

Flow Improvement in Hydrofractured Reservoirs

IMA Summer Program for Graduate Students
Mathematical Modeling

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Section I: Introduction and Assumptions

Introduction

A common problem in the oil industry is the discovery of oil or gas in tight (low permeability) formations. This problem is often surmounted by hydrofracturing the well. In this process, the borehole is perforated along a vertical segment at the same depths as the gas bearing strata, and a fracturing fluid is pumped down the borehole at extremely high pressures. This high-pressure fluid fractures the formation, usually creating a (roughly) straight fracture centered at the borehole. The fracture is usually held open by “packers” - sand or gravel that flows into the fracture along with the fluid. Typically the vertical extent of the fracture roughly matches the top and bottom of the gas bearing strata (see figures 1a and 1b), that is, the top and bottom of the permeable layer. Although these fractures are very thin, usually no more than a few millimeters across, the permeability within the fractures is radically larger than the permeability of the surrounding formation. We will consider how the productivity (flow rate through borehole) changes due to hydrofracturing.

Advances in hydrofracturing techniques promise to yield wider, longer, and more permeable fractures. In fact, it appears possible to generate multiple fractures. So we will also consider how increasing the number of fractures changes the productivity of the well.

Assumptions

In modeling this problem, we make the following assumptions:

- The pay zone is cylindrical.
- The temperature within the pay zone and fracture is constant.
- The atmospheric pressure is so small in comparison to the pressure inside the pay zone that we consider it to be zero.
- The pressure on the edge of the pay zone is constant.
- The fracture is straight and centered at the borehole.
- The top and bottom of the pay zone are impermeable, and we will have no variation in the vertical direction.
- This is a steady-state problem.
- We consider the gas to be ideal.
- We consider the problem to be governed by Darcy’s law.

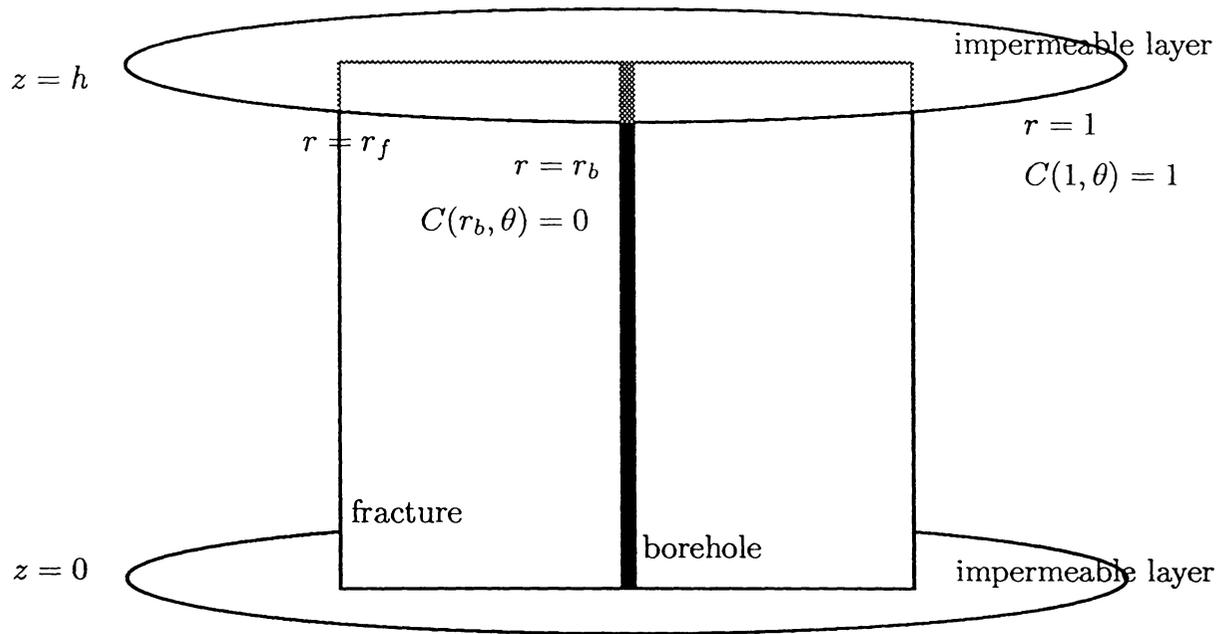


Figure 1a. Nondimensional schematic of pay zone, side view.

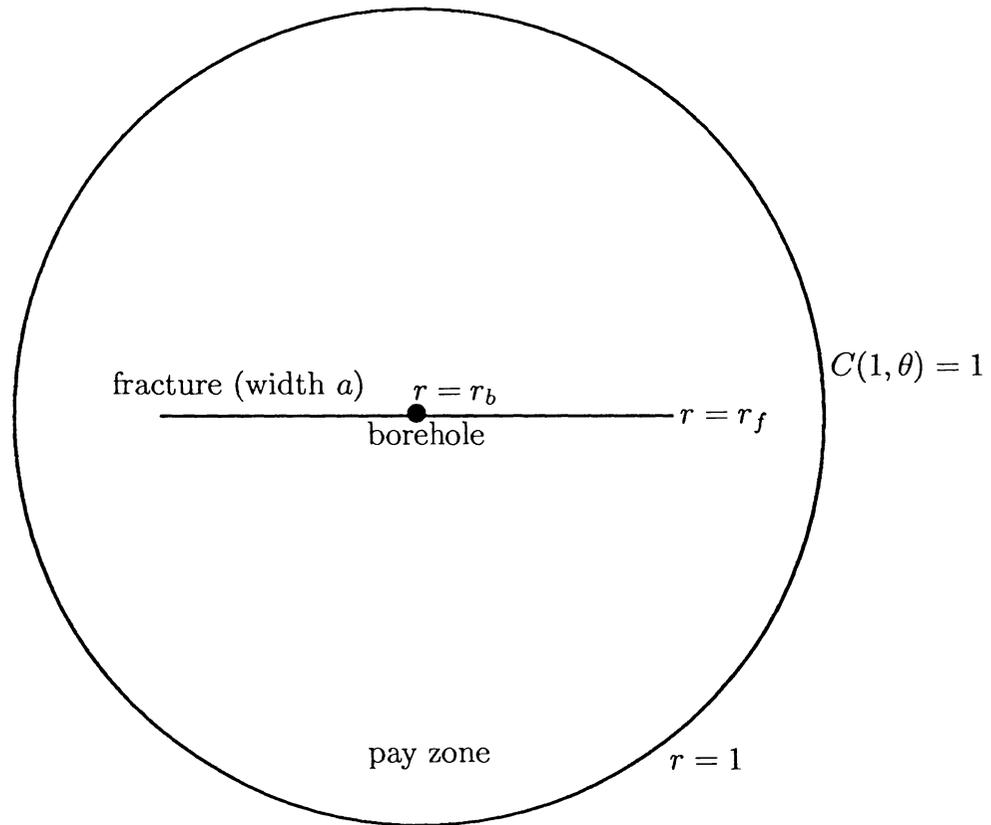


Figure 1b. Nondimensional schematic of pay zone, top view.

Section II: Nomenclature

- a : width of fracture (perturbation model), value 2×10^{-3} m.
 $a_j(\tilde{r})$: width of fracture j (variational model), units m.
 A_j : constants used in equations ($j = 1, 2, 3, 4, 5$).
 $\tilde{C}_0(\mathbf{x}), \tilde{C}_f(\mathbf{x})$: concentration of gas at position \mathbf{x} in the pay zone, fracture. Units mol/m².
 $\tilde{C}_0^{(o)}(\mathbf{x}), \tilde{C}_0^{(u)}(\mathbf{x}),$
 $\tilde{C}_0^{(v)}(\mathbf{x})$: outer, uniform, and variational approximations to $\tilde{C}_0(\mathbf{x})$. Units mol/m².
 $\tilde{C}_f^{(j)}(\mathbf{x})$: j th term in the perturbation expansion for $\tilde{C}_f(\mathbf{x})$. Units mol/m².
 d : constant used in conclusion.
 D : diffusion coefficient for gas (units m²/sec), defined as

$$D \equiv \frac{RTk}{\mu}. \quad (2.1)$$

- $E_1(s)$: the first exponential integral at value s , defined by

$$E_1(s) = \int_s^\infty \frac{e^{-t}}{t} dt = \int_1^\infty \frac{e^{-st}}{t} dt. \quad (2.2)$$

- $f(r, \theta)$: nondimensional test function (variational model).
 h : height of the pay zone. Units m.
 j : indexing variable.
 $\tilde{\mathbf{J}}(\mathbf{x})$: flux of gas at position \mathbf{x} . Units mol/(m·sec).
 k : permeability constant in Darcy's law. Units mol/m².
 m : constant.
 n : constant.
 N : number of fractures.
 $\tilde{p}(\mathbf{x})$: pressure of gas at position \mathbf{x} . Units kg/(m·sec²).
 \tilde{P}_0 : pressure of gas at $\tilde{r} = \tilde{r}_0$. Units kg/(m·sec²).
 q : nondimensional flux constant.
 \tilde{Q} : flux of gas (units mol/sec) through borehole, defined as

$$\tilde{Q} = -\tilde{r} \int_0^{2\pi} \tilde{\mathbf{J}}(\mathbf{x}) d\theta. \quad (2.3)$$

Note that the radius of the circle around which you integrate this is arbitrary due to Gauss' law.

- \tilde{r} : radius from axis of pay zone. Units m.
 \tilde{r}_0 : radius of pay zone, value 300 m.

- \tilde{r}_b : radius of borehole, value 5×10^{-2} m.
 \tilde{r}_f : radius of fracture, value 50 m.
 R : gas law constant, value 8.314×10^3 kg·m²/(sec²·K).
 \Re : the real part of a complex expression.
 s : dummy variable.
 t : integration variable.
 T : temperature of gas, units K.
 $u(\hat{x}, \hat{\zeta})$: difference of nondimensional concentration in inner boundary layer from minimum value.
 $v(\lambda, \hat{\zeta})$: Fourier cosine transform of u , given by the transform pair

$$v(\lambda, \hat{\zeta}) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(\hat{x}, \hat{\zeta}) \cos(\lambda \hat{x}) d\hat{x} \quad (2.4a)$$

$$u(\hat{x}, \hat{\zeta}) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} v(\lambda, \hat{\zeta}) \cos(\lambda \hat{x}) d\lambda. \quad (2.4b)$$

- \tilde{x} : distance along fracture axis from axis of pay zone. Units m.
 \tilde{y} : distance (in fracture) from fracture axis. Units m.
 \tilde{z} : distance along axis of pay zone. Units m.
 α : ratio of radius of pay zone to radius of fracture, defined by $\alpha = \tilde{r}_0/\tilde{r}_f$.
 β : nondimensional parameter, defined by $\beta = 1/\varepsilon_0$.
 γ : Euler's constant, value 0.577.
 ε_0 : perturbation expansion parameter, defined by

$$\varepsilon_0 \equiv \frac{aD_f}{2\tilde{r}_0D_0} = 3 \times 10^{-2}. \quad (2.5)$$

- ε_f : perturbation expansion parameter, defined by

$$\varepsilon_f \equiv \frac{a}{2\tilde{r}_0} = 3 \times 10^{-6}. \quad (2.6)$$

- η : small nondimensional parameter used in equations.
 λ : Fourier transform variable.
 μ : dynamic viscosity of gas, units kg/(m·sec).
 $\tau(r)$: function (units m) used in variational method, defined by

$$\tau(r) \equiv \sum_{j=1}^N 2\beta_j(r). \quad (2.7)$$

- θ : angle measured from fracture axis.
 Υ : function to be minimized in variational method.
 ζ : nondimensionalized distance (in pay zone) from fracture axis.
 Ω : the cylindrical pay zone region.

Nondimensionalized variables will have no tildes. The subscript 0 refers to the rock region, while the subscript f refers to the fracture. Parenthesized superscripts indicate terms in the perturbation expansion. Hats refer to inner expansion variables.

Section III: Governing Equations

We begin by writing the form of Darcy's law which we wish to use:

$$\tilde{\mathbf{J}} = -\frac{k}{\mu}\nabla\tilde{p}. \quad (3.1)$$

In addition, we will use Fick's diffusion law:

$$\frac{\partial\tilde{C}}{\partial t} = -\nabla\cdot\tilde{\mathbf{J}}, \quad (3.2)$$

and the ideal gas law:

$$\tilde{p} = \tilde{C}RT, \quad (3.3)$$

where we assume T to be constant in our region. Combining equations (3.1)-(3.3), we have the following expression for \tilde{C} :

$$\begin{aligned} \frac{\partial\tilde{C}}{\partial t} &= -\nabla\cdot\left(-\frac{k}{\mu}\nabla\tilde{p}\right) \\ &= \frac{k}{\mu}\nabla^2(\tilde{C}RT) \\ &= D\nabla^2\tilde{C}. \end{aligned} \quad (3.4)$$

Note that equation (3.4) holds in either the fracture or the pay zone.

We now assume that the only quantity which fundamentally changes from the pay zone to the fracture is k , and hence D . We assume the pay zone to be a cylinder with height h , inner radius \tilde{r}_b (the radius of the bore hole), and outer radius \tilde{r}_0 . We assume that the pressure on the outer radius of the cylinder is constant, *i.e.*,

$$\tilde{p}_0(\tilde{r}_0, \theta, \tilde{z}) = P_0. \quad (3.5)$$

The pressure at the bore hole is atmospheric pressure, which is so much smaller than P_0 that we assume it to be 0. Hence, we have

$$\tilde{p}_0(\tilde{r}_b, \theta, \tilde{z}) = 0. \quad (3.6)$$

We assume as well that the top and bottom ends of our tube are impermeable. Then, since neither equation (3.5) nor (3.6) depend on \tilde{z} , and since no flow can come in from above or below our tube, we conclude that our entire solution is independent of \tilde{z} and reduce our problem to a two-dimensional system. We also are looking for steady-state solutions, so we set our time derivatives equal to 0.

Writing equation (3.4) in polar coordinates in the pay zone, we have

$$D_0 \left(\frac{\partial^2 \tilde{C}_0}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{C}_0}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2} \frac{\partial^2 \tilde{C}_0}{\partial \theta^2} \right) = 0. \quad (3.7)$$

We assume the fracture to be so thin that we may assume it to be in rectangular coordinates, so we rewrite equation (3.4) in that region:

$$D_f \left(\frac{\partial^2 \tilde{C}_f}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{C}_f}{\partial \tilde{y}^2} \right) = 0. \quad (3.8)$$

Now we nondimensionalize our equations. We nondimensionalize \tilde{r} , \tilde{x} , and all our associated constant radii by \tilde{r}_0 , our pay zone radius. We normalize our concentrations by the maximal concentration at the boundary, which is given by (3.3) and (3.5) to be P_0/RT . We nondimensionalize \tilde{y} by $a/2$, half the width of our fracture. Summarizing, we have the following:

$$r = \frac{\tilde{r}}{\tilde{r}_0}, \quad x = \frac{\tilde{x}}{\tilde{r}_0}, \quad r_f = \frac{\tilde{r}_f}{\tilde{r}_0}, \quad r_b = \frac{\tilde{r}_b}{\tilde{r}_0}, \quad (3.9a)$$

$$C_0 = \frac{\tilde{C}_0 RT}{P_0}, \quad C_f = \frac{\tilde{C}_f RT}{P_0}, \quad y = \frac{2\tilde{y}}{a}. \quad (3.9b)$$

Using equations (3.9) in (3.7), we have

$$\begin{aligned} \frac{P_0 D_0}{RT \tilde{r}_0^2} \left(\frac{\partial^2 C_0}{\partial r^2} + \frac{1}{r} \frac{\partial C_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C_0}{\partial \theta^2} \right) &= 0 \\ \frac{\partial^2 C_0}{\partial r^2} + \frac{1}{r} \frac{\partial C_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C_0}{\partial \theta^2} &= 0. \end{aligned} \quad (3.10)$$

Using equations (3.9) in (3.8), we have

$$\begin{aligned} \frac{P_0 D_f}{RT} \left(\frac{1}{\tilde{r}_0^2} \frac{\partial^2 C_f}{\partial x^2} + \frac{4}{a^2} \frac{\partial^2 C_f}{\partial y^2} \right) &= 0 \\ \epsilon_f^2 \frac{\partial^2 C_f}{\partial x^2} + \frac{\partial^2 C_f}{\partial y^2} &= 0. \end{aligned} \quad (3.11)$$

Now we need to complete our boundary conditions for our system of equations (3.10) and (3.11). In the rest of this report, if no range is listed for an independent variable, it is assumed to be the entire range. We have let $y = 0$ be the centerline of our fracture, so since the problem is symmetric, we have

$$\frac{\partial C_f}{\partial y}(x, 0) = 0, \quad (3.12)$$

and we only need to solve in the region $0 \leq y \leq 1$. In addition, since the fracture splits in two directions, we know that the concentration in the pay zone must be symmetric in each of the four quadrants; hence we only need to solve in the region $0 \leq \theta \leq \pi/2$ and we have

$$\frac{\partial C_0}{\partial \theta} \left(r, \frac{\pi}{2} \right) = 0 \quad (3.13a)$$

$$\frac{\partial C_0}{\partial \theta} (r, 0) = 0, \quad r_f \leq r \leq 1. \quad (3.13b)$$

Along the boundary between the fracture and the rock we want the concentration to be continuous. We approximate the top of the fracture in the pay zone coordinates by $(r, \tan^{-1}(\alpha \varepsilon_f))$ for $r_b \leq r \leq r_f$. However, since ε_f is small, $\tan^{-1}(\alpha \varepsilon_f) \approx \alpha \varepsilon_f$ and we have

$$C_0(r, \alpha \varepsilon_f) = C_f(x, 1), \quad r_b \leq r \leq r_f. \quad (3.14a)$$

Similarly, along the end of the fracture we have

$$C_0(r_f, \theta) = C_f(r_f, y), \quad 0 \leq \theta \leq \alpha \varepsilon_f. \quad (3.14b)$$

In addition, we want the normal flux to be continuous there. First we note that $\tilde{\mathbf{J}} = -D\nabla\tilde{C}$. Therefore, to balance along the top of the fracture we have to set the angular flux into the fracture equal to the flux in the y direction inside the fracture, which we write as

$$\begin{aligned} -\frac{D_0}{\tilde{r}} \frac{\partial \tilde{C}_0}{\partial \theta} (\tilde{r}, \alpha \varepsilon_f) &= -D_f \frac{\partial \tilde{C}_f}{\partial \tilde{y}} \left(\tilde{x}, \frac{a}{2} \right), \quad \tilde{r}_b \leq \tilde{r} \leq \tilde{r}_f \\ \frac{D_0 P_0}{\tilde{r}_0 RT} \frac{1}{r} \frac{\partial C_0}{\partial \theta} (r, \alpha \varepsilon_f) &= \frac{2D_f P_0}{a RT} \frac{\partial C_f}{\partial y} (x, 1), \quad r_b \leq r \leq r_f \\ \frac{\beta \varepsilon_f^2}{r} \frac{\partial C_0}{\partial \theta} (r, \alpha \varepsilon_f) &= \frac{\partial C_f}{\partial y} (x, 1), \quad r_b \leq r \leq r_f. \end{aligned} \quad (3.15a)$$

To balance along the end of the fracture we have to set the radial flux into the fracture equal to the flux in the x direction inside the fracture, which we write as

$$\begin{aligned} -D_0 \frac{\partial \tilde{C}_0}{\partial \tilde{r}} (\tilde{r}_f, \theta) &= -D_f \frac{\partial \tilde{C}_f}{\partial \tilde{x}} (\tilde{r}_f, \tilde{y}), \quad 0 \leq \theta \leq \alpha \varepsilon_f \\ \frac{D_0 P_0}{\tilde{r}_0 RT} \frac{\partial C_0}{\partial r} (r_f, \theta) &= \frac{D_f P_0}{\tilde{r}_0 RT} \frac{\partial C_f}{\partial x} (r_f, y), \quad 0 \leq \theta \leq \alpha \varepsilon_f \\ \beta \varepsilon_f \frac{\partial C_0}{\partial r} (r_f, \theta) &= \frac{\partial C_f}{\partial x} (r_f, y), \quad 0 \leq \theta \leq \alpha \varepsilon_f. \end{aligned} \quad (3.15b)$$

We end this section by nondimensionalizing equations (3.5) and (3.6) and converting them to concentration values:

$$C_0(1, \theta) = 1 \quad (3.16)$$

$$C_0(r_b, \theta) = 0. \quad (3.17)$$

Since the fracture also abuts the bore hole, we have

$$C_f(r_b, y) = 0. \quad (3.18)$$

Section IV: No Fracture

The first model we consider is that of gas flow with no fracture in the pay zone. Hence we only have region 0 to consider, and there is no fracture to break radial symmetry. Hence equation (3.10) with its remaining relevant boundary conditions (3.16) and (3.17) become

$$\frac{d^2 C_0}{dr^2} + \frac{1}{r} \frac{dC_0}{dr} = 0 \quad (4.1)$$

$$C_0(1) = 1 \quad (4.2)$$

$$C_0(r_b) = 0. \quad (4.3)$$

The solution of equation (4.1) is $C_0(r) = A_1 + A_2 \log r$, where the A_j are constants. Solving for them, we have

$$C_0(r) = \frac{\log(r/r_b)}{\log(1/r_b)}. \quad (4.4)$$

Now we wish to know the flux through the borehole. The flux is only in the radial direction, so we have

$$\tilde{Q} = \tilde{r}_b \int_0^{2\pi} D_0 \frac{\partial \tilde{C}_0}{\partial \tilde{r}}(\tilde{r}_b) d\theta.$$

Nondimensionalizing, we have the following:

$$\begin{aligned} \frac{\tilde{Q}RT}{D_0 P_0} &\equiv Q = r_b \int_0^{2\pi} \frac{\partial C_0}{\partial r}(r_b) d\theta \\ &= -\frac{2\pi}{\log(r_b)}. \end{aligned} \quad (4.5)$$

Section V: Fracture Included

Now we wish to examine the more difficult case of the full set of equations derived in section III. Since $\varepsilon_f \ll 1$ and ε_f only appears in conjunction with C_f terms, we postulate the following perturbation expansion in ε_f :

$$C_f = \sum_{j=0}^{\infty} \varepsilon_f^{jn} C_f^{(j)}, \quad (5.1)$$

where n is as yet undetermined. Substituting equation (5.1) into the equations with ε_f explicitly in them, namely (3.11), (3.15a), and (3.15b), we have the following:

$$\sum_{j=0}^{\infty} \varepsilon_f^{jn} \left[\varepsilon_f^2 \frac{\partial^2 C_f^{(j)}}{\partial x^2} + \frac{\partial^2 C_f^{(j)}}{\partial y^2} \right] = 0 \quad (5.2)$$

$$\frac{\beta \varepsilon_f^2}{r} \frac{\partial C_0}{\partial \theta}(r, \alpha \varepsilon_f) = \sum_{j=0}^{\infty} \varepsilon_f^{jn} \frac{\partial C_f^{(j)}}{\partial y}(x, 1), \quad r_b \leq r \leq r_f \quad (5.3)$$

$$\beta \varepsilon_f \frac{\partial C_0}{\partial r}(r_f, \theta) = \sum_{j=0}^{\infty} \varepsilon_f^{jn} \frac{\partial C_f^{(j)}}{\partial x}(r_f, y), \quad 0 \leq \theta \leq \alpha \varepsilon_f. \quad (5.4)$$

We see that equation (5.4) provides the strongest criterion for n , namely that $n = 1$. For the remaining equations in our system, we simply match term-by-term. We now begin our consideration of our system of equations in the fracture. From equations (5.3) and (5.4) we see that to obtain boundary conditions for C_0 , we have to solve our equations up to order ε_f^2 .

Order 0 in Fracture

Using the fact that $n = 1$ and taking the zeroth order terms from (5.2), (3.12), (5.3), and (3.18), we have

$$\frac{\partial^2 C_f^{(0)}}{\partial y^2} = 0 \quad (5.5)$$

$$\frac{\partial C_f^{(0)}}{\partial y}(x, 0) = 0 \quad (5.6)$$

$$\frac{\partial C_f^{(0)}}{\partial y}(x, 1) = 0 \quad (5.7)$$

$$C_f^{(0)}(r_b, y) = 0. \quad (5.8)$$

The solution of equations (5.5)-(5.8) is $C_f^{(0)}(x, y) = C_f^{(0)}(x)$. Note that there is now no variance in the y direction, and hence no variance across $0 \leq \theta \leq \alpha \varepsilon_f$. Since $\varepsilon_f \ll 1$, we then approximate our boundary conditions there by $\theta = 0$ and allow (5.4) to become a point boundary condition. This causes the rest of the zeroth-order equations [namely (3.14a), (3.14b), and (5.4)] to become

$$C_f^{(0)}(x) = C_0(r, 0), \quad r_b \leq r \leq r_f \quad (5.9)$$

$$C_f^{(0)}(r_f) = C_0(r_f, 0) \quad (5.10)$$

$$C_f^{(0)'}(r_f) = 0. \quad (5.11)$$

Order 1 in Fracture

Now taking the first order terms of our fracture equations (5.2), (3.12), and (5.3), we have

$$\frac{\partial^2 C_f^{(1)}}{\partial y^2} = 0 \quad (5.12)$$

$$\frac{\partial C_f^{(1)}}{\partial y}(x, 0) = 0 \quad (5.13)$$

$$\frac{\partial C_f^{(1)}}{\partial y}(x, 1) = 0. \quad (5.14)$$

The solution of equations (5.12)-(5.14) is $C_f^{(1)}(x, y) = C_f^{(1)}(x)$. However, since equation (3.14a) becomes $C_f^{(1)}(x, 1) = 0$, we have $C_f^{(1)}(x, y) \equiv 0$. Hence the first-order term of equation (5.4), which is the real equation of interest for this order, becomes

$$\frac{\partial C_0}{\partial r}(r_f, 0) = 0. \quad (5.15)$$

Order 2 in Fracture

Now taking the second order terms of our fracture equations (5.2), (3.12), and (3.14a), we have

$$\frac{\partial^2 C_f^{(2)}}{\partial y^2} + C_f^{(0)''} = 0 \quad (5.16)$$

$$\frac{\partial C_f^{(2)}}{\partial y}(x, 0) = 0 \quad (5.17)$$

$$C_f^{(2)}(x, 1) = 0. \quad (5.18)$$

The solution of equations (5.16)-(5.18) is

$$C_f^{(2)}(x, y) = C_f^{(0)''}(x) \frac{1-y^2}{2}. \quad (5.19)$$

Also, equations (3.18), (3.14b), (5.3), and (5.4) become

$$C_f^{(2)}(r_b, y) = 0 \quad (5.20)$$

$$C_f^{(2)}(r_f, y) = 0 \quad (5.21)$$

$$\frac{\beta}{r} \frac{\partial C_0}{\partial \theta}(r, 0) = \frac{\partial C_f^{(2)}}{\partial y}(x, 1), \quad r_b \leq r \leq r_f \quad (5.22)$$

$$\frac{\partial C_f^{(2)}}{\partial x}(r_f, y) = 0. \quad (5.23)$$

Note that equation (5.23) is a third boundary condition for a two-point boundary value problem; hence it becomes a consistency condition. Using equation (5.19) in equations (5.20)-(5.23), we have the following system:

$$\frac{\beta}{r} \frac{\partial C_0}{\partial \theta}(r, 0) = -C_f^{(0)''}(x), \quad r_b \leq r \leq r_f \quad (5.24)$$

$$C_f^{(0)''}(r_b) = C_f^{(0)''}(r_f) = C_f^{(0)'''}(r_f) = 0. \quad (5.25)$$

However, since $r = x$ on $\theta = 0$, (5.24) is also a boundary condition for equation (3.10). Hence, we now have a solvable system of equations for C_0 :

$$\frac{\partial^2 C_0}{\partial r^2} + \frac{1}{r} \frac{\partial C_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C_0}{\partial \theta^2} = 0 \quad (5.26)$$

$$C_0(1, \theta) = 1 \quad (5.27)$$

$$C_0(r_b, \theta) = 0 \quad (5.28)$$

$$\frac{\partial C_0}{\partial \theta} \left(r, \frac{\pi}{2} \right) = 0 \quad (5.29)$$

$$\frac{\beta}{r} \frac{\partial C_0}{\partial \theta}(r, 0) = -\frac{\partial^2 C_0}{\partial r^2}(r, 0), \quad r_b \leq r \leq r_f \quad (5.30)$$

$$\frac{\partial C_0}{\partial \theta}(r, 0) = 0, \quad r_f \leq r \leq 1 \quad (5.31)$$

$$\frac{\partial C_0}{\partial r}(r_f, 0) = 0. \quad (5.32)$$

Section VI: Singular Perturbation Solution

Outer Solution

Now we wish to solve our system of equations (5.26)-(5.32) using a singular perturbation approach. We begin by assuming the following perturbation expansion of C_0 in ε_0 :

$$C_0 = C_0^{(o)} + o(1), \quad (6.1)$$

where the superscript (o) indicates that this is the outer solution. Using equation (6.1) in equation (5.30), we have the following:

$$\frac{\partial C_0^{(o)}}{\partial \theta}(r, 0) = -r\varepsilon_0 \frac{\partial^2 C_0^{(o)}}{\partial r^2}(r, 0), \quad r_b \leq r \leq r_f. \quad (6.2)$$

(Since $\varepsilon_f \ll \varepsilon_0$, our assumption that we may approximate the boundary condition by $\theta = 0$ still holds true.) Now letting $\varepsilon_0 \rightarrow 0$ and combining (6.2) with (5.31), we have a new system for $C_0^{(o)}$:

$$\frac{\partial^2 C_0^{(o)}}{\partial r^2} + \frac{1}{r} \frac{\partial C_0^{(o)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C_0^{(o)}}{\partial \theta^2} = 0 \quad (6.3)$$

$$C_0^{(o)}(1, \theta) = 1 \quad (6.4)$$

$$C_0^{(o)}(r_b, \theta) = 0 \quad (6.5)$$

$$\frac{\partial C_0^{(o)}}{\partial \theta} \left(r, \frac{\pi}{2} \right) = 0 \quad (6.6)$$

$$\frac{\partial C_0^{(o)}}{\partial \theta}(r, 0) = 0 \quad (6.7)$$

$$\frac{\partial C_0^{(o)}}{\partial r}(r_f, 0) = 0. \quad (6.8)$$

However, we now note that there is no longer any θ dependence in the problem and our solution immediately reduces to (4.4):

$$C_0^{(o)}(r, \theta) = \frac{\log(r/r_b)}{\log(1/r_b)}. \quad (6.9)$$

To check our assumption, we plug in our solution (6.9) into (6.2):

$$0 = \frac{\varepsilon_0}{r \log(1/r_b)}. \quad (6.10)$$

Unfortunately, $r_b = 1.67 \times 10^{-4} = O(\varepsilon_0^2)$. Hence the right-hand side is no longer negligible in a region near the bore hole. Thus, we must try a singular perturbation approach by introducing an inner expansion variable scaling where the two terms in equation (6.2) are of the same order. Our outer expansion still holds true for the outer region, so we lose one boundary condition and our outer solution becomes the following:

$$C_0^{(o)}(r, \theta) = 1 + A_3 \log r, \quad (6.11)$$

where A_3 is yet to be determined.

Inner Solution

We expect $C_0^{(o)}$ to be $O(1)$ in the region, so we introduce the following scalings in the independent variables only:

$$\hat{r} = \frac{r}{\varepsilon_0^m}, \quad \hat{\theta} = \frac{\theta}{\varepsilon_0^n}, \quad \hat{C}_0(\hat{r}, \hat{\theta}) = C_0^{(o)}(r, \theta), \quad m \geq 0, \quad n \geq 0. \quad (6.12)$$

Then, for the proper choices of m and n , these variables will have the unique quality that

$$\lim_{\hat{r} \rightarrow \infty} \hat{C}_0(\hat{r}, \hat{\theta}) = \lim_{r \rightarrow r_b} C_0^{(o)}(r, \theta). \quad (6.13)$$

We also approximate our small radius r_b , which is $O(\varepsilon_0^2)$, by the origin. Using that fact and equations (6.12) in (6.3), we have the following:

$$\varepsilon_0^{-2m} \frac{\partial^2 \hat{C}_0}{\partial \hat{r}^2} + \varepsilon_0^{-2m} \frac{1}{\hat{r}} \frac{\partial \hat{C}_0}{\partial \hat{r}} + \varepsilon_0^{-2m-2n} \frac{1}{\hat{r}^2} \frac{\partial^2 \hat{C}_0}{\partial \hat{\theta}^2} = 0, \quad 0 \leq \hat{r} < \infty. \quad (6.14)$$

Note that if $n > 0$, then the leading order in equation (6.14) implies that \hat{C}_0 is a linear function of $\hat{\theta}$, which we cannot match to our outer expansion $C_0^{(o)}$. Hence we find that $n = 0$, and we will drop the hat on θ in the rest of the section.

Next we introduce our scalings into equation (6.2):

$$\frac{\partial \hat{C}_0}{\partial \theta}(\hat{r}, 0) = -\hat{r} \varepsilon_0^{1-m} \frac{\partial^2 \hat{C}_0}{\partial \hat{r}^2}(\hat{r}, 0), \quad 0 \leq \hat{r} < \infty. \quad (6.15)$$

This implies that $m = 1$. Now, in order to make the problem easier to solve, we convert from (\hat{r}, θ) to $(\hat{x}, \hat{\zeta})$, where the standard transformations are used:

$$\hat{C}_0(\hat{r}, \theta) = \hat{C} \left(\sqrt{\hat{x}^2 + \hat{\zeta}^2}, \tan^{-1} \left(\frac{\hat{\zeta}}{\hat{x}} \right) \right). \quad (6.16)$$

Doing so, equation (6.14) becomes the following:

$$\frac{\partial^2 \hat{C}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{C}}{\partial \hat{\zeta}^2} = 0, \quad 0 \leq \hat{x} < \infty, \quad 0 < \hat{\zeta} < \infty. \quad (6.17)$$

Here the $\hat{\zeta}$ has a strict inequality since we have the differential equation (6.15) for a boundary condition there. Since we are approximating r_b by the origin, equation (6.5) becomes

$$\hat{C}(0, 0) = 0. \quad (6.18)$$

Continuing with our boundary conditions, we have from equation (6.6) that

$$\frac{\partial \hat{C}}{\partial \hat{x}}(0, \hat{\zeta}) = 0, \quad \hat{\zeta} > 0. \quad (6.19a)$$

Note that equation (6.19a) has a strict inequality. This is because at the fracture (which we have now taken to be the line $\hat{\zeta} = 0$), the flux is *not* 0. In fact, taking the derivative of equation (3.14a) with respect to f on the left-hand side and with respect to x on the right-hand side (since $r = x$ on the line $\theta = 0$, which is our approximate fracture boundary), we have

$$\frac{\partial \tilde{C}_f}{\partial \tilde{x}}(0, 0) = \frac{\partial \tilde{C}_0}{\partial \tilde{r}}(0, 0).$$

Then nondimensionalizing and solving together, we have

$$\begin{aligned} \frac{\partial C_0}{\partial r}(0, 0) &= \frac{\partial C_f}{\partial x}(0, 0) \\ \frac{1}{\varepsilon_0} \frac{\partial \hat{C}_0}{\partial \hat{r}}(0, 0) &= \frac{\partial C_f}{\partial x}(0, 0) \\ \frac{\partial \hat{C}}{\partial \hat{x}}(0, 0) &= \varepsilon_0 \frac{\partial C_f}{\partial x}(0, 0) \equiv q. \end{aligned} \quad (6.19b)$$

In our solution, we solve as if $q = O(1)$, since we now expect a large flux at the borehole.

Continuing with our boundary conditions, from equation (6.15) we have

$$\frac{\partial \hat{C}}{\partial \hat{\zeta}}(\hat{x}, 0) = -\frac{\partial^2 \hat{C}}{\partial \hat{x}^2}(\hat{x}, 0). \quad (6.20)$$

Our matching condition (6.13) gives us the following:

$$\hat{C}(\infty, \hat{\zeta}) = C_0^{(o)}(r_b) \quad (6.21a)$$

$$\hat{C}(\hat{x}, \infty) = C_0^{(o)}(r_b). \quad (6.21b)$$

We wish to solve equations (6.17)-(6.21) using a Fourier cosine transform method. To use that method, however, our boundary conditions at \hat{x} and $\hat{\zeta}$ equal to ∞ must be 0. Hence we introduce the following transformation:

$$\hat{C}(\hat{x}, \hat{\zeta}) = C_0^{(o)}(r_b) [1 - u(\hat{x}, \hat{\zeta})]. \quad (6.22)$$

Using equation (6.22) in equations (6.17), we have the following:

$$\frac{\partial^2 u}{\partial \hat{x}^2} + \frac{\partial^2 u}{\partial \hat{\zeta}^2} = 0, \quad 0 \leq \hat{x} \leq \infty, \quad 0 < \hat{\zeta} < \infty. \quad (6.23)$$

Equation (6.21a) becomes

$$u(\infty, \hat{\zeta}) = 0, \quad (6.24)$$

while equation (6.19a) becomes

$$\frac{\partial u}{\partial \hat{x}}(0, \hat{\zeta}) = 0, \quad \hat{\zeta} > 0. \quad (6.25)$$

Continuing to rewrite our boundary conditions, equation (6.20) becomes

$$\frac{\partial u}{\partial \hat{\zeta}}(\hat{x}, 0) + \frac{\partial^2 u}{\partial \hat{x}^2}(\hat{x}, 0) = 0, \quad (6.26)$$

equation (6.19b) becomes

$$\frac{\partial u}{\partial \hat{x}}(0, 0) = -\frac{q}{C_0^{(o)}(r_b)}, \quad (6.27)$$

and equation (6.21b) becomes

$$u(\hat{x}, \infty) = 0. \quad (6.28)$$

Lastly, we may rewrite our boundary condition at the origin given by (6.18) as

$$u(0, 0) = 1. \quad (6.29)$$

Applying the Fourier cosine transform to equation (6.23) subject to (6.24) and (6.25), we have the following, where we assume λ to be a constant:

$$-\lambda^2 v + v''(\hat{\zeta}) = 0. \quad (6.30)$$

Applying the Fourier cosine transform to equation (6.26) subject to (6.24) and (6.27), we have

$$\frac{q}{C_0^{(o)}(r_b)} \sqrt{\frac{2}{\pi}} - \lambda^2 v(0) + v'(0) = 0. \quad (6.31)$$

Equation (6.28) now becomes

$$v(\infty) = 0, \quad (6.32)$$

and equation (6.29) is not used. Solving (6.30) subject to (6.31) and (6.32), we have

$$v(\lambda, \hat{\zeta}) = \frac{qe^{-\lambda\hat{\zeta}}}{\lambda(\lambda+1)C_0^{(o)}(r_b)} \sqrt{\frac{2}{\pi}},$$

from which we have

$$u(\hat{x}, \hat{\zeta}) = \frac{2q}{\pi C_0^{(o)}(r_b)} \int_0^\infty \frac{e^{-\lambda\hat{\zeta}} \cos(\lambda\hat{x})}{\lambda(\lambda+1)} d\lambda. \quad (6.33)$$

We note that the integral diverges. We move into the complex plane and treat the lower limit formally.

$$\begin{aligned} u(\hat{x}, \hat{\zeta}) &= \frac{2q}{\pi C_0^{(o)}(r_b)} \Re \left[\lim_{\eta \rightarrow 0} \int_\eta^\infty \frac{e^{-\lambda\hat{\zeta} + i\lambda\hat{x}}}{\lambda} d\lambda - \int_0^\infty \frac{e^{-\lambda\hat{\zeta} + i\lambda\hat{x}}}{\lambda+1} d\lambda \right] \\ &= \frac{2q}{\pi C_0^{(o)}(r_b)} \Re \left[\lim_{\eta \rightarrow 0} \int_{\eta(\hat{\zeta} - i\hat{x})}^\infty \frac{e^{-t}}{t} dt - e^{\hat{\zeta} - i\hat{x}} \int_1^\infty \frac{e^{-t(\hat{\zeta} - i\hat{x})}}{t} dt \right]. \end{aligned}$$

Using equations (2.2), which define the exponential integral, we have the following:

$$u(\hat{x}, \hat{\zeta}) = \frac{2q}{C_0^{(o)}(r_b)\pi} \Re \left\{ \lim_{\eta \rightarrow 0} E_1[\eta(\hat{\zeta} - i\hat{x})] - e^{\hat{\zeta} - i\hat{x}} E_1(\hat{\zeta} - i\hat{x}) \right\} \quad (6.34)$$

At this juncture it is appropriate to write down asymptotic expansions for $E_1(s)$:

$$E_1(s) \sim -\gamma - \log s, \quad s \rightarrow 0; \quad E_1(s) \sim \frac{e^{-s}}{s}, \quad s \rightarrow \infty. \quad (6.35)$$

Rewriting u in terms of $\hat{C}_0(\hat{r}, \theta)$, we have

$$\hat{C}_0(\hat{r}, \theta) = C_0^{(o)}(r_b) - \frac{2q}{\pi} \Re \left[\lim_{\eta \rightarrow 0} E_1(-\eta i \hat{r} e^{i\theta}) - \exp(-i \hat{r} e^{i\theta}) E_1(-i \hat{r} e^{i\theta}) \right]. \quad (6.36)$$

Again we note that our first integral, now represented by

$$\lim_{\eta \rightarrow 0} E_1(-\eta i \hat{r} e^{i\theta})$$

is divergent. For our problem this is not physically reasonable. Since we wished to use a cosine transform, we extended the \hat{x} and $\hat{\zeta}$ to a semi-infinite range. However, \hat{x} and $\hat{\zeta}$ are only $O(\varepsilon_0^{-1})$ at our boundary $r = 1$. A more correct but more complicated method would involve a discrete eigenfunction expansion. In this case, the difference between subsequent eigenvalues would be $O(\varepsilon_0)$ because our region is $O(\varepsilon_0^{-1})$. Since the problem is not isotropic in the \hat{x} direction, the first eigenvalue λ_1 would be greater than zero. This would indicate that our limit η (which corresponds to λ_1) remains small, but never reaches 0.

Using (6.35), we know that

$$\lim_{\eta \rightarrow 0} E_1(-\eta i \hat{r} e^{i\theta}) = -\log \hat{r} - \gamma - \log \eta.$$

We use the above in equation (6.36) to obtain a new representation for our inner solution:

$$\hat{C}_0(\hat{r}, \theta) = \frac{2q}{\pi} \left\{ \log \hat{r} + A_4 + \Re \left[\exp(-i\hat{r}e^{i\theta}) E_1(-i\hat{r}e^{i\theta}) \right] \right\}, \quad (6.37)$$

where $A_4 = \gamma + \log \eta - \pi C_0^{(o)}(r_b)/2q$.

Boundary and Matching Conditions

We now need to solve our matching conditions while satisfying the boundary condition $\hat{C}_0 = 0$ at the borehole. We begin with the condition at the borehole. Using (6.35) again, we see that equation (6.37) becomes

$$\hat{C}_0(\hat{r}, 0) = \frac{2q}{\pi} (A_4 - \gamma). \quad (6.38)$$

Setting the above equal to 0, we have $A_4 = \gamma$. Using that fact and equation (6.35) in equation (6.37), we see that

$$\hat{C}_0(\hat{r}, \theta) \sim \frac{2q}{\pi} (\log \hat{r} + \gamma) \text{ as } \hat{r} \rightarrow \infty \quad (6.39a)$$

Writing the outer solution (6.11) in terms of the inner variable \hat{r} , we have

$$C_0^{(o)}(\hat{r}, \theta) = 1 + A_3 \log \varepsilon_0 \hat{r} \quad (6.39b)$$

Matching the inner solution (6.39a) and the outer solution (6.39b) as $\hat{r} \rightarrow \infty$ yields

$$A_3 = \frac{2q}{\pi}, \quad \frac{2q\gamma}{\pi} = 1 + A_3 \log \varepsilon_0 \quad (6.40)$$

Solving the two parts of (6.40) together, we have

$$q = \frac{\pi}{2(\gamma - \log \varepsilon_0)}, \quad A_3 = \frac{1}{\gamma - \log \varepsilon_0} \quad (6.41)$$

Using equations (6.41) in equations (6.37) and (6.39b), we now have our inner and outer solutions:

$$\hat{C}_0(\hat{r}, \theta) = \frac{1}{\gamma - \log \varepsilon_0} \left\{ \log \hat{r} + \gamma + \Re \left[\exp(-i\hat{r}e^{i\theta}) E_1(-i\hat{r}e^{i\theta}) \right] \right\} \quad (6.42a)$$

$$C_0^{(o)}(\hat{r}, \theta) = 1 + \frac{\log r}{\gamma - \log \varepsilon_0}. \quad (6.42b)$$

However, note that using equation (6.35) as $\hat{r} \rightarrow \infty$, the inner solution reduces to the outer solution. Hence, (6.42a) is the uniformly valid solution. We rewrite it in terms of the outer variables and denote it by the subscript (u) for uniform.

$$C_0^{(u)}(r, \theta) = \frac{\log(r/\varepsilon_0 e^{-\gamma})}{\log(1/\varepsilon_0 e^{-\gamma})} + \frac{1}{\log(1/\varepsilon_0 e^{-\gamma})} \Re \left[\exp\left(-\frac{ir e^{i\theta}}{\varepsilon_0}\right) E_1\left(-\frac{ir e^{i\theta}}{\varepsilon_0}\right) \right]. \quad (6.43)$$

Verification of Uniform Solution

Now, since we have performed several steps in our analysis without rigorous mathematical justification, we check that equation (6.43) satisfies our system of equations (5.26)-(5.32) to $O(\varepsilon_0)$. To show that it satisfies Laplace's equation, we note that if we move into the complex plane, the logarithm is an analytic function everywhere except at the origin. However, equation (6.38) shows us that the singularity in the logarithm is exactly canceled by the singularity for the E_1 term, and the function is analytic there as well. So (5.26) is satisfied.

As $r \rightarrow 1$, the second term in equation (6.43) decays algebraically while the first term goes to 1, so we have $C_0^{(u)}(1, \theta) = 1 + O(\varepsilon_0)$. As $r \rightarrow r_b$, which we have assumed to be small, our asymptotics in equation (6.38) show that we yield the correct solution to $O(\varepsilon_0)$. For equations (5.29)-(5.32), we calculate the necessary derivatives using equation (6.42a):

$$\frac{\partial C_0^{(u)}}{\partial \theta} = \Re \left[\hat{r} e^{i\theta} \exp(-i\hat{r} e^{i\theta}) E_1(-i\hat{r} e^{i\theta}) - i \right] \quad (6.44)$$

$$\frac{\partial C_0^{(u)}}{\partial \hat{r}} = \Re \left[-i e^{i\theta} \exp(-i\hat{r} e^{i\theta}) E_1(-i\hat{r} e^{i\theta}) \right] \quad (6.45)$$

$$\hat{r} \frac{\partial^2 C_0^{(u)}}{\partial \hat{r}^2} = -\hat{r} \Re \left[e^{2i\theta} \exp(-i\hat{r} e^{i\theta}) E_1(-i\hat{r} e^{i\theta}) - \frac{i e^{i\theta}}{\hat{r}} \right]. \quad (6.46)$$

Using equation (6.44) at $\theta = \pi/2$ to satisfy (5.29), we have

$$\Re [i\hat{r} \exp(\hat{r}) E_1(\hat{r}) - i] = 0,$$

which is trivially satisfied. Using equation (6.15) with $m = 1$ instead of (5.30), and then using equations (6.44) and (6.46) at $\theta = 0$, we have

$$\Re [\hat{r} \exp(-i\hat{r}) E_1(-i\hat{r})] = \hat{r} \Re \left[\exp(-i\hat{r}) E_1(-i\hat{r}) - \frac{i}{\hat{r}} \right],$$

which is again trivially satisfied. We use equation (6.44) at $\theta = 0$ for equation (5.31):

$$\Re [\hat{r} \exp(-i\hat{r}) E_1(-i\hat{r})] = 0.$$

Asymptotically expanding the above for large \hat{r} using equation (6.35), we have

$$\Re[-i + O(\varepsilon_0)] = 0.$$

For equation (5.32), we use (6.45) evaluated at $\theta = 0$:

$$\Re[-i \exp(-i\hat{r}) E_1(-i\hat{r})] = 0.$$

Once again asymptotically expanding for large \hat{r} using equation (6.35), we have

$$O(\varepsilon_0) = 0.$$

Hence we have satisfied our entire system of equations, and (6.42) is the solution of our problem to $O(\varepsilon_0)$.

Flux Calculation

As a first guess, we surmise that all the flux is coming from the fracture, so we integrate our flux there over the width of both fractures, yielding

$$\begin{aligned} \tilde{Q} &= 2aD_f \frac{\partial \tilde{C}_f}{\partial \tilde{x}}(0, 0) \\ &= \frac{2aP_0 D_f q}{RT\tilde{r}_0\varepsilon_0} \\ Q &= \frac{a\pi}{\tilde{r}_0\varepsilon_f \log(1/\varepsilon_0 e^{-\gamma})} \\ &= \frac{2\pi}{\log(1/\varepsilon_0 e^{-\gamma})}. \end{aligned} \tag{6.47}$$

To actually calculate the flux, we integrate around $r = 1$:

$$\begin{aligned} Q &= r \int_0^{2\pi} \frac{1}{r \log(1/\varepsilon_0 e^{-\gamma})} d\theta \Big|_{r=1} \\ &= \frac{2\pi}{\log(1/\varepsilon_0 e^{-\gamma})}. \end{aligned} \tag{6.48}$$

Since the two equations agree, we see that our intuition was correct and that to leading order all the flux comes from the fracture.

Section VII: Variational Principle Solution

For this section, each fracture only extends in one radial direction; hence, in the notation of this section, the perturbation computations were for two fractures.

From the perturbation analysis we determined that the zeroth order approximation C_0 is the solution to the problem:

$$\nabla^2 C_0(r, \theta) = 0, \quad C_0(1, \theta) = 1, \quad C_0(r_b, \theta) = 0, \quad (7.1)$$

with the additional boundary condition that

$$\frac{\partial}{\partial r} \left[\beta_j(r) \frac{\partial C_0}{\partial r} \right] + \frac{1}{r} \frac{\partial C_0}{\partial \theta} = 0 \quad (7.2)$$

along the j th fracture at $\theta = \theta_j$ and

$$\beta_j(r) = \frac{D_f a_j(r)}{2D_0 \tilde{r}_0}, \quad (7.3)$$

where $a_j(r)$ is the width of the j th fracture as a function of r and is set equal to zero from the end of the fracture up to $r = 1$. We shall now construct a variational formulation of the same problem. We define the functional Υ by

$$\Upsilon(C) = \int_{\Omega} |\nabla C|^2 d\Omega + \sum_{j=1}^N 2 \int_{r_b}^1 \beta_j(r) \left(\frac{\partial C}{\partial r} \right)^2 dr \Bigg|_{\theta=\theta_j}, \quad (7.4)$$

where N is the number of fractures. We were able to prove that our initial problem is equivalent to the following variational formulation:

$$\min_{C(r_b, \theta)=0, C(1, \theta)=1} \Upsilon(C) = Q. \quad (7.5)$$

The proof is easily obtained by differentiating $\Upsilon(C_0 + \delta f)$ with respect to δ at $\delta = 0$, where $f(r, \theta)$ is an arbitrary perturbation of the minimizing function C_0 which satisfies $f = 0$ at $r = r_b$ and $r = 1$. The derivative has to be equal to zero for all f , *i.e.*,

$$\frac{d}{d\delta} [\Upsilon(C_0 + \delta f)] \Bigg|_{\delta=0} = \int_{\Omega} \nabla C_0 \cdot \nabla f d\Omega + 2 \sum_{j=1}^N \int_{r_b}^1 \beta_j(r) \frac{\partial C_0}{\partial r} \frac{\partial f}{\partial r} dr \Bigg|_{\theta=\theta_j}.$$

Integrating by parts, we obtain

$$-2 \sum_{j=1}^m \int_{r_b}^1 f \left[\frac{1}{r} \frac{\partial C_0}{\partial \theta} + \frac{\partial}{\partial r} \left(\beta_j \frac{\partial C_0}{\partial r} \right) \right] dr \Big|_{\theta=\theta_j} - \int_{\Omega} f \nabla^2 C_0 d\Omega. \quad (7.6)$$

Since (7.7) has to be equal to zero for arbitrary P , C_0 must be the solution to

$$\nabla^2 C_0 = 0 \text{ in } \Omega$$

with the boundary conditions

$$C_0(r_b, \theta) = 0, \quad C_0(1, \theta) = 1$$

and

$$\frac{1}{r} \frac{\partial C_0}{\partial \theta} + \frac{\partial}{\partial r} \left(\beta_j \frac{\partial C_0}{\partial r} \right) = 0, \quad \theta = \theta_j.$$

We note here that the value of Υ at its actual minimum $\Upsilon(C_0)$ is

$$\Upsilon(C_0) = \int_0^{2\pi} \frac{\partial C_0(1, \theta)}{\partial r} d\theta = Q. \quad (7.7)$$

The proof is a straightforward, but messy, calculation. In order to obtain an approximate value for $\Upsilon(C_0)$, we restrict the class of functions over which we are minimizing to functions which depend on r only. When we introduce $C(r) = C_0(r) + \delta f(r)$ into the variational problem, we obtain the following ordinary differential equation :

$$2\pi \frac{d}{dr} \left(r \frac{dC_0}{dr} \right) + \frac{d}{dr} \left[\tau(r) \frac{dC_0}{dr} \right] = 0, \quad r_b < r < 1$$

$$C_0(r_b) = 0, \quad C_0(1) = 1,$$

where

$$\tau(r) = \sum_{j=1}^N 2\beta_j(r). \quad (7.8)$$

The solution is easily obtained and is found to be

$$C_0(r) = A_5 \int_{r_b}^r \frac{dt}{2\pi t + \tau(t)}, \quad (7.9a)$$

where

$$A_5 = \left[\int_{r_b}^1 \frac{dt}{2\pi t + \tau(t)} \right]^{-1}. \quad (7.9b)$$

Let us now compute the total flux Q . Since $\tau(1) = 0$,

$$Q = \int_0^{2\pi} \frac{dC}{dr}(1, \theta) d\theta = \int_0^{2\pi} \frac{A_5 d\theta}{2\pi + \tau(1)} = A_5. \quad (7.10)$$

Thus,

$$Q = \left[\int_{r_b}^1 \frac{dt}{2\pi t + \tau(t)} \right]^{-1}. \quad (7.11)$$

We now consider the particular case when all cracks have the same length r_f , and the same constant value β . Thus,

$$\tau(r) = \sum_{j=1}^m 2\beta_j(r) = 2m\beta, \quad (7.12)$$

and Q is given by

$$Q = 2\pi \left[\log \left(\frac{r_f + \tau/2\pi}{(r_b + \tau/2\pi)r_f} \right) \right]^{-1}. \quad (7.13)$$

We then normalize Q by the flow rate with no fracture, which is given by (4.5). In figure 7a, we plot the normalized flow rate with respect to $\tau(r)$. In this special case, $\tau(r) = 2m\beta$. In figure 7b, we plot the normalized flow rate, which is proportional to the gross revenue, versus the number of fractures for a constant value of $\beta = 1/30$. From these graphs we conclude that it is highly profitable to hydrofracture the porous medium with up to four cracks.

Fig 7a : Hydrofracture flow rate

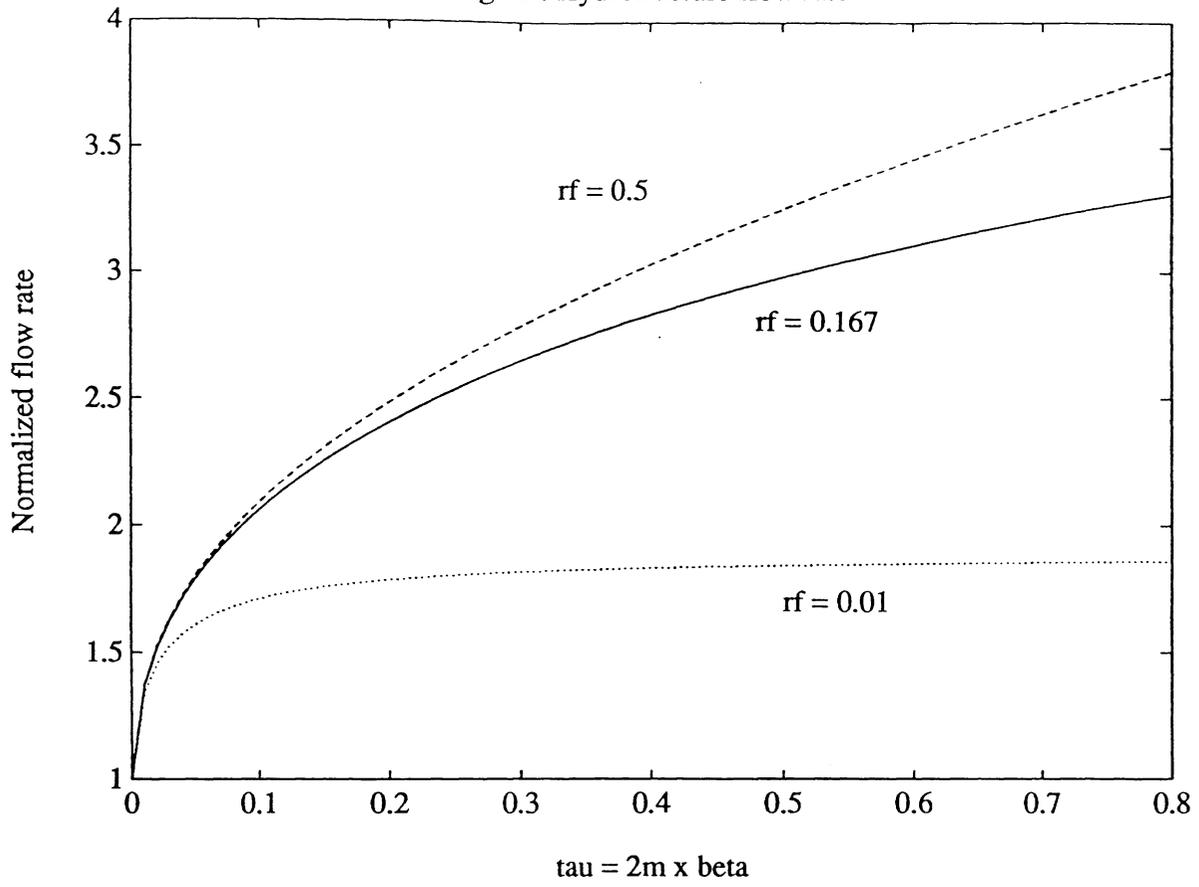
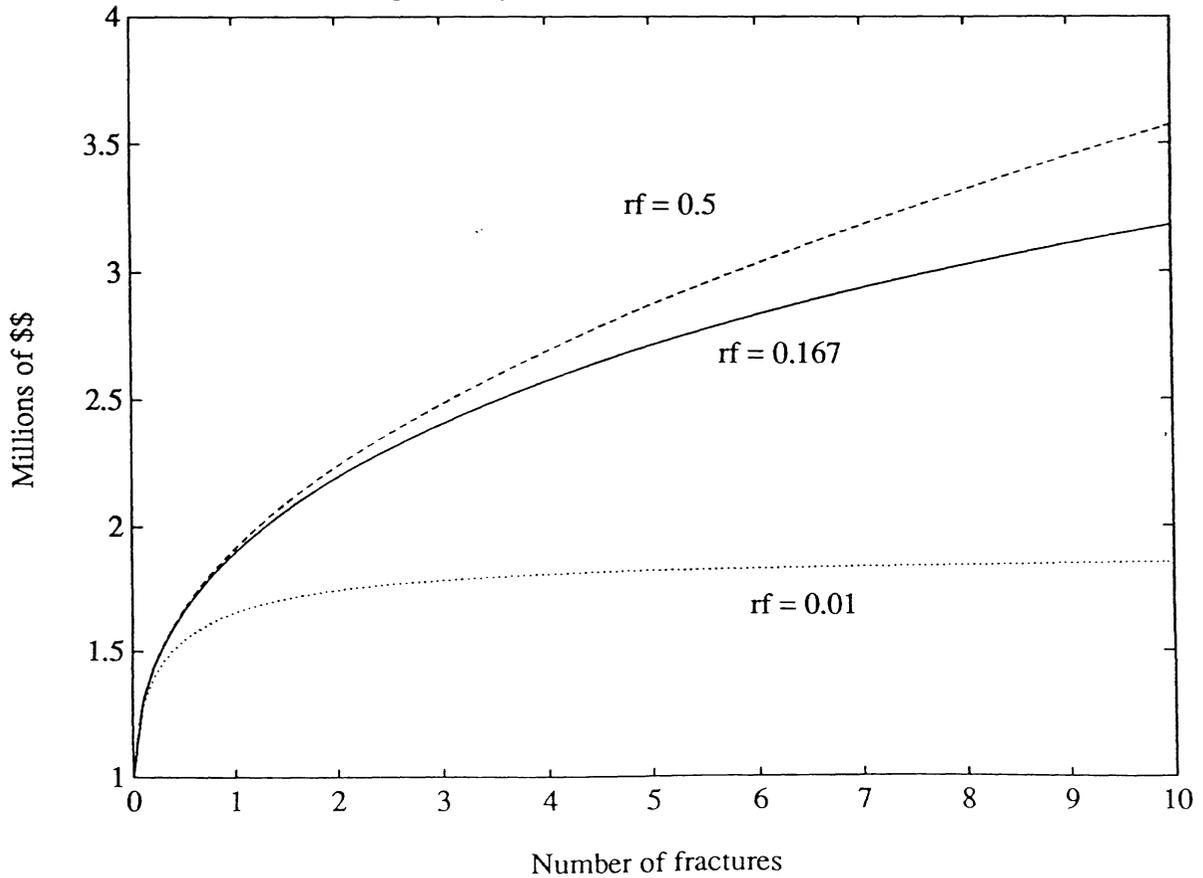


Fig. 7b : Hydrofracture flow rate, $\beta = 1/30$



Section VIII: Conclusions and Future Research

Conclusions

In this report, we have studied a steady-state model for gas flow in a hydrofractured oil field. Without the fracture, equation (4.4) becomes

$$C_0(r) = \frac{\log(r/r_b)}{\log(1/r_b)}, \quad (8.1)$$

so the effective radius of the borehole is $r_b = 1.7 \times 10^{-4}$. However, when we introduce our fracture, the new flow for moderate r is given by equation (6.43), and is

$$C_0^{(u)}(r, \theta) \sim \frac{\log(r/\varepsilon_0 e^{-\gamma})}{\log(1/\varepsilon_0 e^{-\gamma})} \quad (8.2)$$

Note that here our effective radius is $\varepsilon_0 e^{-\gamma} = 1.7 \times 10^{-2}$, an increase of two orders of magnitude. Hence, it is easy to see the usefulness of producing hydrofractures.

Note from equation We employed a perturbation analysis to find out the flow rate through the borehole with two fractures:

$$Q = \frac{2\pi}{\gamma - \log \varepsilon_0},$$

where Euler's constant $\gamma = 0.5771$.

The above result shows how the amount of gas flowing out of the borehole depends on the width and permeability of the fracture.

Using a variational principle, we obtained an approximation to the flow rate with multiple fractures. In the case of two fractures and taking the radius of the borehole to be approximately zero, the variational method gives us the flow rate

$$Q = \frac{2\pi}{d - \log \varepsilon_0},$$

where $d = \log(\pi/2) + \log(1 + 2\varepsilon_0/\pi r_f)$. The second term in d would be generated by calculating the perturbation result to higher order. Discarding the second term in d , we get $d = \gamma - 0.125$, so the two results agree very well, seldom differing by more than 6%. Thus, we conclude that the variational method has given us a surprisingly good approximation considering the crudeness of our trial functions.

Considering the case of multiple fractures, especially the case of an odd number of fractures, we would encounter great difficulties using a perturbation method since we would not be able to use the symmetry property to solve the Laplacian problem in our

perturbation analysis. Therefore, the variational method is superior in solving multi-fracture problems where asymptotic solutions cannot be obtained. We also believe the variational method will render a better approximation when the number of fractures is increased.

Future Research

In the future, it will be important to extend our steady-state model to a transient one. In the transient model, the pressure will change with time, from which we can extract the relation between the flow rate of gas in the borehole and the pressure in the fractured field. This will then allow one to determine the actual fracture widths and permeabilities from the measured flow rate vs. pressure curve.

Another important area to work on is the extraction rate in fields with many interacting oil and gas wells. The work will be to model several hydrofractured wells with interactions between each other (see figure 8a).

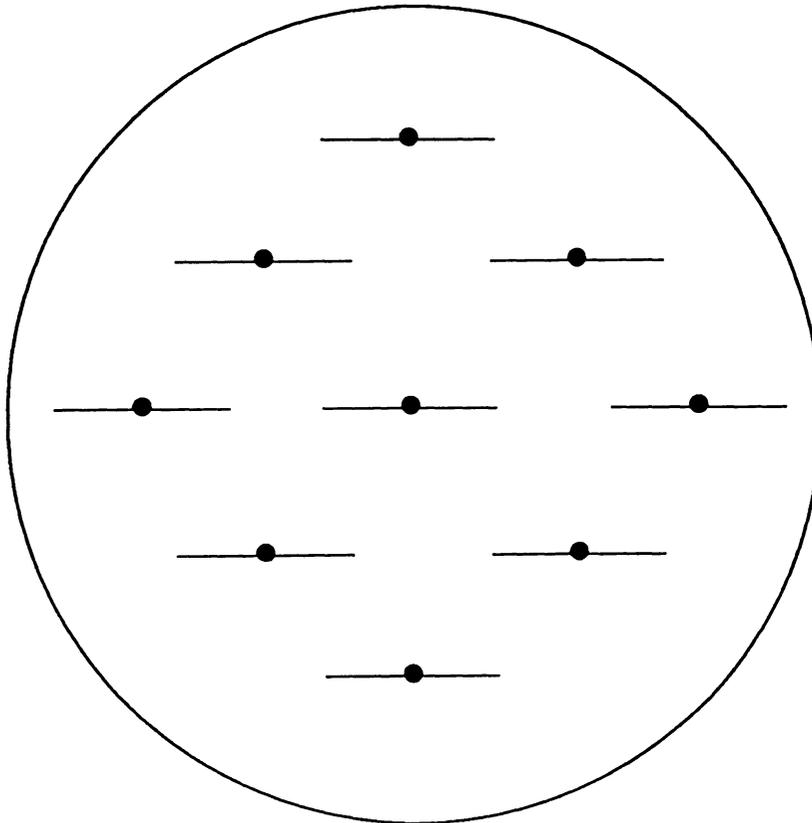


Figure 8a. Pay zone with multiple fractures, top view.