\!]}$. More precisely, the borrower would like to know if it is optimal to refinance at t_k in the sense of (14), where $r_{[\![k]\!]}$ enters into the problem *via* m_2. Unfortunately, this cannot be known deterministically because the F functions in (14) depend on future values of f_t.

Algorithm 1: Finding optimal refinancing threshold rate $r^{\mathrm{O}}_{[\![k]\!]}$ for month k and risk-free rate $f_{[\![k]\!]}$.

Input: Model parameters ($p_0, r_0, \theta_1, \theta_2, \kappa_1, \kappa_2$ (reversion rate), $\sigma_1, \sigma_2, \rho, N, \Delta t$), month k and risk-free rate $f_{[\![k]\!]}$, number n of sets of paths.

Output: Optimal refinancing threshold rate $r^{\mathrm{O}}_{[\![k]\!]}$ for month k and risk-free rate $f_{[\![k]\!]}$.

begin
- $r^{\mathrm{L}}_{[\![k]\!]} \leftarrow 0$ `// ` $r^{\mathrm{L}}_{[\![k]\!]}$ `is the lower bound of the searching range.`
- $r^{\mathrm{H}}_{[\![k]\!]} \leftarrow r_0$ `// ` $r^{\mathrm{H}}_{[\![k]\!]}$ `is the upper bound of the searching range.`
- $r^{\mathrm{O}}_{[\![k]\!]} \leftarrow 0$
- $P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]}) \leftarrow 0$
- **while** $(P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]}) < 90.2\%$ **or** $P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]}) > 90.4\%)$ **and** $(r^{\mathrm{H}}_{[\![k]\!]} - r^{\mathrm{L}}_{[\![k]\!]} > 0.00001)$
- **do**
  - $r^{\mathrm{O}}_{[\![k]\!]} \leftarrow (r^{\mathrm{L}}_{[\![k]\!]} + r^{\mathrm{H}}_{[\![k]\!]})/2$
  - Launch n sets of paths for period from month k to month N
  - Compute $P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]})$ by simulations
  - **if** $P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]}) < 90.2\%$ **then**
    - $r^{\mathrm{H}}_{[\![k]\!]} \leftarrow r^{\mathrm{O}}_{[\![k]\!]}$
  - **else if** $P(k, f_{[\![k]\!]}, r^{\mathrm{O}}_{[\![k]\!]}) > 90.4\%$ **then**
    - $r^{\mathrm{L}}_{[\![k]\!]} \leftarrow r^{\mathrm{O}}_{[\![k]\!]}$
- **return** $r^{\mathrm{O}}_{[\![k]\!]}$

However, using Monte Carlo techniques, we can compute a *probability* that it is optimal to refinance at the end of month k. In particular, given $\{f_{[\![k]\!]}, r_{[\![k]\!]}\}$ (which are known at $t = t_k$), we generate n sequences $\{f_t, r_t\}$ for $t_k \le t \le t_N$ using (2). Then we compute the number n_k of sequences such that

$$F(k)^{-1} M_k = \min\left\{F(k)^{-1} M_k,\, F(k)^{-1} M_{k+1}, \ldots, F(k)^{-1} M_N\right\}. \tag{15}$$

We then define $P(k, f_{[\![k]\!]}, r_{[\![k]\!]})$, the probability that it is optimal to refinance t_k given $\{f_{[\![k]\!]}, r_{[\![k]\!]}\}$, to be $P(k, f_{[\![k]\!]}, r_{[\![k]\!]}) = n_k/n$.

We wish to know the *threshold rate* r_* for optimal refinancing. We define r_* by stating that if $r_{[\![k]\!]} < r_*$, month k will most likely be optimal in the sense of $P(k, f_{[\![k]\!]}, r_{[\![k]\!]})$.

We find r_* using an algorithm motivated by Gan et al. (2012). In particular, we set the optimal probability range to be [90.2%, 90.4%]. If $P(k, f_{[\![k]\!]}, r_{[\![k]\!]})$ falls within this range, we say that $r_* = r_{[\![k]\!]}$. We use an iterative bisection algorithm to find r_*. Motivated by our discussion in Sect. 4.1, we set the initial interval to $[0, r_0]$. For a given month k and a risk-free rate $f_{[\![k]\!]}$, Algorithm 1 summarizes the bisection searching we use in determining the optimal threshold rate.

Table 1 Parameter meanings and their values used in the reported test

Parameter	Meaning	Value
p_0	Initial principal of the loan	100,000
r_0	Original contract mortgage rate	0.05
θ_1	Long-term mean value of the mortgage rate	0.05
κ_1	Mean-reverting rate of the mortgage rate	0.1
σ_1	Volatility of the mortgage rate	0.002
θ_2	Long-term mean value of the risk-free rate	0.03
κ_2	Mean-reverting rate of the risk-free rate	0.1
σ_2	Volatility of the risk-free rate	0.001
ρ	Correlation between the mortgage rate and the risk-free rate	0.8
N	Number of months in the contract mortgage period	240
n	Number of paths generated in the simulation to compute the probability	10,000
Δt	Time interval between two adjacent points on a generated path of interest rates	1/365

6 Numerical Output and Theoretical Calibrations

Although we did a number of simulation tests using different parameter settings, here we present the result from one of the tests. The meanings of the parameters and their values used in this test are summarised in Table 1. Figure 2 shows plots created from the reported data.

The three-dimensional surface in Fig. 2a shows the variation of r_* under different f_0 and refinancing months. The curves in Fig. 2b show that r_* increases as the refinance month moves towards the end of the contract period. On the other hand, for fixed k, the monotonicity of r_* as a function of f_0 is not apparent, as shown in Fig. 2c, d. Note that this result is obtained under the assumptions we made in Sect. 3.1. If transaction costs had been considered, the shape of the curves would be different.

In particular, when transaction costs are included, early refinancing may be no longer a general optimal choice for the case 3 discussed in Sect. 4.1. In addition, the monotonicity of r_* as a function of N as indicated by Fig. 2b may no longer hold when transaction costs are accounted for. As shown by the curves, the initial risk-free rate f_0 has impact on r_* only when refinancing takes place early in the contract period. When refinancing takes place close to the end of the period, f_0 has almost no effect on r_*. This pattern is demonstrated more clearly by the plots in Fig. 2c, d, where the refinancing takes place at month 1 and month 217, respectively. To make a fair comparison, in both Fig. 2c, d we set the r_*-range equal to 0.005. Fig. 2c shows that, roughly, r_* increases as f_0 increases. But f_0 almost has no effect on r_* in Fig. 2d.

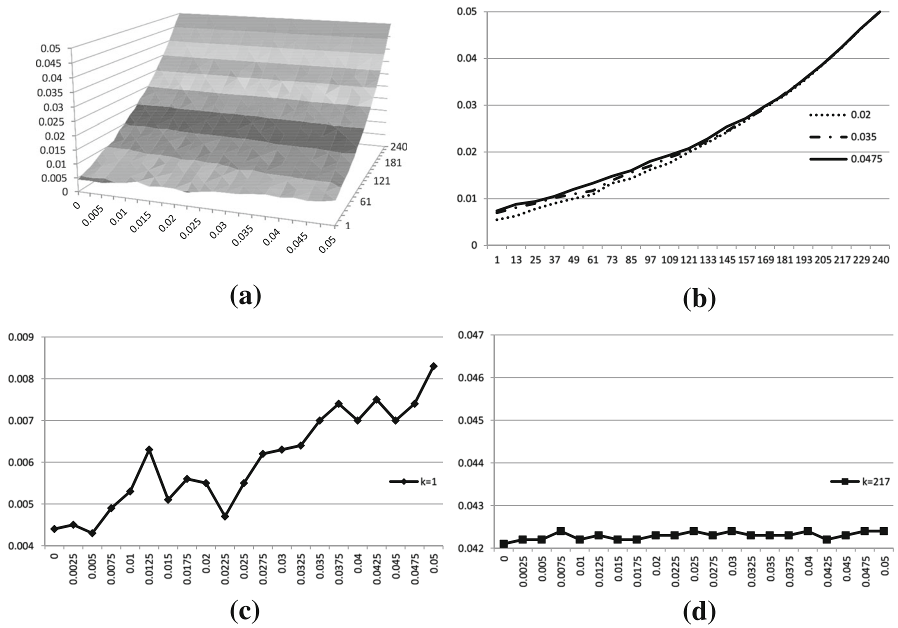

Fig. 2 Optimal refinancing threshold rate plots from the reported data. **a** Threshold rate versus f_0 and k. **b** Threshold rate versus N with different f_0. **c** Threshold rate versus f_0 when $k = 1$. **d** Threshold rate versus f_0 when $k = 217$

7 Conclusion

This work examines the mortgage borrower's optimal refinancing strategy under the restriction that only one refinancing opportunity is allowed across the duration of a mortgage loan. Using Monte Carlo simulations we find the optimal refinancing time is more likely to appear at the early stage of the contract, if such an optimal time exists at all. Optimal refinancing curves as a function in time are generated, the properties of which are analysed and interpreted financially.

The current paper overcomes several weaknesses seen in the earlier treatment of a similar problem in Gan et al. (2012). In addition to the theoretical fortifications in economic analysis and numerical enhancement, the current work provides a complete and rigorous stochastic optimization formulation of the problem, including the generalisation of the one-dimensional problem to a more business realistic two-dimensional problem in f_t and r_t. One of the possible future directions is to boost the speed and efficiency of the Monte Carlo simulation by implementing various error reduction techniques. In addition, it is worthwhile to attempt the cases where a refinancing fee is charged or multiple refinancings are allowed. It is anticipated, for instance, that an early refinancing may no longer be an optimal choice in general, especially when the transaction fee is charged in proportion to the remaining loan balance.

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