A Model for Venous Blood Flow in the Legs

IMA Summer Program for Graduate Students
Mathematical Modeling

Week 1, Group 1
August 3-7, 1992

Mary Brewster, Tutor

Theresa A. Bright
David A. Edwards
Lori Freitag
Cristina Greig
Marcus Grote
Scott James
Rahul Kulkarni
Arthur Ralfs
Lynn Schreyer Bennethum
Volker Schulz

Section I: Introduction and Assumptions

Introduction

Blood flow through the veins in the leg is driven by the external pressure of muscles contracting. This pressure acts to collapse the vein, forcing the blood up the vein back towards the heart. When a person stands still, there is little contraction and valves are needed within the leg to prevent blood from rushing back down the length of the leg and "pooling" at the feet. This could be lethal within a matter of a few minutes or a few hours [**2**]. The average human being has fifteen valves in the leg to aid in circulation by preventing blood from going very far down the leg, so that even small muscle contractions are sufficient to maintain circulation. It is possible, however, for these valves to become damaged. The latest medical tools available are artificial valves that could be implanted in their place, and this model has been created to assist surgeons in determining how many valves should be implanted along a vein in the leg to attain an acceptable level of circulatory efficiency.

Assumptions

Due to the variety of variables involved and the time constraint of the seminar, assumptions had to be made involving various physical properties of this system, the leg. Probably the largest assumption is that we only need consider one-dimensional flow. This is reasonable since compared to the amount of blood moving up and down the leg, there is actually very little motion that is not parallel to the walls. Since we are looking at a vein, blood flow can be considered laminar and axisymmetric [**1**]. The flow of blood through the capillaries has damped out the pulses found in arteries. We shall also only consider the effects of contractions of the thigh and calf muscles since they are the largest muscles in the leg. A standard contraction will be parabolic in shape (smooth, even contraction) and we will leave the strength and length of contractions and time between contractions to be random, therefore including patients with differing levels of activity. We shall assume a maximum pressure value in the foot of 90 mmHg when a person is standing completely still and all valves are open [**2**]. For computing efficiency, pressure within the vein of 150 mmH_2O will be considered optimal [**1**].

As a starting point, we assume that the defect lies in a single major vein which runs vertically through the length of the leg from the ankle to the hip. This vein is considered to be a collapsible tube which carries approximately a third of the blood out of the leg. We will assume a distribution of capillaries along the length of the leg, proportional to the muscle tissue mass. The local cross-sectional area of the vein is assumed to be a function of distance from the foot and time. Longitudinal uniformities in the vein wall are assumed to be such that a local "tube law" is applicable at each cross-section [**4**]. The particular form of the tube law that we use was motivated by experimental data and the requirement that a zero transmural pressure difference corresponds to the neutral cross-sectional area. An additional assumption made was that the local cross-sectional area cannot exceed the neutral value by more than a factor of two.

To close off the system, one valve was placed below the ankle and one at the hip. The behaviour of each valve is independent of the others and we will assume there is no breakage. Each valve will be considered to be completely open or completely closed. Readings show the density of blood hardly varies, implying incompressibility. Due to the high shear rate within the blood flow, we will assume blood acts as a Newtonian fluid [5], so that the standard Navier-Stokes equations can be used.

Section II: Nomenclature

In order to model our problem of venous blood flow in the leg, we begin by modeling the rest state as a cylinder. We then introduce the following quantities:

$a_{\pm}$: Riemann invariants of the hyperbolic system.

$\tilde{A}(\tilde{z}, \tilde{t})$: cross-sectional area of the vein at height $\tilde{z}$ and time $\tilde{t}$. To be directly calculated from equations. Units cm^2.

A_0: rest cross-sectional area of the vein, value $\pi/4$ cm^2.

B: matrix operator used in computer simulation.

$c_{\pm}$: eigenvalues of B.

$\mathbf{F}$: vector used in computer simulation.

$\tilde{g}_P(|\tilde{P}_{\text{ext}}|)$, $\tilde{g}_d(\tilde{t}_d)$, $\tilde{g}_{\Delta}(\Delta\tilde{t}_i)$: probability density functions used to determine argument parameters.

$\tilde{G}_P(|\tilde{P}_{\text{ext}}|)$, $\tilde{G}_d(\tilde{t}_d)$, $\tilde{G}_{\Delta}(\Delta\tilde{t}_i)$: probability mass functions used to determine argument parameters.

k: conversion factor used in dimensional tube law, value 1 mmHg.

l: length of the vein, value 100 cm.

$\tilde{p}(\tilde{z}, \tilde{t})$: internal pressure on venous wall caused by fluid dynamics inside vein. To be calculated directly from equations. Units $\text{g}\cdot\text{cm}/\text{sec}^2$.

$\tilde{p}_a$: ankle pressure (used for efficiency calculations), value 150 mmH_2O.

$\tilde{P}_{\text{ext}}(\tilde{z}, \tilde{t})$: external pressure on venous wall caused by muscular contraction. Units $\text{g}\cdot\text{cm}/\text{sec}^2$.

$\tilde{p}_h$: hip pressure boundary condition (needed for computational purposes), value 111 mmH_2O.

$\tilde{q}$: nonmuscular influx density of blood through capillaries. Units cm^2/sec.

$\tilde{Q}_c(\tilde{z})$: influx density of blood through capillaries. Units cm^2/sec. Given input function.

$\tilde{q}_f$: maximum amplitude of foot blood influx. Units cm^3/sec.

$\tilde{Q}_f(\tilde{t})$: influx of blood through valve at foot as a result of foot pumping action. Units cm^3/sec. Given input function.

$\tilde{Q}_r(\tilde{z})$: average influx density of blood, namely

$$\frac{1}{l}\int_0^l \tilde{Q}_c(\tilde{z})\,d\tilde{z}.$$

$\tilde{r}$: radial measurement of vein interior. Units: cm.

R: rest radius of the vein, value 0.5 cm.

$\tilde{t}$: time. Units: sec.

$\tilde{t}_d$: duration of muscular compression. Units: sec.

$\tilde{t}_i$: time at which ith muscular compression begins. Units: sec.

$\tilde{T}_i$: time at which ith foot input begins. Units: sec.

$\tilde{u}(\tilde{z},\tilde{t})$: *averaged* vertical velocity of blood at height $\tilde{z}$ and time $\tilde{t}$, namely

$$\frac{1}{\tilde{A}(\tilde{z},\tilde{t})}\int_{\tilde{A}(\tilde{z},\tilde{t})} U(\tilde{r},\tilde{z},\tilde{t})\,d\tilde{A}.$$

$\tilde{U}(\tilde{r},\tilde{z},\tilde{t})$: vertical velocity of blood at radius $\tilde{r}$, height $\tilde{z}$ and time $\tilde{t}$.

$\mathbf{x}_{\pm}$: left eigenvectors of B.

$\mathbf{y}$: vector of nondimensionalized area and velocity.

$\tilde{z}$: height above foot valve input level. Units: cm.

$\tilde{z}_i$: height above foot valve input level of ankle ($i = 1$), top part of calf ($i = 2$), lower part of thigh ($i = 3$), and upper part of thigh ($i = 4$). Units: cm.

α: nondimensional parameter used in equations:

$$\alpha = \frac{lQ_r^2}{gA_0^2}.$$

β: nondimensional paramenter used in equations:

$$\beta = \frac{8\pi l Q_r \nu}{gA_0^2}.$$

γ: derivative of the nondimensional tube law with respect to nondimensionzalized area.

ν: viscosity of blood, value 0.006 cm^2/sec.

ρ: density of blood, value 1 g/cm^3.

Nondimensionalized variables will have no tildes.

Section III: Governing Equations

The governing equations used are:

1. the tube law:

$$\tilde{P}_{ext}(\tilde{z},\tilde{t}) - \tilde{p}(\tilde{z},\tilde{t}) = k\left(\frac{1}{A^{3/2}} - \frac{1}{2-A}\right) \tag{3.1}$$

2. the conservation of mass (continuity equation):

$$\text{flow in} = \text{increase in volume} + \text{flow out} \tag{3.2}$$

3. the Navier-Stokes equation for an incompressible fluid with no bulk viscosity:

$$\frac{\partial \mathbf{v}}{\partial \tilde{t}} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\tilde{p} + \nu\nabla^2\mathbf{v} + \mathbf{g}. \tag{3.3}$$

The conservation of mass on an infinitesimal slice is:

$$\int\limits_{\tilde{A}(\tilde{z},\tilde{t})} \tilde{U}(\tilde{r},\tilde{z},\tilde{t})\,d\tilde{A} + \tilde{Q}_c(\tilde{z})\Delta\tilde{z} = \frac{\partial\tilde{A}(\tilde{z},\tilde{t})}{\partial\tilde{t}}\Delta\tilde{z} + \int\limits_{\tilde{A}(\tilde{z}+\Delta\tilde{z},\tilde{t})} \tilde{U}(\tilde{r},\tilde{z}+\Delta\tilde{z},\tilde{t})\,d\tilde{A}. \tag{3.4}$$

Dividing by $\Delta\tilde{z}$ and letting $\Delta\tilde{z}$ tend to zero yields

$$\frac{\partial}{\partial\tilde{z}}\left[\int\limits_{\tilde{A}(\tilde{z},\tilde{t})} U(\tilde{r},\tilde{z},\tilde{t})\,d\tilde{A}\right] + \frac{\partial\tilde{A}}{\partial\tilde{t}} = \tilde{Q}_c(\tilde{z})$$

and by the definition of $\tilde{u}$ we get

$$\frac{\partial(\tilde{A}\tilde{u})}{\partial\tilde{z}} + \frac{\partial\tilde{A}}{\partial\tilde{t}} = \tilde{Q}_c(\tilde{z}). \tag{3.5}$$

Assuming the velocity field has the form

$$\mathbf{v} = 0\mathbf{e}_{\tilde{r}} + 0\mathbf{e}_{\theta} + \tilde{U}(\tilde{r},\tilde{z},\tilde{t})\mathbf{e}_{\tilde{z}}$$

the Navier-Stokes equation becomes

$$\left(\frac{\partial\tilde{U}}{\partial\tilde{t}} + \tilde{U}\frac{\partial\tilde{U}}{\partial\tilde{z}}\right) = -\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial\tilde{z}} + \nu\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{U}}{\partial\tilde{r}}\right) + \frac{\partial^2\tilde{U}}{\partial\tilde{z}^2}\right] - g. \tag{3.6}$$

To eliminate the dependence of the solution on $\tilde{r}$, we assume that the velocity profile at any time is not too different from that in a horizontal cylinder, *e.g.*,

$$\tilde{U}(\tilde{r},\tilde{z},\tilde{t}) = 2\left[1-\left(\frac{\tilde{r}}{R}\right)^2\right]\tilde{u}(\tilde{z},\tilde{t})$$

where the velocity profile has been normalized so that the "mean" velocity is $\tilde{u}(\tilde{z},\tilde{t})$ (see nomenclature). This estimate for a profile is rough and empirical data could provide a more accurate profile.

Using our parabolic profile, the Navier-Stokes equation becomes

$$2\left[1-\left(\frac{\tilde{r}}{R}\right)^2\right]\left\{\frac{\partial \tilde{u}}{\partial \tilde{t}} + 2\left[1-\left(\frac{\tilde{r}}{R}\right)^2\right]\tilde{u}\frac{\partial \tilde{u}}{\partial z}\right\}$$
$$= -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{z}} + \nu\left\{-\frac{8}{R^2}\tilde{u} + 2\left[1-\left(\frac{\tilde{r}}{R}\right)^2\right]\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}\right\} - g.$$

Averaging the equation over the rest cross-sectional area, e.g. integrating over a disk of radius R and dividing by πR^2, yields the final form of the Navier Stokes equation:

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{4}{3}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{z}} - \frac{8}{R^2}\nu\tilde{u} + \nu\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} - g. \tag{3.7}$$

Now we proceed to nondimensionalize our equations. To normalize $\tilde{z}$, we choose l, the length of our vein. To normalize $\tilde{u}$, we calculate the velocity of the fluid given only the capillary input, which we call u_0:

$$u_0 = \frac{1}{A_0}\int_0^l \tilde{Q}_c(\tilde{z})\,d\tilde{z}.$$

Defining

$$Q_r = \frac{1}{l}\int_0^l \tilde{Q}_c(\tilde{z})\,d\tilde{z},$$

which is the average value of $\tilde{Q}_c$ along the vein, we have

$$u_0 = \frac{lQ_r}{A_0}.$$

Since we are concerned with the velocity in the $\tilde{z}$-direction, we normalize $\tilde{t}$ by l/u_0. We normalize $\tilde{A}$ by A_0, which is the area when the vein is at rest. Summarizing, we have the following:

$$z = \frac{\tilde{z}}{l}, \qquad u(z,t) = \frac{\tilde{u}(\tilde{z},\tilde{t})A_0}{lQ_r}, \qquad t = \frac{\tilde{t}Q_r}{A_0}, \qquad A = \frac{\tilde{A}}{A_0}. \tag{3.8}$$

We normalize pressure by the maximum hydrostatic pressure of the vein with no valves:

$$P_{\text{ext}}(z,t) = \frac{\tilde{P}_{\text{ext}}(\tilde{z},\tilde{t})}{\rho g l}, \qquad p(z,t) = \frac{\tilde{p}(\tilde{z},\tilde{t})}{\rho g l}$$

$$p_a = \frac{\tilde{p}_a}{\rho g l} = 0.15, \qquad p_h = \frac{\tilde{p}_h}{\rho g l} = 0.11. \tag{3.9}$$

Now normalizing equation (3.5) gives us the following:

$$\frac{A_0 \cdot l Q_r}{A_0 \cdot l}\frac{\partial (Au)}{\partial z} + \frac{A_0 \cdot Q_r}{A_0}\frac{\partial A}{\partial t} = \tilde{Q}_c(z)$$

$$\frac{\partial (Au)}{\partial z} + \frac{\partial A}{\partial t} = \frac{\tilde{Q}_c(z)}{Q_r} \equiv Q_c(z). \tag{3.10}$$

Normalizing equation (3.7), we have

$$\frac{lQ_r \cdot Q_r}{A_0 \cdot A_0}\frac{\partial u}{\partial t} + \frac{4l^2 Q_r^2}{3A_0^2 l} u \frac{\partial u}{\partial z} = -\frac{\rho g l}{\rho l}\frac{\partial p}{\partial z} - \frac{8lQ_r \cdot \pi}{A_0 \cdot A_0}\nu u + \frac{\nu l Q_r}{A_0 \cdot l^2}\frac{\partial^2 u}{\partial z^2} - g$$

$$\alpha\left(\frac{\partial u}{\partial t} + \frac{4}{3}u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} - \beta u + \frac{\nu Q_r}{g l A_0}\frac{\partial^2 u}{\partial z^2} - 1, \tag{3.11}$$

where α and β are defined in the nomenclature. Lastly, normalizing the "tube law:"

$$P_{\text{ext}}(z,t) - p(z,t) = \frac{(1.3332 \times 10^3 \text{g})}{\rho g l \cdot \text{cm} \cdot \text{sec}^2}\left(\frac{1}{A^{3/2}} - \frac{1}{2-A}\right)$$

$$P_{\text{ext}}(z,t) - p(z,t) = 1.36 \times 10^{-2}\left(\frac{1}{A^{3/2}} - \frac{1}{2-A}\right). \tag{3.12}$$

We set our boundaries as follows:

- $z = 0$: valve at foot
- $0 < z < z_1$: no contraction of muscle (foot to ankle)
- $z_1 < z < z_2$: contraction possible from calf muscle
- $z_2 < z < z_3$: no contraction of muscle (knee)
- $z_3 < z < z_4$: contraction possible from thigh muscle
- $z_4 < z < 1$: no contraction of muscle (above thigh)
- $z = 1$: valve at thigh.

Section IV: Input

In order to continue our analysis further, we must determine the input functions $\tilde{P}_{\text{ext}}$, $\tilde{Q}_f$, and $\tilde{Q}_c$. In addition, we must determine the $\tilde{z}_i$. Taking measurements from an average person, we have

$$\tilde{z}_1 = 8 \text{ cm}, \qquad \tilde{z}_2 = 40 \text{ cm}, \qquad \tilde{z}_3 = 54 \text{ cm}, \qquad \tilde{z}_4 = 92 \text{ cm}.$$

Normalizing by l, we immediately have the following:

$$z_1 = 0.08, \qquad z_2 = 0.4, \qquad z_3 = 0.54, \qquad z_4 = 0.92. \tag{4.1}$$

First we consider Q_r. We have found that the aorta, which has an area of 2.5 cm^2, pumps blood at the rate of 33 cm/sec. Since all blood passes through the aorta, we then have a flux rate for the entire of the system. We estimate that 1/6 of the total blood is found in the leg, and that 10% of that amount is in the foot. Hence, we have a flux rate for the leg.

$$\frac{2.5 \text{ cm}^2 \cdot 33 \text{ cm} \cdot 90\%}{\text{sec} \cdot 6} = 12.4 \text{ cm}^3/\text{sec}$$

We assume a system of three veins traversing the entire leg. We assume the blood flows into them equally, so to find our *average* density, we divide by the length of the leg to find

$$Q_r = \frac{12.4 \text{ cm}^3}{100 \text{ cm} \cdot 3} = 4.12 \times 10^{-2} \text{ cm}^2/\text{sec}.$$

Hence we have $Q_r/A_0 = 5.25 \times 10^{-2} \text{ sec}^{-1}$.

Next we consider $\tilde{P}_{\text{ext}}$. Since the muscle tissue is approximately quadratically distributed throughout the muscle, we assume that the pressure exerted by the muscle is quadratically distributed in space. Also, we assume that the pressure is quadratically distributed in time as well, since it does take some "warm-up" and "cool-down" time for the muscle. Hence, we have the following model:

$$\tilde{P}_{\text{ext}} = \frac{16|\tilde{P}_{\text{ext}}|}{\tilde{t}_d^2(\tilde{z}_i - \tilde{z}_{i+1})^2}(\tilde{z}-\tilde{z}_i)(\tilde{z}_{i+1}-\tilde{z})(\tilde{t}-\tilde{t}_j)(\tilde{t}_j+\tilde{t}_d-\tilde{t}), \quad \tilde{z}_i < \tilde{z} < \tilde{z}_{i+1}, \quad \tilde{t}_j < \tilde{t} < \tilde{t}_j+\tilde{t}_d.$$

Here $i = 1$ or 3, and $j = 1, 2, 3, \ldots$. Note that the peak value of this function comes at the middle of the muscle halfway through the duration $\tilde{t}_d$, and has value $|\tilde{P}_{\text{ext}}|$. Note also that since all of our length and time measurements appear in ratios, we may immediately nondimensionalize with only our pressure nondimensionalization, which yields the following:

$$P_{\text{ext}} = \frac{16|P_{\text{ext}}|}{t_d^2(z_i - z_{i+1})^2}(z-z_i)(z_{i+1}-z)(t-t_j)(t_j+t_d-t), \quad z_i < z < z_{i+1}, \quad t_j < t < t_j+t_d, \tag{4.2}$$

where $|P_{\text{ext}}| = |\tilde{P}_{\text{ext}}|/\rho g l$.

Now we must determine the other parameters in the equation. We are given in [5] that pressures in the muscle over 200 mmHg are common. We assume that the thigh and calf muscles work independently. We want no negative pressures exerted and we want maximal pressures of no more than 400 mmHg, so we assume a pressure amplitude probability density distribution of the following form:

$$\tilde{g}_P(|\tilde{P}_{\text{ext}}|) = \frac{6|\tilde{P}_{\text{ext}}|(400 \text{ mmHg} - |\tilde{P}_{\text{ext}}|)}{(400 \text{ mmHg})^3}, \quad 0 \text{ mmHg} < |\tilde{P}_{\text{ext}}| < 400 \text{ mmHg}.$$

Note the above has a maximum at 200 mmHg. For the computer implementation, it will be more helpful to have the probability mass function, which is

$$\tilde{G}_P(|\tilde{P}_{\text{ext}}|) = \frac{2|\tilde{P}_{\text{ext}}|^2(600 \text{ mmHg} - |\tilde{P}_{\text{ext}}|)}{(400 \text{ mmHg})^3}, \quad 0 \text{ mmHg} < |\tilde{P}_{\text{ext}}| < 400 \text{ mmHg}.$$

Nondimensionalizing, we have the following:

$$G_a(|P_{\text{ext}}|) = 1.25 \times 10^{-2}|P_{\text{ext}}|^2(8.15 - |P_{\text{ext}}|), \quad 0 < |P_{\text{ext}}| < 5.43. \tag{4.3}$$

For the probability functions that determine our input conditions, we select a random number from between 0 and 1. This is the value of the probability mass function, since we are drawing from a uniform distribution. We then set our equations corresponding to the probability mass function for the desired quantity equal to our random number. We use Newton's method to solve for the root.

Next we work on the actual duration of the compression, $\tilde{t}_d$. We assume that the average compression can take no less than 0.5 sec, and probably averages around 8 sec. Hence, we assume the following exponential distribution:

$$\tilde{g}_d(\tilde{t}_d) = \frac{\tilde{t}_d - 0.5 \text{ sec}}{(7.5 \text{ sec})^2} \exp\left(-\frac{\tilde{t}_d - 0.5 \text{ sec}}{7.5 \text{ sec}}\right), \quad \tilde{t}_d > 0.5 \text{ sec}.$$

Once again, we find the probability mass function:

$$\tilde{G}_d(\tilde{t}_d) = 1 - \frac{\tilde{t}_d + 7 \text{ sec}}{7.5 \text{ sec}} \exp\left(-\frac{\tilde{t}_d - 0.5 \text{ sec}}{7.5 \text{ sec}}\right), \quad \tilde{t}_d > 0.5 \text{ sec}.$$

Nondimensionalizing, we have the following:

$$G_d(t_d) = 1 - \frac{t_d + 3.68 \times 10^{-1}}{3.94 \times 10^{-1}} \exp\left(-\frac{\tilde{t}_d - 2.62 \times 10^{-2}}{3.94 \times 10^{-1}}\right), \quad t_d > 2.62 \times 10^{-2}. \tag{4.4}$$

For the duration between the compressions $\Delta\tilde{t}_j \equiv \tilde{t}_j - \tilde{t}_{j-1}$, we assume that it takes at least 1 second for the body to recover. Once again we assume a mean of 8 seconds, which comes from the model of someone slowly shifting from leg to leg. Hence we once again use an exponential distribution:

$$\tilde{g}_\Delta(\Delta\tilde{t}_j) = \frac{\Delta\tilde{t}_j - 1 \text{ sec}}{49 \text{ sec}^2} \exp\left(-\frac{\Delta\tilde{t}_j - 1 \text{ sec}}{7 \text{ sec}}\right), \quad \Delta\tilde{t}_j > 1.$$

Once again we find the probability mass function, which is

$$\tilde{G}_\Delta(\Delta\tilde{t}_j) = 1 - \frac{\Delta\tilde{t}_j + 6 \text{ sec}}{7 \text{ sec}} \exp\left(-\frac{\Delta\tilde{t}_j - 1 \text{ sec}}{7 \text{ sec}}\right), \quad \Delta\tilde{t}_j > 1.$$

Nondimensionalizing, we have

$$G_\Delta(\Delta t_j) = 1 - \frac{\Delta t_j + 3.15 \times 10^{-1}}{3.68 \times 10^{-1}} \exp\left(-\frac{\Delta t_j - 5.25 \times 10^{-2}}{3.68 \times 10^{-1}}\right), \quad \Delta t_j > 5.25 \times 10^{-2}. \tag{4.5}$$

Next we work with $\tilde{Q}_c(\tilde{z})$. We postulate that there is some sort of steady capillary flow in the entire leg. We assume that the muscles exert more force in the center in some sort of quadratic fashion (see below), so since capillaries flow to muscle tissue, we should expect some sort of quadratic function rising from that steady flow. Since the surface area of the thigh is approximately 3 times that of the calf, we expect that the peak flow in the thigh will be 3 times that of the peak flow in the calf, which we postulate to be about 10 times that of the non-muscular areas.

Using this approach, we see that we have the following problems to consider. Letting $\tilde{q}$ be the nonmuscular capillary flow, we have the following equation:

$$\tilde{q}l + \int_{\tilde{z}_1}^{\tilde{z}_2} \frac{36\tilde{q}(\tilde{z} - \tilde{z}_1)(\tilde{z}_2 - \tilde{z})}{(\tilde{z}_1 - \tilde{z}_2)^2}\, d\tilde{z} + \int_{\tilde{z}_3}^{\tilde{z}_4} \frac{116\tilde{q}(\tilde{z} - \tilde{z}_3)(\tilde{z}_4 - \tilde{z})}{(\tilde{z}_3 - \tilde{z}_4)^2}\, d\tilde{z} = Q_r l.$$

Nondimensionalizing immediately, we have that

$$1 + 6(z_2 - z_1) + \frac{58(z_4 - z_3)}{3} = \frac{Q_r}{\tilde{q}}.$$

Using our numbers, we have

$$\frac{\tilde{q}}{Q_r} = \frac{1}{10.27} = 9.74 \times 10^{-2}.$$

This is our nondimensionalized basic flux rate. Hence, we have the following functional form for the normalized flux density $Q_c(z)$:

$$Q_c(z) = \begin{cases} 9.74 \times 10^{-2}, & 0 \le z \le 0.08 \\ 9.74 \times 10^{-2} + 34.24(z - 0.08)(0.40 - z), & 0.08 \le z \le 0.40 \\ 9.74 \times 10^{-2}, & 0.40 \le z \le 0.54 \\ 9.74 \times 10^{-2} + 78.25(z - 0.54)(0.92 - z), & 0.54 \le z \le 0.92 \\ 9.74 \times 10^{-2}, & 0.92 \le z \le 1. \end{cases} \tag{4.6}$$

Lastly, we work with $\tilde{Q}_f(\tilde{t})$. Using the arguments above when we found Q_r, we have a flow rate into the foot:

$$\frac{2.5\ \text{cm}^2 \cdot 33\ \text{cm} \cdot 10\%}{\text{sec} \cdot 6} = 1.38\ \text{cm}^3/\text{sec}.$$

We consider the foot to be a reservoir from which we can always siphon a small fraction of its blood. We assume *average* spacing of 8 seconds between steps to get an average volume of blood that should be pumped out of the foot in one step. Now, there are 3 veins, so we have

$$\frac{1.38\ \text{cm}^3 \cdot 8}{\text{sec} \cdot 3} = 3.67\ \text{cm}^3.$$

We hypothesize that an average step lasts for approximately 1/2 second. We also postulate a parabolic shape for the rate. Hence, letting $\tilde{q}_f$ be some constant, we require that

$$\int_0^{0.5\ \text{sec}} \frac{16\tilde{q}_f\tilde{t}(0.5\ \text{sec} - \tilde{t})}{\text{sec}^2} = 3.67\ \text{cm}^3$$

Using our normalization, we have

$$\frac{\tilde{q}_f \cdot \text{sec}}{3lQ_r} = \frac{3.67\ \text{cm}^3 \cdot \text{sec}}{1.236 \times 10^{-1}\ \text{cm}^2 \cdot 100\ \text{cm}}$$

$$\frac{\tilde{q}_f}{lQ_r} = 0.297.$$

This once again gives our normalized flow rate. Then our normalized function becomes the following:

$$Q_f(t) = 4.75(t - T_i)(0.5 + T_i - t), \tag{4.7}$$

where T_i $(i = 1, 2, 3, \ldots)$ is the start time of the ith pump. To model the ΔT_i, we use the same function as for the Δt_i, since we are assuming shifting from leg to leg, which would not change the lag time. Hence we have

$$G_T(\Delta T_i) = 1 - \frac{\Delta T_i + 3.15 \times 10^{-1}}{3.68 \times 10^{-1}} \exp\left(-\frac{\Delta T_i - 5.25 \times 10^{-2}}{3.68 \times 10^{-1}}\right), \quad \Delta T_i > 5.25 \times 10^{-2}. \tag{4.8}$$

Now that we have calculated our parameters, we may actually attach values to equation (3.11). Doing so, we have the following:

$$\frac{100\ \text{cm} \cdot (4.12 \times 10^{-2}\ \text{cm}^2)^2 \cdot \text{sec}^2 \cdot 256}{\text{sec}^2 \cdot 981\ \text{cm} \cdot \pi^4 \text{cm}^4}\left(\frac{\partial u}{\partial t} + \frac{4}{3}u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z}$$
$$- \frac{8\pi \cdot 100\ \text{cm} \cdot 4.12 \times 10^{-2}\ \text{cm}^2 \cdot 6 \times 10^{-3}\ \text{cm}^2 \cdot \text{sec}^2 \cdot 256}{\text{sec}^2 \cdot 981\ \text{cm} \cdot \cdot \pi^4\ \text{cm}^4} u$$
$$+ \frac{6 \times 10^{-3}\ \text{cm}^2 \cdot 4.12 \times 10^{-2}\ \text{cm}^2 \cdot 16}{981\ \text{cm} \cdot 100\ \text{cm} \cdot \pi^2\ \text{cm}^2} \frac{\partial^2 u}{\partial z^2} - 1$$

$$4.55 \times 10^{-4} \left(\frac{\partial u}{\partial t} + \frac{4}{3} u \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial z} - 1.66 \times 10^{-3} u + 4.09 \times 10^{-9} \frac{\partial^2 u}{\partial z^2} - 1. \qquad (4.9)$$

As a first guess, we consider the diffusive term in equation (4.9) to be insignificant, so we have the following equation:

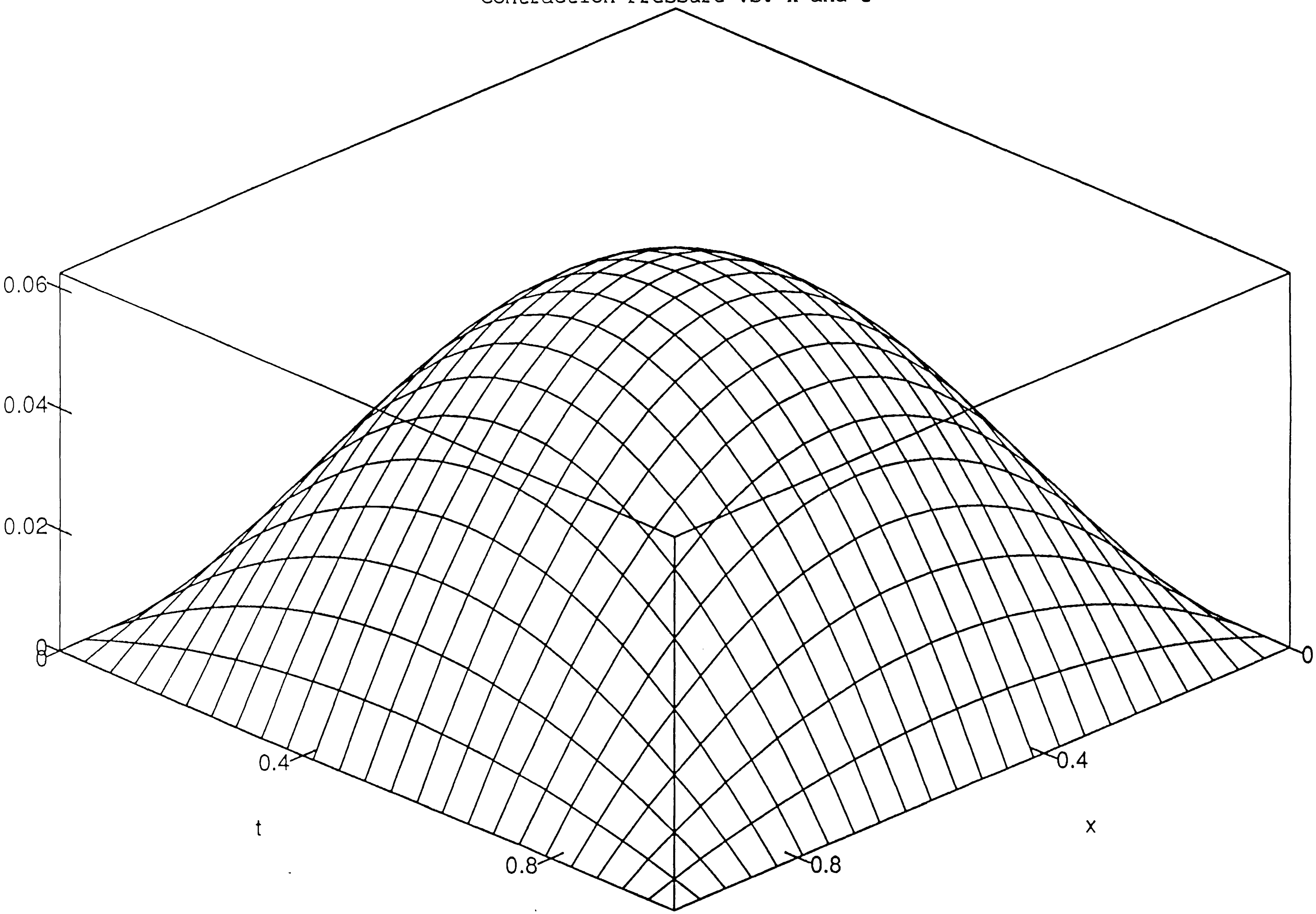
$$4.55 \times 10^{-4} \left(\frac{\partial u}{\partial t} + \frac{4}{3} u \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial z} - 1.66 \times 10^{-3} u - 1. \qquad (4.10)$$

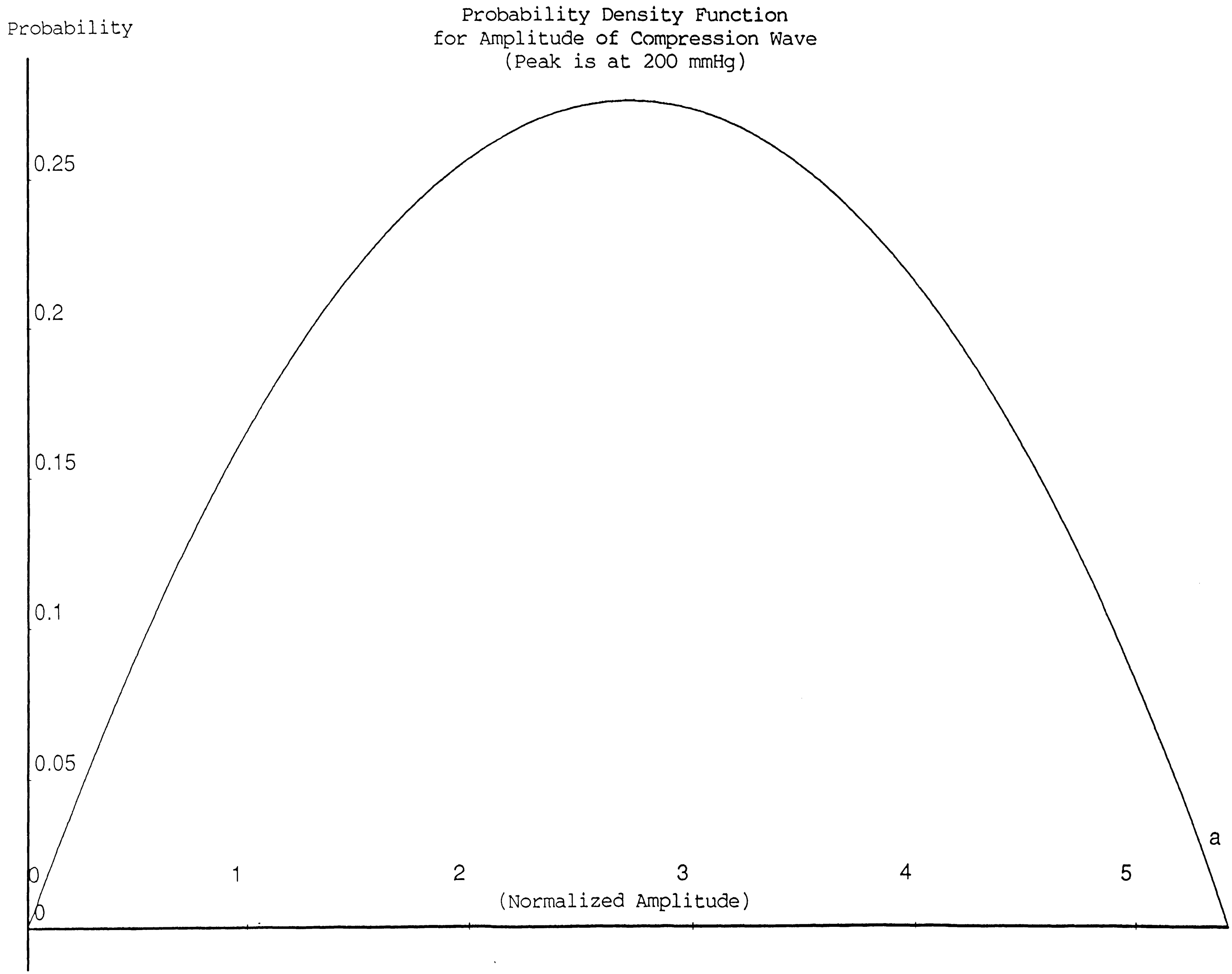
The system of three equations [(3.10), (3.12), and (4.10)] with their attendant input functions is what we are trying to solve.

Lastly, we have the conditions at the valves. If a valve is closed, trivially $u = 0$ above and below it. This is in essence two conditions. If a valve is open, both u and P must be continuous across it. Once again, we have two conditions.

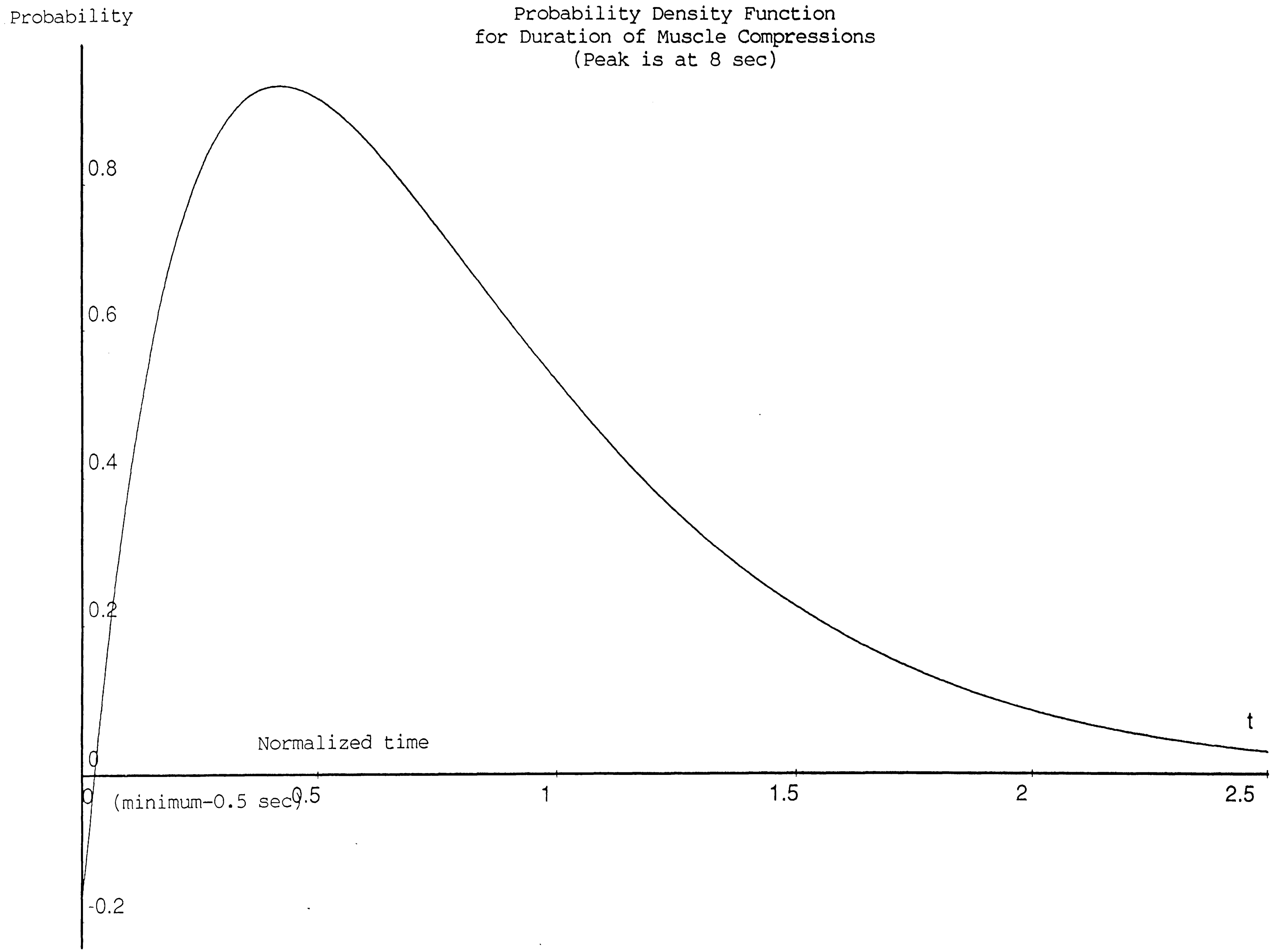
To determine if an open valve should be closed, we check the value of u at the valve. If it is less than 0, we close the valve, since we want no backflow. To determine if a closed valve should be opened, we check the pressure below and above the valve. If the pressure below the valve is greater than the pressure above the valve, then we open it.

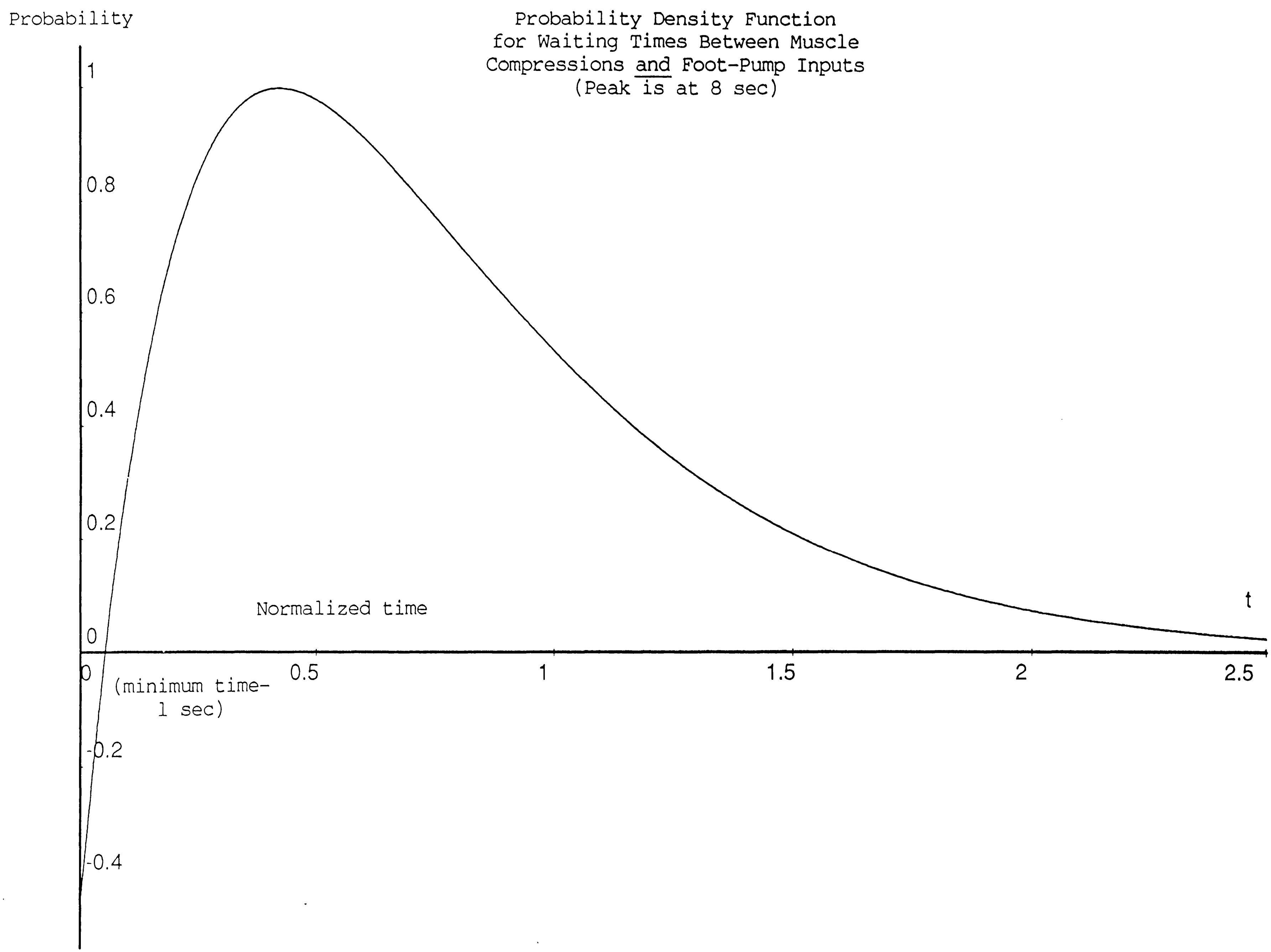
Approximate Shape of Muscle
Contraction Pressure vs. x and t

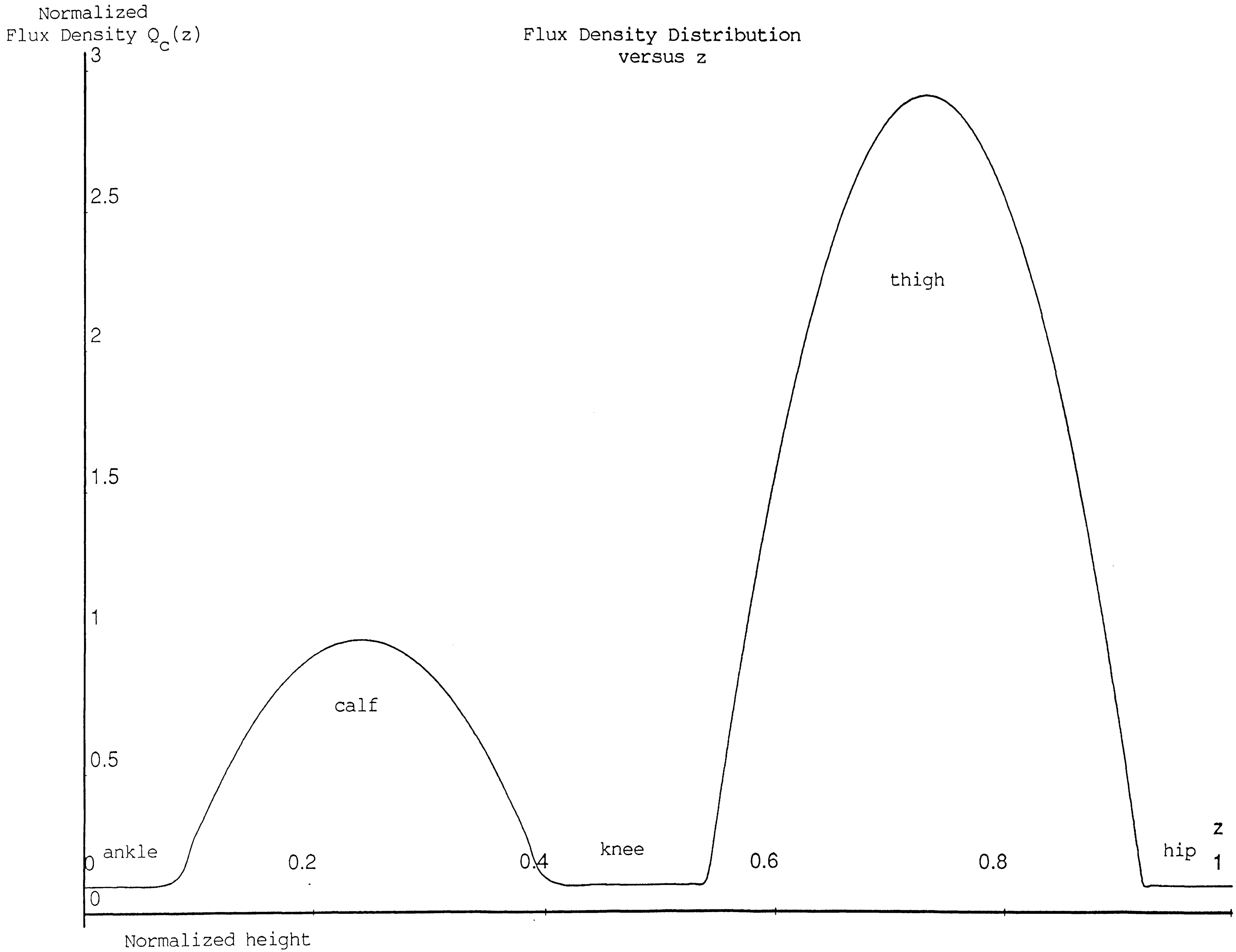




Probability Density Function
for Amplitude of Compression Wave
(Peak is at 200 mmHg)
Probability
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
a
(Normalized Amplitude)


Probability
Probability Density Function
for Duration of Muscle Compressions
(Peak is at 8 sec)
0.8
0.6
0.4
0.2
0
-0.2
Normalized time
t
0 (minimum-0.5 sec)
0.5
1
1.5
2
2.5


Probability
Probability Density Function
for Waiting Times Between Muscle
Compressions and Foot-Pump Inputs
(Peak is at 8 sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
Normalized time
t
0
(minimum time-
1 sec)
0.5
1
1.5
2
2.5


Normalized
Flux Density Q_c(z)
Flux Density Distribution
versus z
3
2.5
2
1.5
1
0.5
0
calf
thigh
ankle
knee
hip
0.2
0.4
0.6
0.8
1
z
Normalized height

Section V: Computer Simulation

The Characteristic Method

In order to solve our problem numerically, we use the characteristic method [**6**]. First we replace p in equation (3.11) using the tube law (3.12) and divide the whole equation by α. Then, neglecting our diffusive term and letting $\mathbf{y} = (A, u)$, our system becomes

$$\frac{\partial \mathbf{y}}{\partial t} + B\frac{\partial \mathbf{y}}{\partial z} + \mathbf{F} = 0, \text{ where} \tag{5.1}$$

$$B = \begin{bmatrix} u & A \\ \gamma/\alpha & 4u/3 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} -Q_c(z) \\ \left(\partial P_{\text{ext}}/\partial z + \beta u + 1\right)/\alpha \end{bmatrix}, \text{ and} \tag{5.2a}$$

$$\gamma = k\left(\frac{3}{2A^{5/2}} + \frac{1}{(2-A)^2}\right). \tag{5.2b}$$

Since our system is hyperbolic, B has two real eigenvalues c_+ and c_-, given by

$$c_\pm = \frac{7u}{6} \pm \frac{1}{2}\sqrt{\frac{u^2}{9} + \frac{4A\gamma}{\alpha}} \tag{5.3}$$

with their corresponding left eigenvectors $\mathbf{x}_\pm$. Then the Riemann invariants $a_\pm = \langle \mathbf{x}_\pm, \mathbf{y}\rangle$ are given by

$$\frac{da_\pm}{dt} + \langle \mathbf{x}_\pm, \mathbf{F}\rangle = 0 \text{ on } \frac{dz}{dt} = c_\pm, \tag{5.4}$$

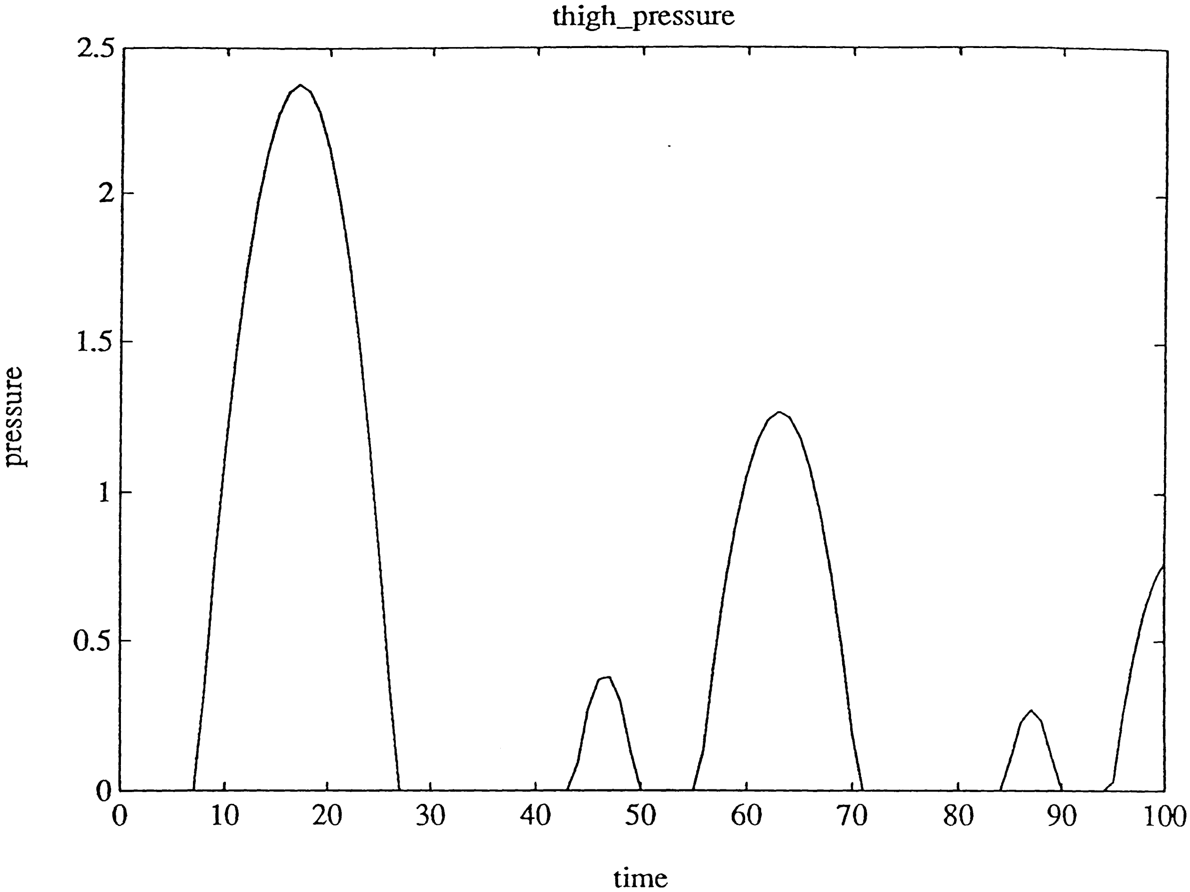
where $|c_\pm|$ is the local "speed of sound."

The leg (z-axis) is subdivided into equal segments. Values are calculated on a grid subdividing each segment. The characteristic method is used to calculate boundary conditions at the valves.

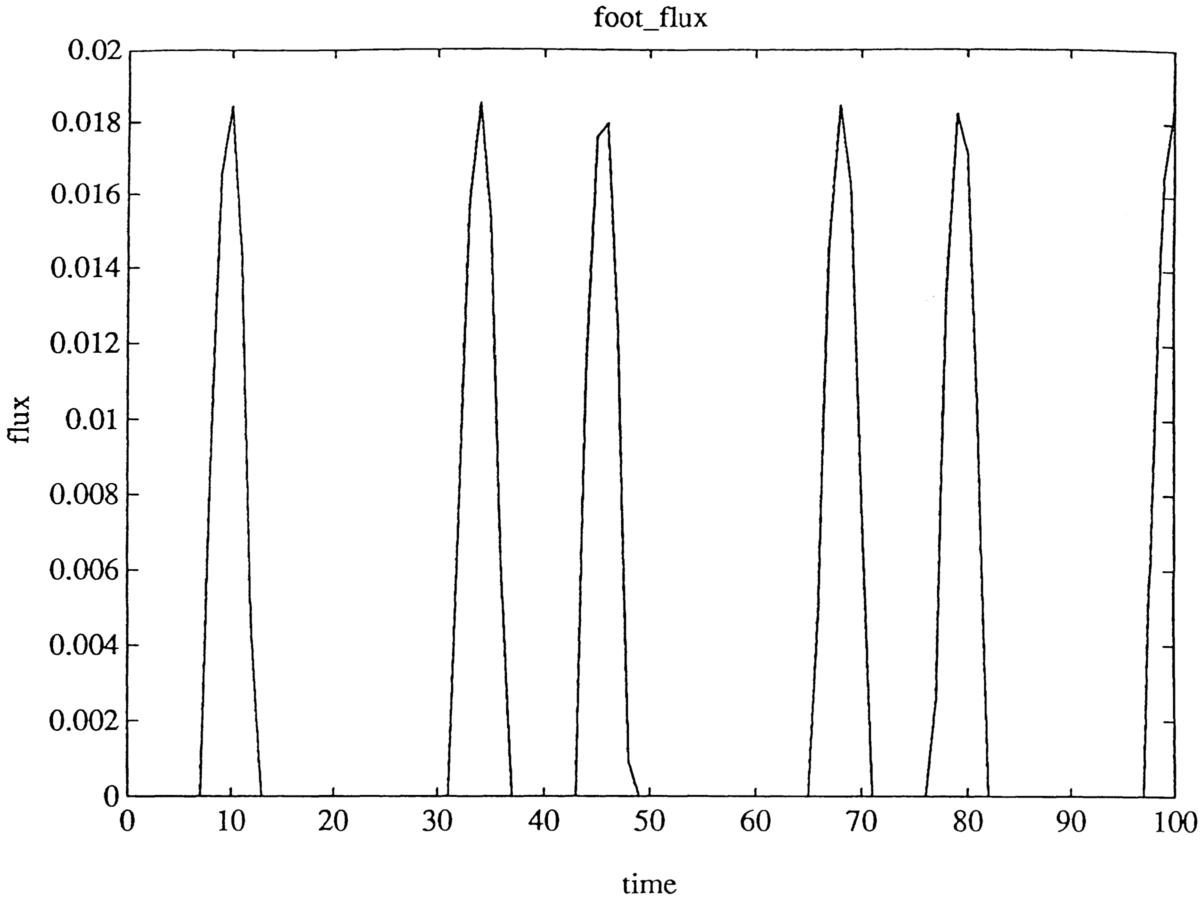
Computing Algorithm

1. Assume we know u, A, and p everywhere.
2. For each open valve, if $u < 0$ at the valve, then close it. For each closed valve, if $p_{\text{below}} > p_{\text{above}}$, open it.
3. Use conservative central differencing in space and explicit Euler in time to update all interior points. An open valve is considered to be an interior point.
4. Use characteristic relations and the boundary condition $u = 0$ to update values at the closed valves.
5. For the boundary condition at the foot, if there is no flux, we use $u = 0$. If there is a flux, we use the characteristic relations and $uA = Q_f$.

6. For the boundary condition at the hip, if the valve is closed, we have $u = 0$, so we can calculate A (from the characteristic relations). Otherwise, the pressure is p_h, and we may calculate u. From the tube law, we calculate A, and from the characteristic relations we obtain u.
7. Compute p everywhere using the tube law.
8. Go to line 1.



Random Input Generated
by Computer Simulation
for Normalized Thigh Pressure



Random Input Generated
by Computer Simulation
for Foot Pump Flux

Section VI: Future Research

There are many areas that could be further explored to attain a more accurate model. The assumption of one-dimensional flow could be expanded to include two or three dimensions. Also, the major veins in the legs are branched together, which needs to be taken into account. Shear rates vary, making it important to consider blood as a non-Newtonian fluid. The values inputed and assumed for maxima and minima are average numbers for an average person, and can vary greatly depending on the individual's health, age, and other factors. There is also wave propagation and possible blood turbulence that will appear in more than one dimension. A more accurate version of the tube law could also be employed for the specific case being considered [**3**,**4**].

Section VII: References

[1] Atman, P. and Dittmer, D. *Biology Data Book*, 2nd ed. Federation of American Societies for Experimental Biology, 1974.

[2] Guyton, A.C. *Textbook of Medical Physiology.* W. B. Saunders Co., 1991.

[3] Moreno, A. H., *et al.* "Mechanics of elistension of dog veins and other very thin-walled tubular structures," *Circ. Res.*, **27** (1970), 1069-1080.

[4] Shapiro, A. "Steady Flow in Collapsible Tubes," *J. Biomech. Eng.*, **99** (1977), 126-147.

[5] Skalak, R. and Chien, S. *Handbook of Bioengineering.* McGraw-Hill Book Company, 1987.

[6] Whitham, G. B. *Linear and Nonlinear Waves.* Wiley, New York, 1974.