Fluid Flow in a Filtering Device

IMA Summer Program for Graduate Students
Mathematical Modeling

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Section I: Introduction and Assumptions

Filters are commonly used, both within industry and in an average household. One filter of considerable importance is the oil filter of a car. This should be replaced at regular intervals to prevent damage to the engine from particles being carried in with the oil.

This model will simulate the flow of a liquid through a filter, and then introduce particles in the liquid to determine the amount of time until the flow through a filter is $1/e$ times its initial flow, at which time we conclude that the filter should be changed.

For simplification purposes, the following assumptions have been made:

- we consider one pleat of the filter and assume it is rectangular in shape
- we assume no flow through the walls W_1, W_2, and W_3 (see figure 1)
- there is symmetry about the x-axis
- the fluid used is incompressible
- the filter is stationary and will not tear
- fluid flows through the filter in the vertical direction only
- we assume symmetric flow about $y = 2$ and $y = -2$ (see figure 1).

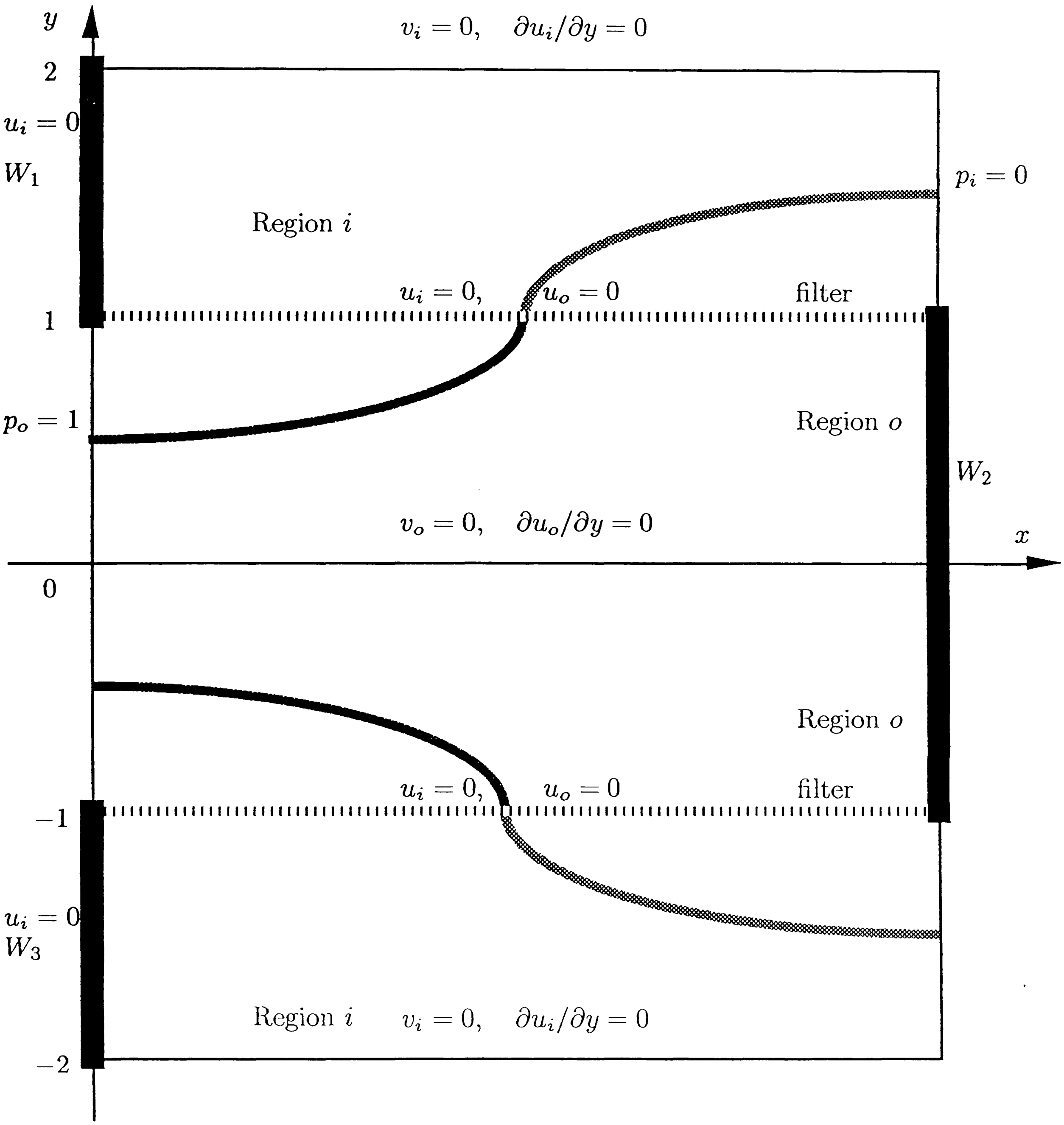


Figure 1. Nondimensional schematic of two-dimensional filter.

Section II: Nomenclature

A_j: integration constants (j=1, 2, 3, 4, 5, 6).

$\tilde{C}(\tilde{x},\tilde{t})$: concentration of particles in filter at position $\tilde{x}$ and time $\tilde{t}$. Units mol/cm.

C_c: concentration at which filter is completely clogged, units mol/cm.

$\tilde{C}_f$: concentration of particles in fluid entering device, units mol/cm^2.

$\tilde{C}_l$: concentration of fluid entering device, units mol/cm^2.

D: nondimensional function used when solving equations.

h: one-half the width of filter pleat. Units cm.

j: indexing variable.

$\tilde{k}(\tilde{C})$: proportionality function (units g/(cm^2·sec) used in permeability law, given in this model by

$$\tilde{k}(\tilde{C}) = \frac{\tilde{k}_0}{C_c - \tilde{C}(\tilde{x},\tilde{t})}. \tag{2.1}$$

$\tilde{k}_0$: proportionality constant [units g·mol/(cm^3·sec)] used in permeability law

L: length of filter. Units cm.

n: indexing variable.

N: discretization parameter used in numerical scheme.

$\tilde{P}_i$: given pressure of fluid upon leaving filter device. Units g/(cm·sec^2).

$\tilde{p}_i(\tilde{x},\tilde{y},\tilde{t})$: pressure of fluid after passing through filter. Units g/(cm·sec^2).

$\tilde{P}_o$: given pressure of fluid upon entering filter device. Units g/(cm·sec^2).

$\tilde{p}_o(\tilde{x},\tilde{y},\tilde{t})$: pressure of fluid before passing through filter. Units g/(cm·sec^2).

$\tilde{q}(\tilde{t})$: flow density through filter [units g/(cm·sec)] at time $\tilde{t}$, defined as

$$\tilde{q}(\tilde{t}) = \rho \int_0^L \tilde{v}_o(\tilde{x},h^-,\tilde{t})\,d\tilde{x}. \tag{2.2}$$

$\tilde{Q}$: total flow (units g/cm) through filter when we dispose of it, defined as

$$\tilde{Q} = \int_0^{\tilde{t}_c} \tilde{q}(\tilde{t})\,d\tilde{t}. \tag{2.3}$$

$\tilde{t}$: time. Units sec.

$\tilde{t}_c(\tilde{k}_0,\tilde{C}_f)$: quantity defined as that time at which $\tilde{q}(\tilde{t}_l) = \tilde{q}(0)/e$. At $\tilde{t}_c$, we change the filter since the rate through it has decayed by the $1/e$ factor. Units sec.

T: fictional nondimensionalization parameter. Units sec.

U: fictional nondimensionalization parameter. Units cm/sec.

$\tilde{u}_i(\tilde{x},\tilde{y},\tilde{t})$: velocity in $\tilde{x}$ direction of fluid after passing through filter. Units cm/sec.

$\tilde{u}_o(\tilde{x}, \tilde{y}, \tilde{t})$: velocity in $\tilde{x}$ direction of fluid before passing through filter. Units cm/sec.
V: fictional nondimensionalization parameter. Units cm/sec.
$\tilde{v}_i(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$: velocity in $\tilde{y}$ direction of fluid after passing through filter. Units cm/sec.
$\tilde{v}_o(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$: velocity in $\tilde{y}$ direction of fluid before passing through filter. Units cm/sec.
$\tilde{\mathbf{v}}(\tilde{x}, \tilde{y}, \tilde{z})$: velocity vector of fluid. Units cm/sec.
$\tilde{w}_i(\tilde{x}, \tilde{y}, \tilde{z})$: velocity in $\tilde{z}$ direction of fluid after passing through filter. Units cm/sec.
$\tilde{w}_o(\tilde{x}, \tilde{y}, \tilde{z})$: velocity in $\tilde{z}$ direction of fluid before passing through filter. Units cm/sec.
W_j: (j =1, 2, 3) inpermeable walls of filter.
$\tilde{x}$: length in direction along filter. Units cm.
$\tilde{y}$: length in direction across filter. Units cm.
$\tilde{z}$: length in height of filter. Units cm.
Z: height of filter. Units cm.
α: nondimensional parameter used in equations:

$$\alpha = \frac{\rho h^4(\tilde{P}_o - \tilde{P}_i)}{\mu^2 L^2}. \tag{2.4}$$

β: ratio of concentration of particles in pre-filtration area to the clogging concentration of the filter:

$$\beta = \frac{\tilde{C}_f h}{C_c}. \tag{2.5}$$

Δt: discretization parameter in t used for numerical scheme.
Δx: discretization parameter in t used for numerical scheme.
γ: nondimensional parameter used in equations.
μ: dynamic viscosity of fluid, units g/(cm·sec).
ν: kinematic viscosity of fluid, units cm^2/sec.
ρ: density of fluid, value g/cm^3.
ζ: aspect ratio of three-dimensional filter device, given by $\zeta = Z/L$.

Nondimensionalized variables will have no tildes.

Section III: Two-Dimensional Governing Equations

We begin by assuming that the filter is a two-dimensional object with length L, width $2h$, and distance $2h$ between filters. We also assume that the concentration of the particles is so low that the density in both regions is the same. We then use the two-dimensional Navier-Stokes equations in rectangular coordinates:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \tag{3.1}$$

$$\rho \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} \right) = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu \left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right) \tag{3.2}$$

$$\rho \left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \mu \left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \right). \tag{3.3}$$

These equations are general and hold for the fluid before and after it passes through the filter, though not in the filter itself.

At the filter, the change in concentration of particles on the filter is equal to the concentration of particles coming in:

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} = \tilde{C}_f \tilde{v}_o(\tilde{x}, h^-). \tag{3.4}$$

Next we proceed to nondimensionalize our equations. We nondimensionalize $\tilde{x}$ by our filter length L and $\tilde{y}$ by half of our filter width, which is h. We nondimensionalize our velocities $\tilde{u}$ and $\tilde{v}$ by U and V, respectively, which are at this time unknown. In addition, we nondimensionalize $\tilde{t}$ by T, which is as of yet unknown. We nondimensionalize $\tilde{C}$ by our clogging concentration C_c. Our pressure values vary between $\tilde{P}_o$ and $\tilde{P}_i$, so we normalize to make them vary between 1 and 0. Summarizing, we have the following:

$$x = \frac{\tilde{x}}{L}, \qquad y = \frac{\tilde{y}}{h}, \qquad u = \frac{\tilde{u}}{U}, \qquad v = \frac{\tilde{v}}{V}, \qquad t = \frac{\tilde{t}}{T}, \qquad C = \frac{\tilde{C}}{C_c}, \qquad p = \frac{\tilde{p} - \tilde{P}_i}{\tilde{P}_o - \tilde{P}_i}. \tag{3.5}$$

Once again, these nondimensionalizations apply to the fluid both before and after it passes through the filter.

Using equations (3.5) in (3.1), we have the following:

$$\frac{U}{L} \frac{\partial u}{\partial x} + \frac{V}{h} \frac{\partial v}{\partial y} = 0.$$

In order to simplify our equations, we let $U/L = V/h$ to yield

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.6}$$

Using equations (3.5) in (3.4), we have

$$\frac{C_c}{T}\frac{\partial C}{\partial t} = \tilde{C}_f V v_o(x, 1^-),$$

and substituting Uh/L for V yields

$$\frac{\partial C}{\partial t} = \frac{\tilde{C}_f U h T}{L C_c} v_o(x, 1^-). \tag{3.7}$$

Since we want $O(1)$ changes in our concentration, we let

$$T = \frac{L C_c}{\tilde{C}_f U h}. \tag{3.8}$$

Then equation (3.7) becomes

$$\frac{\partial C}{\partial t} = v_o(x, 1^-). \tag{3.9}$$

Now nondimensionalizing equation (3.2), we have

$$\rho\left(\frac{\tilde{C}_f U^2 h}{L C_c}\frac{\partial u}{\partial t} + \frac{UV}{h} v\frac{\partial u}{\partial y} + \frac{U^2}{L} u\frac{\partial u}{\partial x}\right) = -\frac{\tilde{P}_o - \tilde{P}_i}{L}\frac{\partial p}{\partial x} + \mu\left(\frac{U}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{U}{h^2}\frac{\partial^2 u}{\partial y^2}\right).$$

Using $U/L = V/h$, we may express everything in terms of U:

$$\frac{\rho U^2}{L}\left(\beta\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x}\right) = -\frac{\tilde{P}_o - \tilde{P}_i}{L}\frac{\partial p}{\partial x} + \frac{\mu U}{h^2}\left(\frac{h^2}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Now we want to set the coefficient of the pressure equal to unity so that we may examine the relative contributions from the other terms. Doing so, we have the following:

$$\frac{\rho U^2}{\tilde{P}_o - \tilde{P}_i}\left(\beta\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \frac{\mu U L}{h^2(\tilde{P}_o - \tilde{P}_i)}\left(\frac{h^2}{L^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Since we will eventually wish to examine oil and other viscous fluids, we want the viscous diffusive terms to be dominant. Hence, we set the coefficient of the diffusive terms equal to unity to yield

$$U = \frac{h^2(\tilde{P}_o - \tilde{P}_i)}{\mu L} \tag{3.10}$$

$$\alpha\left(\beta\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}+\left(\frac{h^2}{L^2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right). \tag{3.11}$$

Now, since $h \ll L$, we see that both the first diffusive term and α are negligible, so equation (3.11) reduces to

$$\frac{\partial p}{\partial x}=\frac{\partial^2 u}{\partial y^2}. \tag{3.12}$$

Now using equations (3.5) in equation (3.3), we have the following:

$$\rho\left(\frac{\tilde{C}_f UVh}{LC_c}\frac{\partial v}{\partial t}+\frac{UV}{L}u\frac{\partial v}{\partial x}+\frac{V^2}{h}v\frac{\partial v}{\partial y}\right)=-\frac{\tilde{P}_o-\tilde{P}_i}{h}\frac{\partial p}{\partial y}+\mu\left(\frac{V}{L^2}\frac{\partial^2 v}{\partial x^2}+\frac{V}{h^2}\frac{\partial^2 v}{\partial y^2}\right).$$

Using our relation between U and V, we have the following:

$$\frac{\rho U^2 h^2}{L^2(\tilde{P}_o-\tilde{P}_i)}\left(\beta\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\mu U}{L(\tilde{P}_o-\tilde{P}_i)}\left(\frac{h^2}{L^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).$$

Using equation (3.10), we have

$$\frac{\alpha h^2}{L^2}\left(\beta\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{h^2}{L^2}\left(\frac{h^2}{L^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).$$

Now, since $h \ll L$, we see that we may neglect all terms but the pressure term and our equation becomes the following:

$$\frac{\partial p}{\partial y}=0. \tag{3.13}$$

Our system of equations (3.6), (3.12), and (3.13) are called the *lubrication equations*. Note that our time dependence has scaled out of all but equation (3.9).

Now we wish to postulate some boundary conditions. As shown in figure 1, we place our axes such that the problem becomes symmetric. We then only need consider the region $0 < x < 1$, $0 < y < 2$. Since our filter is along $y = 1$, we split the problem into two regions. We begin with the region before the fluid reaches the filter ($0 < x < 1$, $0 < y < 1^-$), which we call region o. In this region, we have a pressure condition at the inflow which has scaled to become

$$p_o(0, y, t) = 1. \tag{3.14}$$

We assume that the boundary at $x = 1$ is a wall through which no fluid can pass, so

$$u_o(1, y, t) = 0. \tag{3.15}$$

Since $y = 0$ is a line of symmetry, we have

$$v_o(x, 0, t) = 0 \tag{3.16a}$$

$$\frac{\partial u_o}{\partial y}(x, 0, t) = 0. \tag{3.16b}$$

We assume that the filter is porous only in the vertical direction, hence

$$u_o(x, 1, t) = 0. \tag{3.17a}$$

Also, by doing a simple concentration balance, we see that

$$\tilde{C}_l \tilde{v}_i(\tilde{x}, h^+, \tilde{t})\, d\tilde{t}\, d\tilde{x} = \tilde{C}_l \tilde{v}_o(\tilde{x}, h^-, \tilde{t})\, d\tilde{t}\, d\tilde{x} - \tilde{C}_f \tilde{v}_o(\tilde{x}, h^-, \tilde{t})\, d\tilde{t}\, d\tilde{x}.$$

We nondimensionalize our concentrations by $\tilde{C}_l$. Nondimensionalizing the rest of the equation, we have

$$v_i(x, 1^+, t) = (1 - C_f) v_o(x, 1^-, t). \tag{3.17b}$$

We assume that initially our concentration in the filter is 0, so

$$C(x, 0) = 0. \tag{3.18}$$

We now proceed to solve our equations in region o. From (3.13) we see immediately that p_o is a function of x and t only. Solving equation (3.12) subject to our boundary conditions (3.16b) and (3.17a), we have

$$u_o(x, y, t) = \frac{y^2 - 1}{2} \frac{\partial p_o}{\partial x}. \tag{3.19}$$

The velocity u_o can now satisfy equation (3.15) only if

$$\frac{\partial p_o}{\partial x}(1, t) = 0. \tag{3.20}$$

Using equation (3.19) in equation (3.6), we have the following:

$$\frac{\partial v_o}{\partial y} = -\frac{y^2 - 1}{2} \frac{\partial^2 p_o}{\partial x^2}.$$

Solving the above subject to boundary condition (3.16a), we have

$$v_o(x, y, t) = -\left(\frac{y^3}{6} - \frac{y}{2}\right) \frac{\partial^2 p_o}{\partial x^2}. \tag{3.21}$$

We now define our boundary conditions in region i, where the fluid has already passed through the filter. This is the region $0 < x < 1$, $1^+ < y < 2$. In this case, the pressure boundary condition is specified at $x = 1$:

$$p_i(1, y, t) = 0. \tag{3.22}$$

Now the boundary at $x = 0$ is the wall, so

$$u_i(0, y, t) = 0. \tag{3.23}$$

Since $y = 2$ is also a line of symmetry, we have

$$v_i(x, 2, t) = 0 \tag{3.24a}$$

$$\frac{\partial u_i}{\partial y}(x, 2, t) = 0. \tag{3.24b}$$

We assume that the filter is porous only in the vertical direction, hence

$$u_i(x, 1, t) = 0. \tag{3.25}$$

Solving the equations in this region in a perfectly analogous manner to the way in which we solved the equations in region o, we have

$$u_i(x, y, t) = \frac{(2-y)^2 - 1}{2}\frac{\partial p_i}{\partial x} \tag{3.26}$$

$$\frac{\partial p_i}{\partial x}(0, t) = 0 \tag{3.27}$$

$$v_i(x, y, t) = \left[\frac{(2-y)^3}{6} - \frac{(2-y)}{2}\right]\frac{\partial^2 p_i}{\partial x^2}. \tag{3.28}$$

In addition, we have the following *permeability law*:

$$\tilde{p}_o(\tilde{x}, h^-, \tilde{t}) - \tilde{p}_i(\tilde{x}, h^+, \tilde{t}) = \tilde{k}(\tilde{C})\tilde{v}_o(\tilde{x}, h^-, \tilde{t}).$$

Here we postulate the form of $\tilde{k}$ given in section II. This then states that regardless of the pressure differential, no fluid will flow through the filter when $\tilde{C} = C_c$, the clogging concentration. Nondimensionalizing, we have

$$p_o(x, 1^-, t) - p_i(x, 1^+, t) = \frac{\tilde{k}_0 h^3}{\mu L^2 C_c\left[1 - C(x, t)\right]} v_o(x, 1^-, t) \equiv \frac{k_0}{1 - C(x, t)} v_o(x, 1^-, t). \tag{3.29}$$

Section IV: Steady Flow, No Particles

The first model we consider is that of steady flow with no particles in the fluid. Hence we have $C_f = 0$, and equation (3.4), together with boundary condition (3.18), implies that $C \equiv 0$ for all $t \geq 0$. Thus, there is no time dependence in the problem. Then equation (3.17b) becomes

$$v_i(x, 1^+) = v_o(x, 1^-) = v(x, 1) \tag{4.1}$$

and equation (3.29) becomes

$$p_o(x, 1^-) - p_i(x, 1^+) = k_0 v(x, 1). \tag{4.2}$$

Now we continue our solution of the problem. Since there is no time dependence in the problem, p_o and p_i are functions of x only. Using equations (3.21) and (3.28) in (4.1), we have the following:

$$p_o''(x) = -p_i''(x)$$

$$p_o(x) + p_i(x) = A_1 x + A_2. \tag{4.3}$$

Now using equations (3.21) and (3.28) in equation (4.2), we have

$$p_o(x) - p_i(x) = \frac{k_0}{3} p_o''(x). \tag{4.4}$$

Combining equations (4.3) and (4.4) and solving for $p_o(x)$, we have

$$2p_o(x) - \frac{k_0}{3} p_o''(x) = A_1 x + A_2.$$

$$p_o(x) = \frac{A_1 x + A_2}{2} + A_3 e^{-\gamma x} + A_4 e^{\gamma x}, \tag{4.5a}$$

where $\gamma = \sqrt{6/k_0}$. Using equation (4.3), we immediately see that

$$p_i(x) = \frac{A_1 x + A_2}{2} - A_3 e^{-\gamma x} - A_4 e^{\gamma x} \tag{4.5b}$$

Using our boundary condition (3.14) in (4.5a), we have

$$\frac{A_2}{2} + A_3 + A_4 = 1, \tag{4.6}$$

while equation (3.20) gives us

$$\frac{A_1}{2} - \gamma A_3 e^{-\gamma} + \gamma A_4 e^{\gamma} = 0. \tag{4.7}$$

For our $p_i(x)$ conditions, we use (3.22):

$$\frac{A_1 + A_2}{2} - A_3 e^{-\gamma} - A_4 e^{\gamma} = 0, \tag{4.8}$$

and we use (3.27) to find

$$\frac{A_1}{2} + \gamma A_3 - \gamma A_4 = 0. \tag{4.9}$$

Equations (4.6)-(4.9) are a system of four equations in four unknowns which we may solve using Maple to find

$$A_1 = -\frac{2\gamma(e^{\gamma} - 1)}{D}, \qquad A_2 = 1 + \frac{\gamma(e^{\gamma} - 1)}{D}, \qquad A_3 = \frac{e^{\gamma}}{D}, \qquad A_4 = \frac{1}{D}, \tag{4.10a}$$

$$D = 2(e^{\gamma} + 1) + \gamma(e^{\gamma} - 1). \tag{4.10b}$$

Using equations (4.10) in (4.5), we have

$$p_o(x) = \frac{1}{2} + \frac{\gamma(1 - 2x)(e^{\gamma} - 1)}{2D} + \frac{e^{\gamma(1-x)} + e^{\gamma x}}{D} \tag{4.11a}$$

$$p_i(x) = \frac{1}{2} + \frac{\gamma(1 - 2x)(e^{\gamma} - 1)}{2D} - \frac{e^{\gamma(1-x)} + e^{\gamma x}}{D}. \tag{4.11b}$$

Now we have our flow completely determined. Summarizing, using (4.11a) in (3.19), we have

$$u_o(x, y) = \left[1 - e^{\gamma} + e^{\gamma x} - e^{\gamma(1-x)}\right] \frac{\gamma(y^2 - 1)}{2D}. \tag{4.12}$$

Using equation (4.11b) in equation (3.26), we have

$$u_i(x, y) = \left[1 - e^{\gamma} - e^{\gamma x} + e^{\gamma(1-x)}\right] \frac{\gamma[(2 - y)^2 - 1]}{2D}. \tag{4.13}$$

Using equation (4.11a) in equation (3.21), we have the following:

$$v_o(x, y) = -\frac{\gamma^2}{D} \left[e^{\gamma(1-x)} + e^{\gamma x}\right] \left(\frac{y^3}{6} - \frac{y}{2}\right). \tag{4.14}$$

Using equation (4.11b) in equation (3.28), we have

$$v_i(x, y) = -\frac{\gamma^2}{D} \left[e^{\gamma(1-x)} + e^{\gamma x}\right] \left[\frac{(2 - y)^3}{6} - \frac{(2 - y)}{2}\right]. \tag{4.15}$$

As shown in figure 2, the lower velocity profile at the filter $v_o(x, 1^-)$ is nearly parabolic in shape. This contradicted our first guess at the profile, since we originally thought the flow should be faster at the inlet and decay monotonically as x moved toward the outlet. However, further examination convinced us that our solutions were indeed correct. As will be shown in section V, this implies that our filter will have a higher concentration at *both* ends, rather than a monotonic profile as we first thought.

Now we wish to plot streamlines. In region o, the equations for the streamlines are given by

$$\frac{dy}{dx} = \frac{v_o}{u_o} = -\frac{1}{y^2 - 1}\left(\frac{\partial p_o}{\partial x}\right)^{-1}\frac{\partial^2 p_o}{\partial x^2}\left(\frac{y^3}{3} - y\right),$$

a differential equation which can be separated to yield

$$\log\left(\frac{y^3}{3} - y\right) = -\log\left(\frac{\partial p_o}{\partial x}\right) + A_5,$$

where A_5 is a constant of integration. Then using equation (4.11a) our solution is the following:

$$\frac{y^3}{3} - y = \frac{e^{A_5} D}{\gamma}\left[e^{\gamma x} - e^{\gamma(1-x)} + 1 - e^{\gamma}\right]^{-1}. \tag{4.16}$$

In region i, the equations for the streamlines are given by

$$\frac{dy}{dx} = \frac{v_i}{u_i} = \frac{1}{(2-y)^2 - 1}\left[\frac{(2-y)^3}{3} - (2-y)\right]\left(\frac{\partial p_i}{\partial x}\right)^{-1}\frac{\partial^2 p_i}{\partial x^2},$$

a differential equation which can be separated to yield

$$\log\left[\frac{(2-y)^3}{3} - (2-y)\right] = -\log\left(\frac{\partial p_i}{\partial x}\right) + A_6,$$

where A_6 is a constant of integration. Then using equation (4.11b) our solution is the following:

$$\frac{(2-y)^3}{3} - (2-y) = \frac{e^{A_6} D}{\gamma}\left[e^{\gamma(1-x)} - e^{\gamma x} + 1 - e^{\gamma}\right]^{-1}. \tag{4.17}$$

These streamlines are graphed in figure 3.

Figure 2.

$v_o(x)$ vs. x for Steady Flow Case

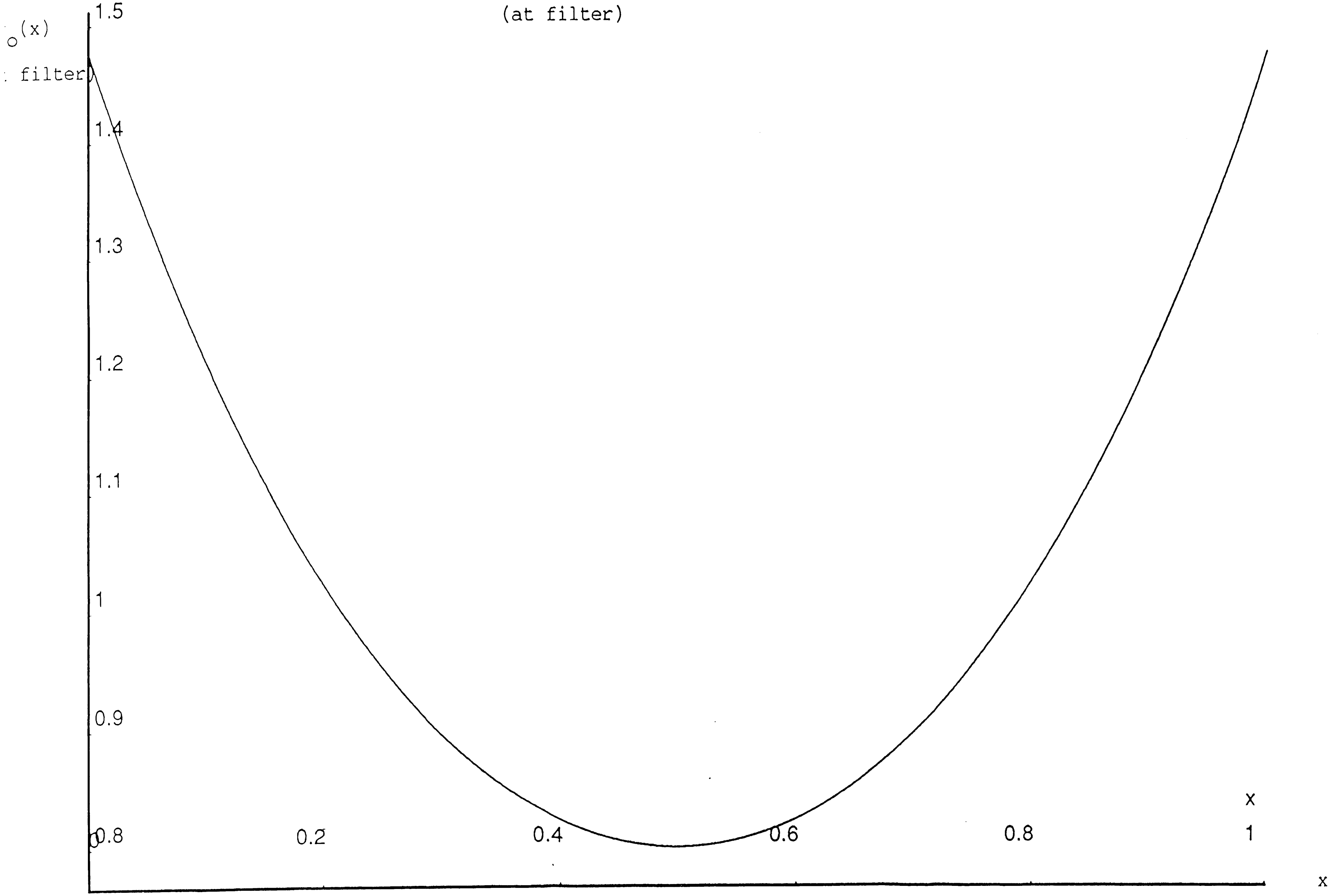
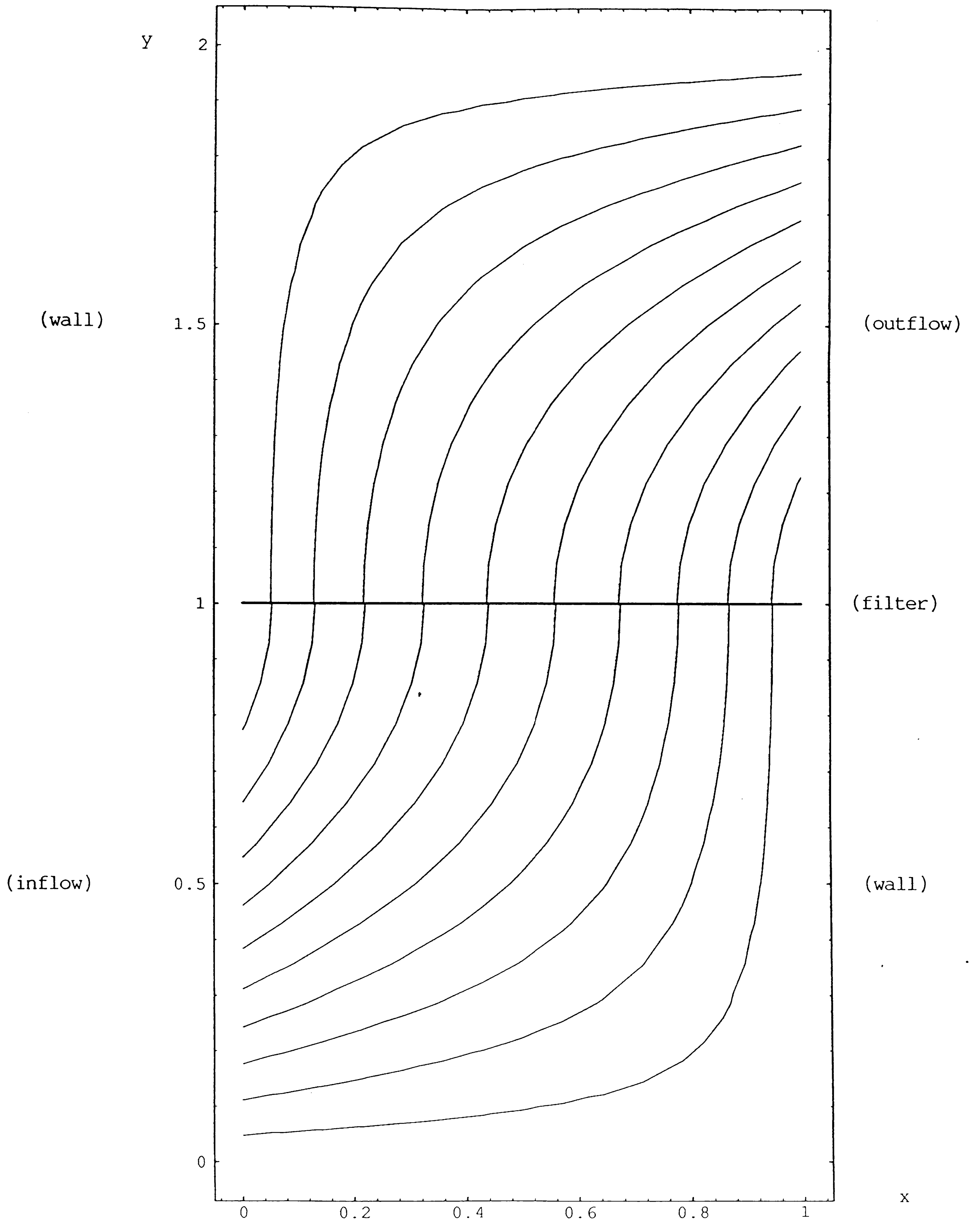


Figure 3.
Streamlines for Steady Flow Case



Section V: 2-Dimensional Flow with Particles

The next model we consider is that of flow with particles in the fluid. Hence we no longer have our simplification that $C_f = 0$, and the problem is time-dependent. However, we may simplify our equations. Using equation (3.21) in equation (3.9), we have the following at $y = 1$:

$$\frac{\partial C}{\partial t} = \frac{1}{3}\frac{\partial^2 p_o}{\partial x^2}. \tag{5.1}$$

Using equations (3.21) and (3.28) in equation (3.17b), we have

$$(1 - C_f)\frac{\partial^2 p_o}{\partial x^2} = -\frac{\partial^2 p_i}{\partial x^2}. \tag{5.2}$$

Using equation (3.21) in equation (3.29), we have the following:

$$p_o(x,t) - p_i(x,t) = \frac{k_0}{3[1 - C(x,t)]}\frac{\partial^2 p_o}{\partial x^2}. \tag{5.3}$$

Our boundary conditions are given by equations (3.14), (3.18), (3.20), (3.22), and (3.27), which we rewrite for easy reference:

$$p_o(0,t) = 1. \tag{5.4}$$

$$\frac{\partial p_o}{\partial x}(1,t) = 0. \tag{5.5}$$

$$p_i(1,t) = 0. \tag{5.6}$$

$$\frac{\partial p_i}{\partial x}(0,t) = 0 \tag{5.7}$$

$$C(x,0) = 0. \tag{5.8}$$

We wish to solve the system of equations numerically; hence we also need initial conditions for p_o and p_i. We assume that the filter has no particles in it at time $t = 0$, so we use our results from section IV:

$$p_o(x,0) = \frac{1}{2} + \frac{\gamma(1-2x)(e^\gamma - 1)}{2D} + \frac{e^{\gamma(1-x)} + e^{\gamma x}}{D} \tag{5.9}$$

$$p_i(x,0) = \frac{1}{2} + \frac{\gamma(1-2x)(e^\gamma - 1)}{2D} - \frac{e^{\gamma(1-x)} + e^{\gamma x}}{D}, \text{ where} \tag{5.10}$$

$$D = 2(e^\gamma + 1) + \gamma(e^\gamma - 1). \tag{5.11}$$

We discretize our x direction by $\Delta x = 1/N$, and our t direction by $\Delta t = (\Delta x)^2$. We choose this Δt so that our scheme, which is first-order in t and second-order in x, has the same size error in both directions. We then introduce a discretization notation by $f(n,j) = f(n\Delta x, j\Delta t)$, where f is p_o, p_i, or C. Four our unknowns, we let $n = 1, \ldots, N$ for p_o [since we know $p_o(0,t)$], $n = 0, \ldots, N-1$ for p_i [since we know $p_i(1,t)$] and $n = 1, \ldots, N$ for C (for reasons which will be discussed later).

We break our scheme into two parts. First, we discretize equation (5.1) and perform an explicit Euler step in time to solve for the concentration at the next time step:

$$C(n,j) = C(n,j-1) + \frac{\Delta t}{3(\Delta x)^2}\left[p_o(n-1,j-1) - 2p_o(n,j-1) + p_o(n+1,j-1)\right], \qquad 1 \le n \le N. \tag{5.12}$$

Note that for $n = N$ we use the Neumann condition; that is, approximating (5.5) with respect to x by a central difference scheme, we see that

$$p_o(N+1,j) = p_o(N-1,j). \tag{5.13}$$

We use equation (5.13) in (5.12) when $n = N$ and throughout the rest of the scheme. We also use its counterpart for p_i, namely

$$p_i(-1,j) = p_i(1,j). \tag{5.14}$$

Next we solve for our new pressures implicitly by discretizing equations (5.2) and (5.3) and solving them together. At interior grid points, we approximate second derivatives using the standard second-order central difference scheme, so that (5.2) becomes

$$(1-C_f)\left[p_o(n-1,j) - 2p_o(n,j) + p_o(n+1,j)\right] + p_i(n-1,j) - 2p_i(n,j) + p_i(n+1,j) = 0, \qquad n = 1, \ldots, N-1. \tag{5.15}$$

In equation (5.15), note that for $n = 1$ we use equation (5.4) so that the right-hand side to our equation is nonzero. We use equation (5.6) for $n = N - 1$, but the right-hand side remains the same. Discretizing (5.3), we have the following:

$$p_i(n,j) - p_o(n,j) + \frac{k_0\left[p_o(n-1,j-1) - 2p_o(n,j-1) + p_o(n+1,j-1)\right]}{3(\Delta x)^2[1 - C(n,j)]} = 0, \qquad n = 1, \ldots, N. \tag{5.16}$$

In equation (5.16), we again use (5.4) for $n = 1$, and for $n = N$ we use (5.6) and equation (5.13).

We now have $2N - 1$ equations in $2N$ unknowns. To obtain the final equation, we construct the second-order forward-difference second-derivative scheme, since we do not know $p_o(-1,j)$. Hence, we have the following expression:

$$(1 - C_f)\left[2p_o(0,j) - 5p_o(1,j) + 4p_o(2,j) - p_o(3,j)\right] + p_i(-1,j) - 2p_i(0,j) + p_i(1,j) = 0. \tag{5.17}$$

For equation (5.17), we use equation (5.14) and (5.4).

Examining equations (5.15)-(5.17), it becomes apparent why we did not calculate $C(0,t)$. Since $C(0,t)$ never appears in our expressions, it wouldn't be used to calculate new pressure conditions from which it in turn would be calculated at the next time step. Hence, if we tried to calculate it, we would get spurious results. However, since the concentration profile is nearly symmetric, we can make a reasonable approximation to $C(0,t)$ by examining $C(1,t)$.

Now that we have expressions for our pressure, we need to check our flow rate. Nondimensionalizing equation (2.2), we have

$$\tilde{q}(t) = \rho L V \int_0^1 v(x,1,t)\,dx.$$

Using equations (3.10) and (3.21)

$$\frac{\tilde{q}(t)\mu L}{\rho h^3(\tilde{P}_o - \tilde{P}_i)} = \frac{1}{3}\int_0^1 \frac{\partial^2 p_o}{\partial x^2}\,dx.$$

Integrating and using equation (5.5), we have

$$\frac{3\tilde{q}(t)\mu L}{\rho h^3(\tilde{P}_o - \tilde{P}_i)} \equiv q(t) = -\frac{\partial p_o}{\partial x}(0,t). \tag{5.18}$$

Since we are taking ratios, we see that t_c is that time at which $q(t_c)/q(0) = e^{-1}$.

We also nondimensionalize equation (2.3) to yield

$$\frac{3\tilde{Q}\mu L}{\rho h^3(\tilde{P}_o - \tilde{P}_i)} = \frac{LC_c}{\tilde{C}_f U h}\int_0^{t_c} q(t)\,dt$$

$$\frac{3\tilde{Q}\tilde{C}_f}{\rho L C_c} \equiv Q = \int_0^{t_c} q(t)\,dt. \tag{5.19}$$

Using our code, we performed several tests, the results of which are graphed on the following pages. The first test we performed was to test the sensitivity of t_c and C to changes in C_f as we varied it from 0 to 0.05. Figure 4 shows that we saw very little change in t_c as C_f varies. Note that we are varying C_f over a small range, so we would expect a small variance. As far as t_c is concerned, the graph shows us that for varying values of C_f the number of *nondimensional* time units is not changing considerably. However, since our scaling in equation (5.8) depends explicitly on C_f, the *dimensional* $\tilde{t}_c$ does vary with respect to C_f.

Figure 5 shows the change in concentration at the outlet of the filtering device as a function of t for varying k_0. Note that as k_0 increases, the concentration approaches its final value more slowly, and that the final value is smaller. This can also be seen in figure 6, which shows $C(x,t_c)$ as a function of x for varying k_0. Note that, as discussed briefly in section IV, due to the shape of our velocity profile, the maximum values of C are at the ends of the filter. As indicated in figure 5, as k_0 increases, the maximum value of our profile decreases. In addition, our profile begins to flatten. This is reasonable, since as $k_0 \to \infty$, equation (3.29) indicates that there would be no flow of particles and hence we would have a flat profile of $C \equiv 0$. Figure 7 shows the maximum concentration at t_c vs. k_0. Note that as you increase k_0, the filter must be thrown away at smaller concentrations.

Figure 8 shows the outer and inner pressures p_o and p_i at t_c as a function of x for various values of k_0. Note that for smaller k_0, a lower pressure gradient is needed to make the fluid flow through, as indicated by equation (3.29). As expected, the pressures are larger near the inlet than at the outlet.

Figure 9 is a graph of $q(t)$ vs. t for varying k_0. Note that as k_0 increases, the flow starts off at a slower rate [as indicated by (3.29)], but takes longer to decay to $1/e$ times its initial value. This variance of t_c with k_0 is indicated explicitly in figure 10. Note that it is nearly linear for large values of k_0. However, figure 11 shows that even though the *time* that you use the filter is longer for larger values of k_0, the *flow* you force through the filter (which is indicated by Q) is smaller.

Hence, which value of k_0 one should use for an efficient filter depends on your definition of the word "efficient." In the case of an oil filter, where a longer life might be more important than a high flow rate, a large k_0 would be indicated. However, in industrial filter applications, such as the filtering of yeast from beer, where large flow rates are economically desirable, a smaller k_0 would be indicated if the machinery could be designed so that the filter, which would have to be changed more often, is readily accessible.

Figure 4.

Critical time t_c and $C(1,t_c)$ vs. C_f

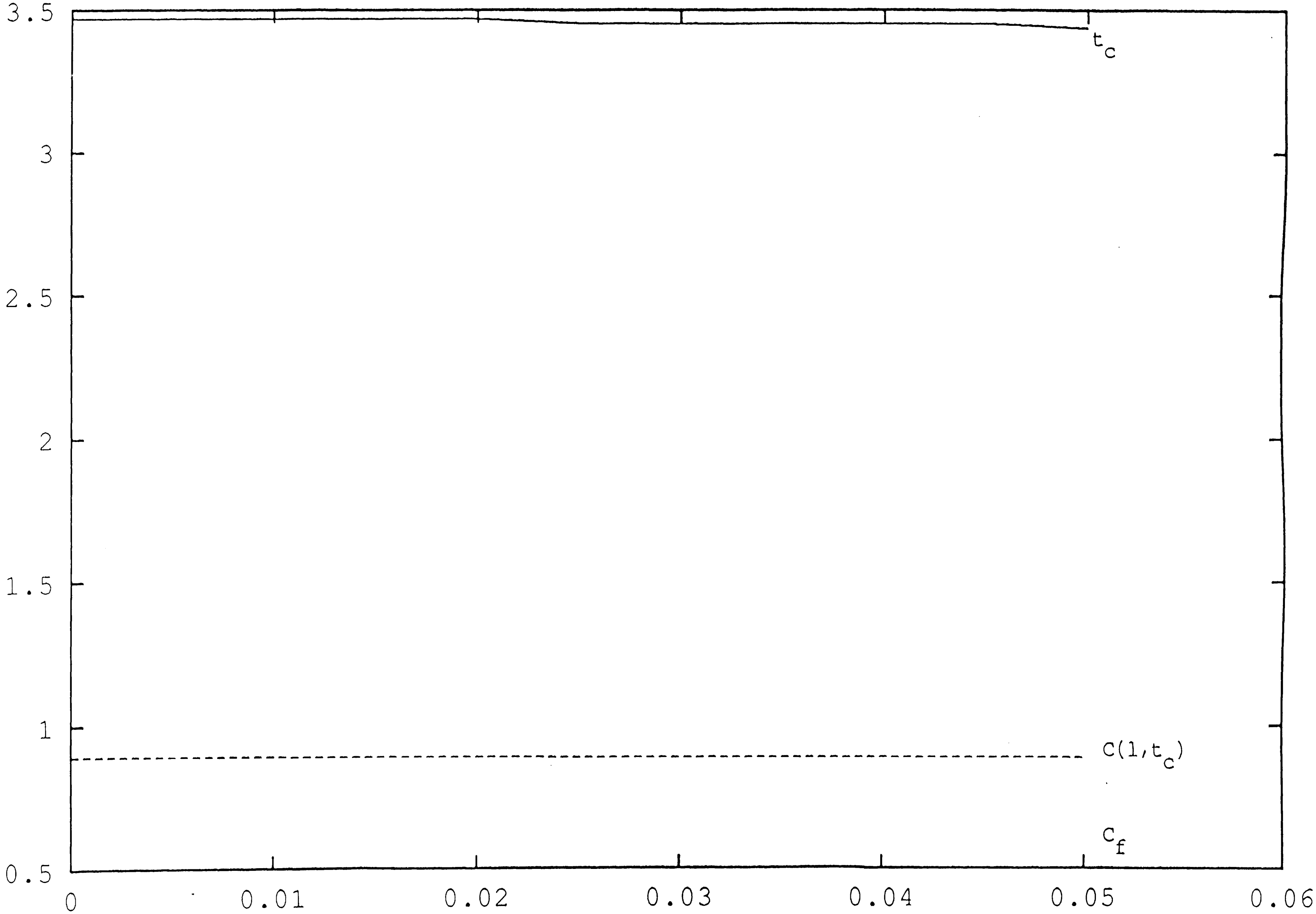


Figure 5.
C(1,t) vs. t for Various Values of k_0

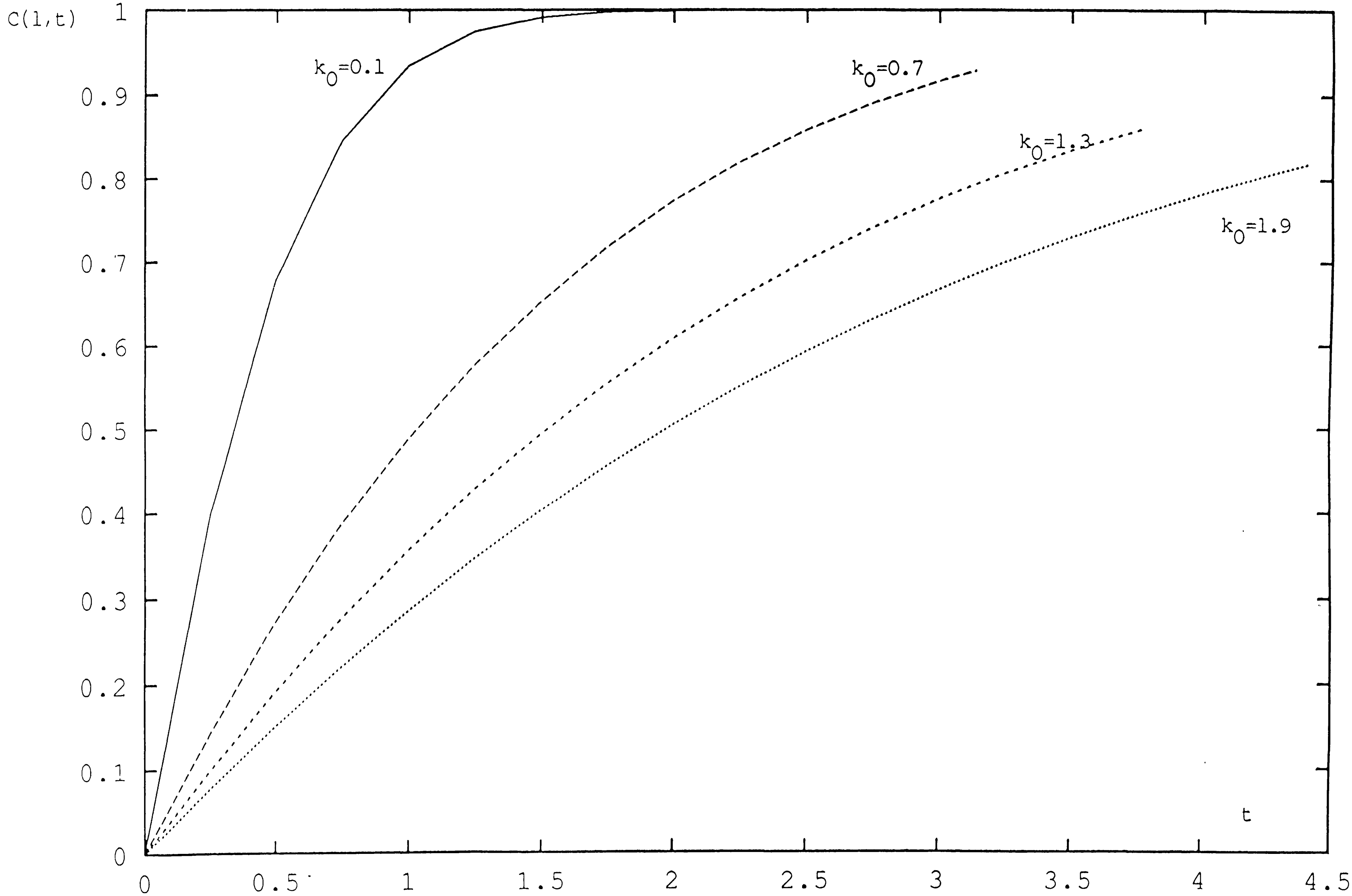


Figure 6.
$C(x,t_c)$ vs. x for Various Values of k_0

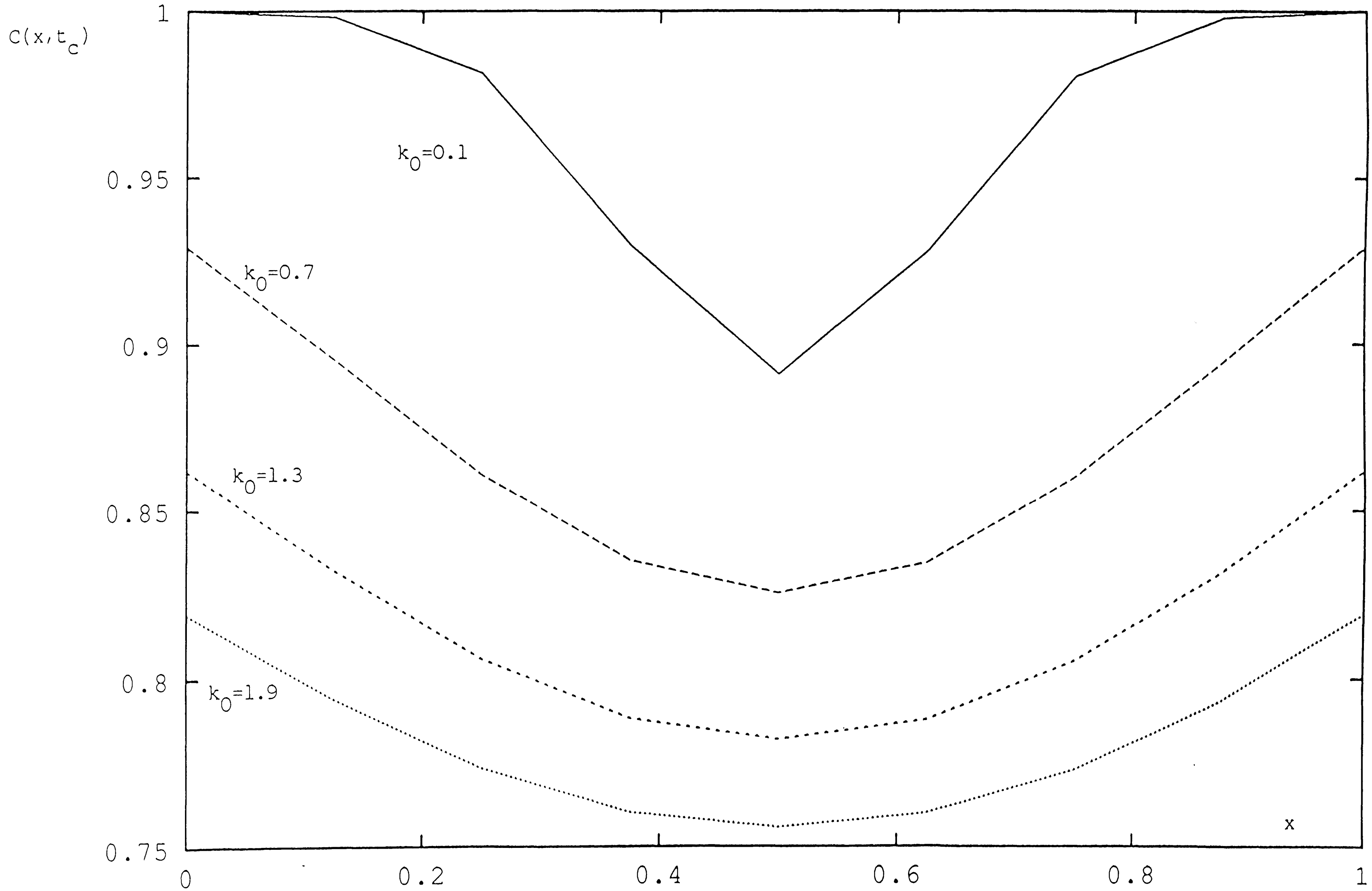


Figure 7.
$C(1, t_c)$ vs. k_0

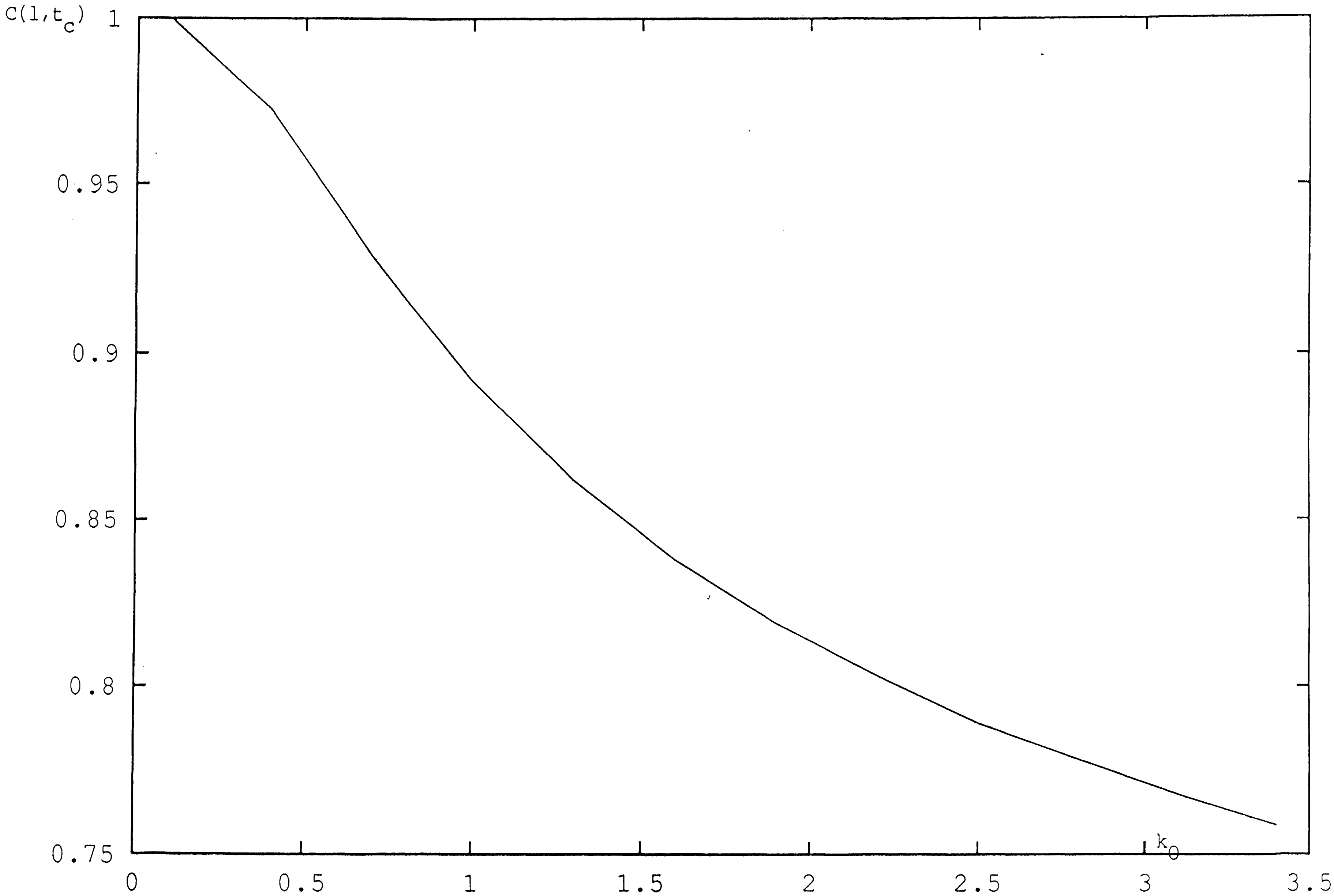


Figure 8.
$p_o(x,t_c)$ and $p_i(x,t_c)$ vs. x for Various Values of k_0

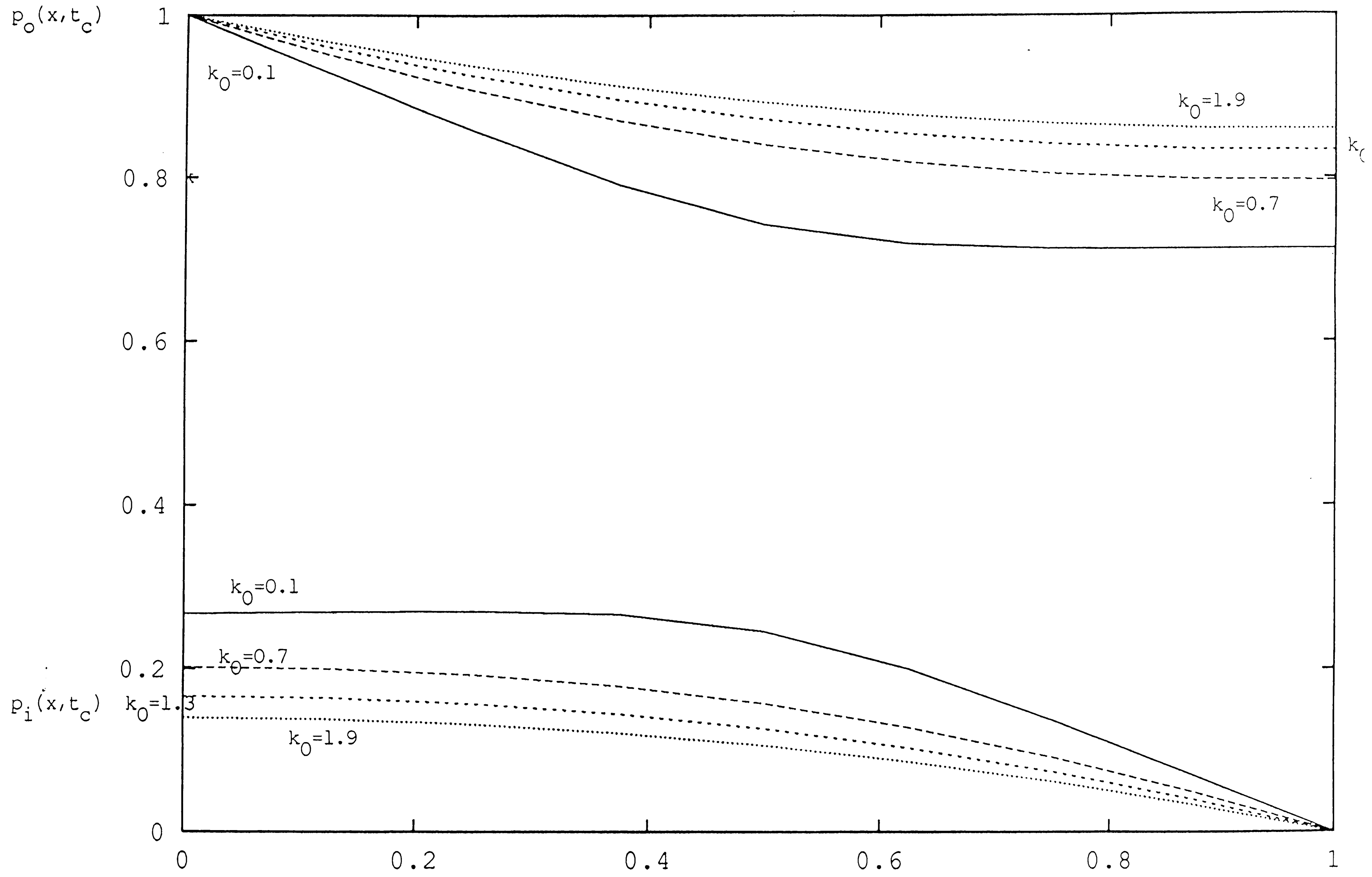


Figure 9.
q(t) vs. t for Various Values of k_0

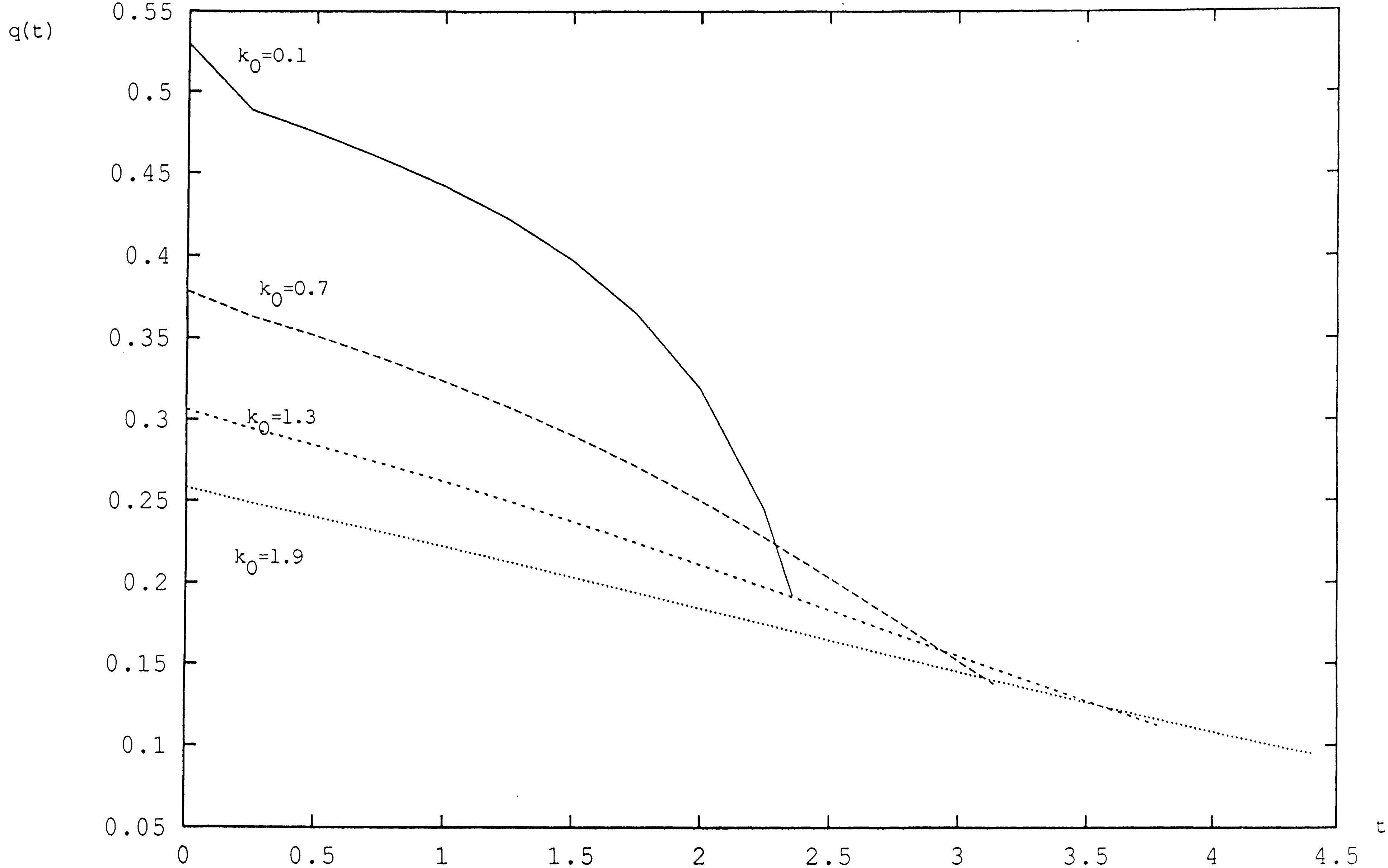


Figure 10.
t_c vs. k_0

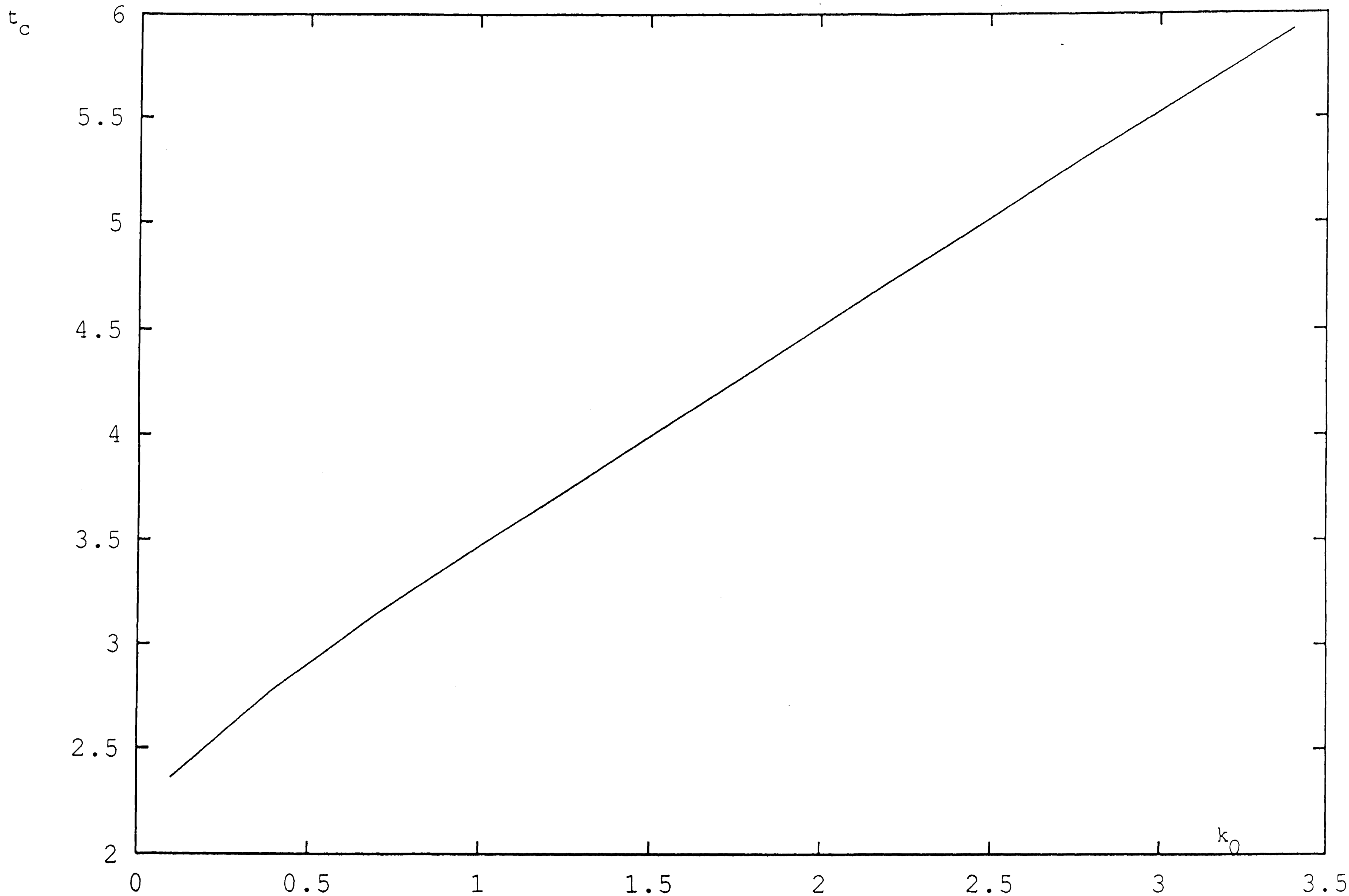
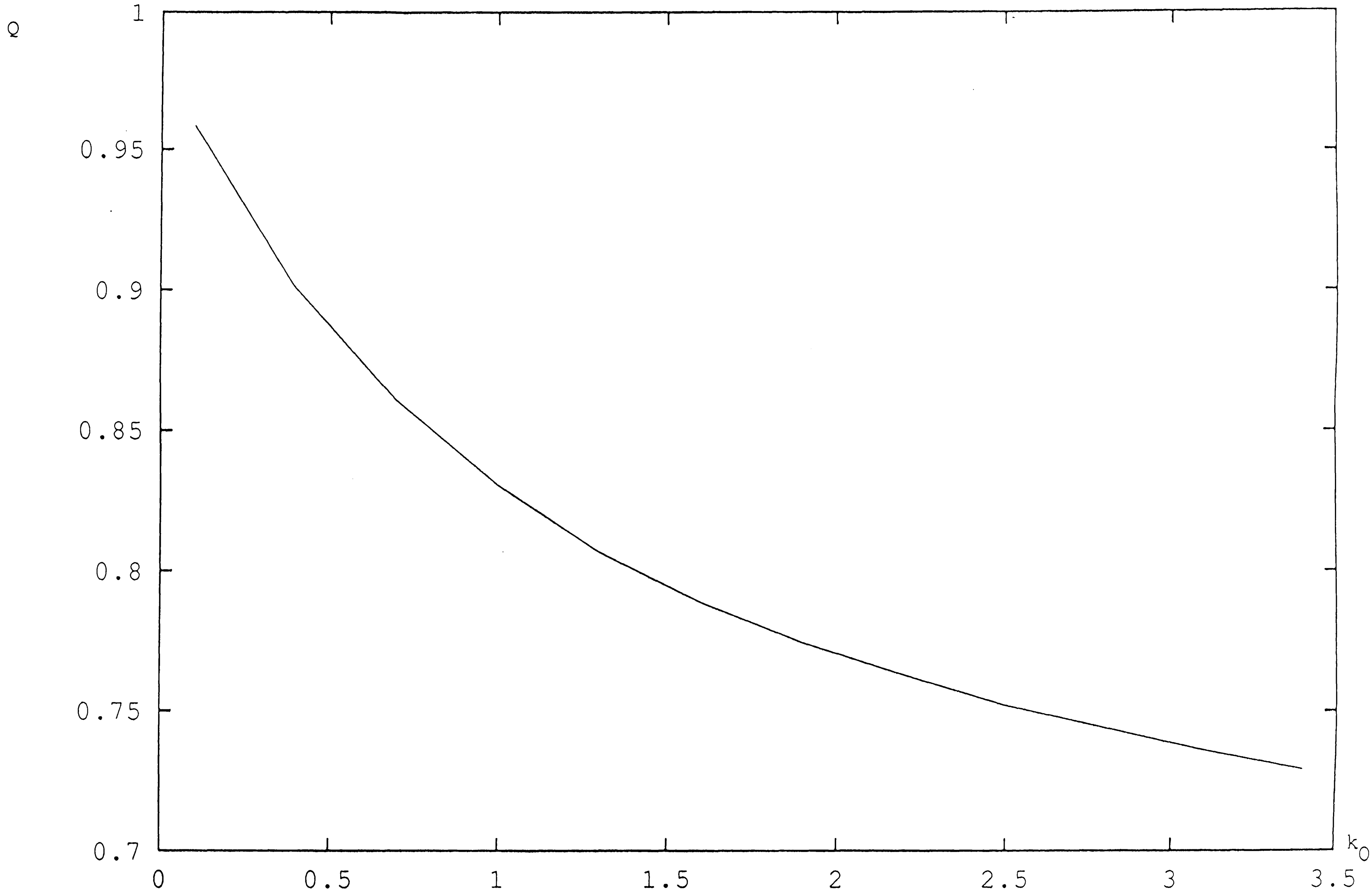


Figure 11.

Q vs. k_0



Section VI: The Three-Dimensional Problem for Steady Flow and No Particles

Next we consider a filter which cannot be reduced to a two-dimensional problem. Here the fluid enters through the top, flows through the filter and leaves through the left side. The perimeter is represented in the figure below by a bold line. The closed walls in the filter unit are represented by brick walls in the figure. As before, this figure actually only shows one portion of a large filter, and the front and back of the figure are planes of *symmetry*.

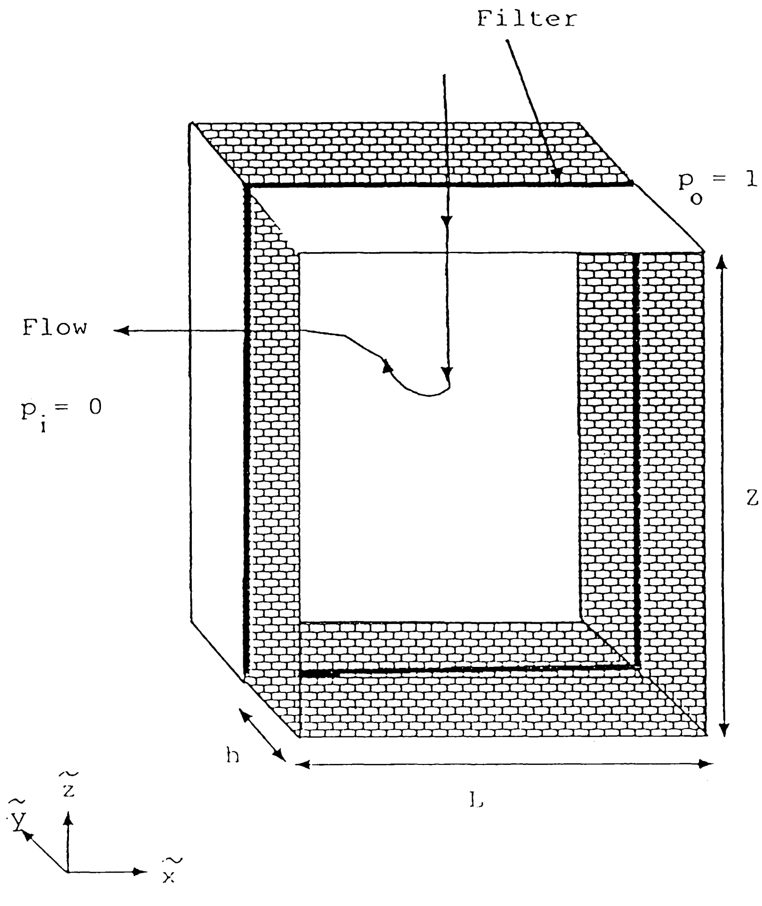


Figure 12. Schematic for three-dimensional problem.

For this problem, we use the three-dimensional Navier-Stokes equations in rectangular coordinates for steady flow. These are

$$\nabla \cdot \tilde{v} = 0 \tag{6.1a}$$

$$(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} = -\frac{1}{\rho}\nabla\tilde{p} + \nu\nabla^2\tilde{\mathbf{v}}\,, \tag{6.1b}$$

where $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v}, \tilde{w})$ and $\tilde{w}$ is the velocity in the $\tilde{z}$ direction. As in the two-dimensional problem, these equations hold for the fluid flow before and after passing through the filter.

Our nondimensionalization process is identical to that of the two-dimensional problem with the addition of

$$z = \frac{\tilde{z}}{L} \qquad \text{and} \qquad w = \frac{\tilde{w}}{U}\,. \tag{6.2}$$

We nondimensionalize $\tilde{z}$ by L since we assume L and Z to be of the same order. Similarly, we expect $\tilde{w}$ to be of the same order as $\tilde{u}$. As before, the nondimensionalization and simplification of equations (6.1) using (6.2) and our other quantities from previous sections lead to the following three-dimensional system of *lubrication equations*:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6.3a}$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} \tag{6.3b}$$

$$\frac{\partial p}{\partial y} = 0 \tag{6.3c}$$

$$\frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2}\,. \tag{6.3d}$$

We consider the region

$$[0,1] \times [0,2] \times [0,\zeta]\,,$$

where ζ is defined in section II. Our problem is again divided into two regions determined by the plane $y = 1$. In region o,

$$[0,1] \times [0,1^-] \times [0,\zeta]\,,$$

we have the following boundary conditions:

$$p_o(x,y,\zeta) = 1\,; \qquad u_o(1,y,z) = 0 \tag{6.4a}$$

$$u_o(0,y,z) = 0\,; \qquad w_o(x,y,0) = 0 \tag{6.4b}$$

$$v_o(x,0,z) = 0\,; \qquad \frac{\partial u_0}{\partial y}(x,0,z) = 0\,; \qquad \frac{\partial w_0}{\partial y}(x,0,z) = 0\,. \tag{6.4c}$$

On the other hand, in region i,

$$[0,1] \times [1^+,2] \times [0,\zeta]\,,$$

the boundary conditions are

$$w_i(x,y,\zeta) = 0\,; \qquad u_i(1,y,z) = 0 \tag{6.5a}$$

$$p_i(0,y,z) = 0\,; \qquad w_i(x,y,0) = 0 \tag{6.5b}$$

$$v_i(x,2,z) = 0\,; \qquad \frac{\partial u_i}{\partial y}(x,2,z) = 0\,; \qquad \frac{\partial w_i}{\partial y}(x,2,z) = 0\,. \tag{6.5c}$$

At the filter our permeability law in its nondimensional form is

$$p_o(x, 1^-, z) - p_i(x, 1^+, z) = k_0 v(x, 1, z)\,. \tag{6.6}$$

Our other boundary conditions at the filter are

$$u(x, 1, z) = 0 \qquad \text{and} \qquad w(x, 1, z) = 0\,. \tag{6.7}$$

Since we are assuming a pure fluid, v must be continuous across the filter, *i.e.*

$$v_o(x, 1^-, z, t) = v_i(x, 1^-, z, t)\,. \tag{6.8}$$

We solve this three-dimensional system in exactly the same manner as in the two-dimensional problem. This gives the following:

$$u_o = \frac{y^2 - 1}{2}\frac{\partial p_o}{\partial x}; \qquad u_i = \frac{(2-y)^2 - 1}{2}\frac{\partial p_i}{\partial x} \tag{6.9}$$

$$w_o = \frac{y^2 - 1}{2}\frac{\partial p_o}{\partial z}; \qquad w_i = \frac{(2-y)^2 - 1}{2}\frac{\partial p_i}{\partial z} \tag{6.10}$$

$$v_o = -\frac{1}{2}\left(\frac{y^3}{3} - y\right)\left(\frac{\partial^2 p_o}{\partial x^2} + \frac{\partial^2 p_o}{\partial z^2}\right) \tag{6.11a}$$

$$v_i = \frac{1}{2}\left[\frac{(2-y)^3}{3} - (2-y)\right]\left(\frac{\partial^2 p_i}{\partial x^2} + \frac{\partial^2 p_i}{\partial z^2}\right)\,. \tag{6.11b}$$

Using the continuity of v at the filter and the permeability law we arrive at the following equations:

$$\frac{\partial^2 p_i}{\partial x^2} + \frac{\partial^2 p_i}{\partial z^2} = -\left(\frac{\partial^2 p_o}{\partial x^2} + \frac{\partial^2 p_o}{\partial z^2}\right) \tag{6.12}$$

$$p_o - p_i = k_0 v_o = \frac{k_0}{3}\left(\frac{\partial^2 p_o}{\partial x^2} + \frac{\partial^2 p_o}{\partial z^2}\right)\,. \tag{6.13}$$

These equations hold in the rectangle $0 < x < 1$ and $0 < z < \zeta$. The boundary conditions are now

$$\frac{\partial p_o}{\partial x}(1, z) = \frac{\partial p_i}{\partial x}(1, z) = 0 \tag{6.14a}$$

$$\frac{\partial p_o}{\partial z}(x, 0) = \frac{\partial p_i}{\partial z}(x, 0) = 0 \tag{6.14b}$$

$$\frac{\partial p_o}{\partial x}(0, z) = \quad p_i(0, z) = 0 \tag{6.14c}$$

$$p_o(x, \zeta) = 1 \qquad \text{and} \qquad \frac{\partial p_i}{\partial z}(x, \zeta) = 0\,. \tag{6.14d}$$

Two attempts were made to solve this system analytically. Neither was successful, as one led to coupled equations and the other to coupled boundary conditions. The most promising method for obtaining a solution seems to be numerical.

Section VII: Future Research

The work presented in this paper may be extended in several directions:

1. A rectangular shape was assumed for the pleats throughout this work. However, the pleat shape is most often not rectangular, but is the result of a folding and crimping process that gives the pleat a characteristic shape and strength, which then determines the behavior. The analysis performed for the rectangular pleat, which assumed a high aspect ratio and slow flow, could be modified to describe the flow in more realistically shaped pleats, such as wedges.
2. The solution obtained for the steady flow in a pleat may be used to calculate the stresses in the filter material. With this information, we may determine deformation in the filter material and whether the yield stress of the material has been exceeded. If the filter undergoes excessive deformation, its performance may be deteriorated, and exceeding the yield stress almost certainly leads to catastrophic failure. These issues are important aspects of filter design.
3. The numerical solution presented in Section V was obtained using *LU* decomposition techniques. The matrix structure, however, is diagonally sparse, and for sufficiently high resolution, iterative techniques will give superior machine performance. In addition, an explicit Eulerian time integration is used. To avoid excessively small time steps to preserve numerical stability, implicit schemes should be considered.
4. The work presented here assumes the particles are trapped within the filter material. There are applications in which the particles do not penetrate the filter material resulting in the accumulation of particulate matter on the filter. This effectively reduces the permeability of the filter and narrows the pleat width available for fluid flow. A more realistic model would combine a nonrectangular pleat configuration as described in number 1 above and incorporate the effects of a growing layer of particulate matter.
5. Finally, the three-dimensional analysis presented in Section VI is woefully incomplete. Further analytical analysis of the steady, no particle flow may give insight about the qualitative behavior of flow in three-dimensional, pleated filter packs. However, for the transient three-dimensional problem, numerical techniques similar to those discussed in Section V and noted in number 3 above would be used.