

Supplemental Materials for:

“Microstructure and Rheology Relationships for Shear Thickening Colloidal Dispersions”

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Summary of integrals dependencies (superscript indicates neutron beam direction):

Radial geometry: (velocity – vorticity)

$$W_0^{\text{gradient}}(q)(B_{0,0}^+(q), -B_{2,0}^+(q), B_{4,0}^+(q))$$

$$W_2^{\text{gradient}}(q)(B_{2,2}^+(q), B_{4,2}^+(q))$$

$$W_4^{\text{gradient}}(q)(B_{4,4}^+(q))$$

$$W_{12}^{\text{gradient}}(q) = 0$$

1-2 flow plane geometry: (velocity – velocity gradient)

$$W_0^{\text{vorticity}}(q)(B_{0,0}^+(q), B_{2,0}^+(q), B_{2,2}^+(q), B_{4,0}^+(q), B_{4,2}^+(q), B_{4,4}^+(q))$$

$$W_2^{\text{vorticity}}(q)(B_{2,0}^+(q), -B_{2,2}^+(q), B_{4,0}^+(q), B_{4,2}^+(q), -B_{4,4}^+(q))$$

$$W_4^{\text{vorticity}}(q)(B_{4,0}^+(q), -B_{4,2}^+(q), B_{4,4}^+(q))$$

$$W_{12}^{\text{vorticity}}(q)(B_{2,1}^+(q), B_{4,1}^+(q))$$

Tangential geometry: (velocity gradient – vorticity)

$$W_0^{\text{velocity}}(q)(B_{0,0}^+(q), B_{2,0}^+(q), B_{2,2}^+(q)B_{4,0}^+(q), -B_{4,2}^+(q), -B_{4,4}^+(q))$$

$$W_2^{\text{velocity}}(q)(B_{2,0}^+(q), -B_{2,2}^+(q), -B_{4,0}^+(q), -B_{4,2}^+(q), B_{4,4}^+(q))$$

$$W_4^{\text{velocity}}(q)(-B_{4,0}^+(q), B_{4,2}^+(q), -B_{4,4}^+(q))$$

$$W_{12}^{\text{velocity}}(q) = 0$$

From Theory the hydrodynamic and thermodynamic shear stresses are: (from Ackerson and Wagner eqns 52 and 62)

$$\tau^{\text{thermodynamic}}(\dot{\gamma}) = -\frac{\rho}{\pi\sqrt{30\pi}} \int \theta^*(q) B_{2,1}^+(q; Pe) q^2 dq$$

$$\tau^{\text{hydrodynamic}}(\dot{\gamma}) / (2\dot{\gamma}\mu) = 1 + \frac{5}{2}\phi(1+\phi) + 2.7\phi^2 + \frac{5}{2}\phi\sqrt{\pi} \int \left[\begin{array}{l} B_{0,0}^+(q; Pe) \left(2\alpha(q) + \frac{4}{15}\zeta_0(q) \right) \\ + B_{2,0}^+(q; Pe) \left(\frac{2}{3\sqrt{5}}\beta(q) - \frac{4\sqrt{2}}{21\sqrt{7}}\zeta_2(q) \right) \\ + B_{2,2}^+(q; Pe) \left(\frac{\sqrt{2}}{\sqrt{15}}\beta(q) + \frac{2\sqrt{2}}{105}\zeta_2(q) \right) \\ + B_{4,0}^+(q; Pe) \left(\frac{-16}{105}\zeta_4(q) \right) \\ + B_{4,2}^+(q; Pe) \left(\frac{4\sqrt{2}}{21\sqrt{5}} \frac{-16}{105}\zeta_4(q) \right) \end{array} \right] q^2 dq$$

And for the thermodynamic and hydrodynamic first and second normal stresses are:

$$N_1^{\text{thermodynamic}}(\dot{\gamma}) = \frac{\rho}{\pi\sqrt{30}\pi} \int \theta^*(q) B_{2,2}^+(q; Pe) - \sqrt{6} B_{2,0}^+(q; Pe) q^2 dq$$

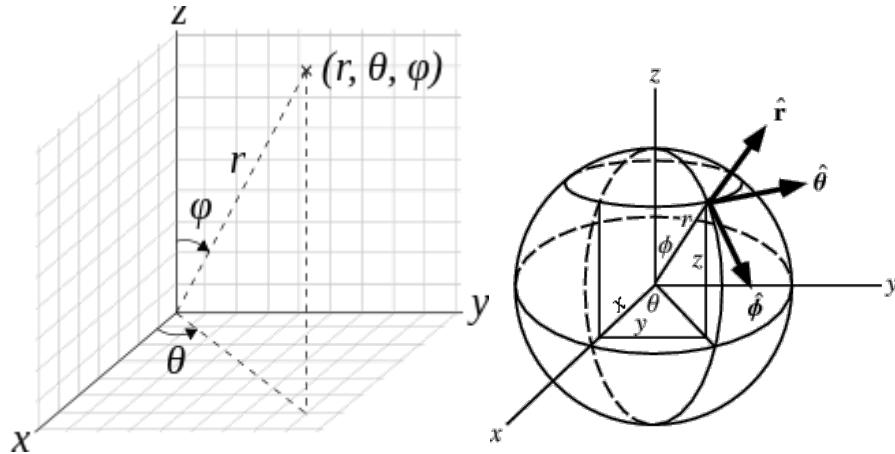
$$N_2^{\text{thermodynamic}}(\dot{\gamma}) = \frac{\rho}{\pi\sqrt{30}\pi} \int \theta^*(q) B_{2,2}^+(q; Pe) + \sqrt{6} B_{2,0}^+(q; Pe) q^2 dq$$

$$N_1^{\text{hydrodynamic}}(\dot{\gamma}) / 2\dot{\gamma}\mu = -5\phi\sqrt{\pi} \int \left[B_{4,1}^+(q; Pe) \left(\frac{5\sqrt{6} + 24\sqrt{5}}{105\sqrt{30}} \zeta_4(q) \right) + B_{4,3}^+(q; Pe) \left(\frac{1}{3\sqrt{35}} \zeta_4(q) \right) \right] q^2 dq$$

$$N_2^{\text{hydrodynamic}}(\dot{\gamma}) / 2\dot{\gamma}\mu = -5\phi\sqrt{\pi} \int \left[B_{2,1}^+(q; Pe) \left(\frac{2\sqrt{2}}{\sqrt{15}} \beta(q) + \frac{2\sqrt{2}}{7\sqrt{15}} \zeta_2(q) \right) + B_{4,1}^+(q; Pe) \left(\frac{5\sqrt{6} - 24\sqrt{5}}{105\sqrt{30}} \zeta_4(q) \right) + B_{4,3}^+(q; Pe) \left(\frac{1}{3\sqrt{35}} \zeta_4(q) \right) \right] q^2 dq$$

For SANS decomposition in 3 planes of shear:

Starting with the spherical harmonic expansion in terms of the spherical coordinate system shown here:



$$\hat{h}(q) = \sum_{l,m} B_{l,m}^+(q) \left(Y_{l,-m}(\Omega_k) + (-1)^m Y_{l,m}(\Omega_k) \right), \quad 1 \geq m \geq 0$$

$$\hat{h}(q) = \sum_{l,m} B_{l,m}^+(q) \left(Y_{l,-m}(\Omega_k) + (-1)^m Y_{l,m}(\Omega_k) \right)$$

$$[l=0, m=0] = B_{0,0}^+(q) \left(Y_{0,-0}(\Omega_k) + (-1)^0 Y_{0,0}(\Omega_k) \right)$$

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{2\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} \right)$$

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(Y_{2,-0}(\Omega_k) + (-1)^0 Y_{2,0}(\Omega_k) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta - 1) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(Y_{2,-1}(\Omega_k) + (-1)^1 Y_{2,1}(\Omega_k) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\frac{1}{2} e^{-i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) - \frac{-1}{2} e^{i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(Y_{2,-2}(\Omega_k) + (-1)^2 Y_{2,2}(\Omega_k) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{4} e^{-2i\phi} \sqrt{\frac{15}{2\pi}} \sin^2(\theta) + \frac{1}{4} e^{2i\phi} \sqrt{\frac{15}{2\pi}} \sin^2(\theta) \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \left(Y_{4,-0}(\Omega_k) + (-1)^0 Y_{4,0}(\Omega_k) \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \left(\frac{3(3 - 30 \cos^2(\theta) + 35 \cos^4(\theta))}{8\sqrt{\pi}} \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(Y_{4,-1}(\Omega_k) + (-1)^1 Y_{4,1}(\Omega_k) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} e^{-i\phi} \cos(\theta) \sin(\theta) (-3 + 7 \cos^2(\theta)) - \frac{-3\sqrt{5}}{8\sqrt{\pi}} e^{i\phi} \cos(\theta) \sin(\theta) (-3 + 7 \cos^2(\theta)) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(Y_{4,-2}(\Omega_k) + (-1)^2 Y_{4,2}(\Omega_k) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{8} e^{-2i\phi} \sqrt{\frac{5}{2\pi}} \sin^2(\theta) (-1 + 7 \cos^2(\theta)) + \frac{3}{8} e^{2i\phi} \sqrt{\frac{5}{2\pi}} \sin^2(\theta) (-1 + 7 \cos^2(\theta)) \right)$$

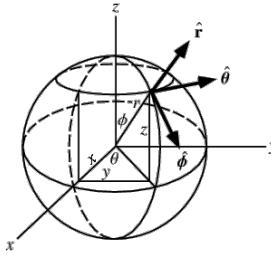
$$[l=4, m=3] = B_{4,3}^+(q) \left(Y_{4,-3}(\Omega_k) + (-1)^3 Y_{4,3}(\Omega_k) \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{8} e^{-3i\phi} \sqrt{\frac{35}{\pi}} \cos(\theta) \sin^3(\theta) - \frac{-3}{8} e^{3i\phi} \sqrt{\frac{35}{\pi}} \cos(\theta) \sin^3(\theta) \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(Y_{4,-4}(\Omega_k) + (-1)^4 Y_{4,4}(\Omega_k) \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{16\sqrt{2\pi}} e^{-4i\phi} \sin^4(\theta) + \frac{3\sqrt{35}}{16\sqrt{2\pi}} e^{4i\phi} \sin^4(\theta) \right)$$

To change from spherical harmonics to Cartesian coordinate frame as defined below we use the following definitions:



$$e^{i\phi} = \frac{x + iy}{\sqrt{x^2 + y^2}} \text{ and } e^{-i\phi} = \frac{x - iy}{\sqrt{x^2 + y^2}}$$

$$e^{2i\phi} = \frac{(x + iy)^2}{x^2 + y^2} = \frac{x^2 + 2ixy - y^2}{x^2 + y^2} \text{ and } e^{-2i\phi} = \frac{(x + iy)^2}{x^2 + y^2} = \frac{x^2 - 2ixy - y^2}{x^2 + y^2}$$

$$e^{3i\phi} = \frac{(x + iy)^3}{(x^2 + y^2)^{3/2}} = \frac{x^3 + 3ix^2y - 3xy^2 - iy^3}{(x^2 + y^2)^{3/2}} \text{ and } e^{-3i\phi} = \frac{(x - iy)^3}{(x^2 + y^2)^{3/2}} = \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{(x^2 + y^2)^{3/2}}$$

$$e^{4i\phi} = \frac{(x + iy)^4}{(x^2 + y^2)^2} = \frac{x^4 + 4ix^3y - 6x^2y^2 - 4xy^3 + y^4}{(x^2 + y^2)^2} \text{ and } e^{-4i\phi} = \frac{(x - iy)^4}{(x^2 + y^2)^2} = \frac{x^4 - 4ix^3y - 6x^2y^2 + 4xy^3 + y^4}{(x^2 + y^2)^2}$$

$$\sin(\theta) = \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}}$$

$$\cos(\theta) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

In Cartesian coordinate system (x, y, z)

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta - 1) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 - 1 \right) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(Y_{2,-1}(\Omega_k) + (-1)^1 Y_{2,1}(\Omega_k) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\frac{1}{2} e^{-i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) - \frac{1}{2} e^{i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\frac{1}{2} e^{-i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) + \frac{1}{2} e^{i\phi} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\cos(\phi) \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \left(\sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}} \right) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{xz}{(x^2 + y^2 + z^2)} \right) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(Y_{2,-2}(\Omega_k) + (-1)^2 Y_{2,2}(\Omega_k) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{4} e^{-2i\phi} \sqrt{\frac{15}{2\pi}} \sin^2(\theta) + \frac{1}{4} e^{2i\phi} \sqrt{\frac{15}{2\pi}} \sin^2(\theta) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} e^{2i\phi} \sin^2(\theta) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \left(\frac{x^2 - y^2}{x^2 + y^2} \right) \left(\sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}} \right)^2 \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \left(\frac{x^2 - y^2}{x^2 + y^2 + z^2} \right) \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \left(Y_{4,-0}(\Omega_k) + (-1)^0 Y_{4,0}(\Omega_k) \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \left(\frac{3(3 - 30 \cos^2(\theta) + 35 \cos^4(\theta))}{8\sqrt{\pi}} \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \frac{3}{8\sqrt{\pi}} \left(3 - 30 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + 35 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^4 \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(Y_{4,-1}(\Omega_k) + (-1)^1 Y_{4,1}(\Omega_k) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} e^{-i\phi} \cos(\theta) \sin(\theta) (-3 + 7 \cos^2(\theta)) - \frac{-3\sqrt{5}}{8\sqrt{\pi}} e^{i\phi} \cos(\theta) \sin(\theta) (-3 + 7 \cos^2(\theta)) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{\pi}} e^{-i\phi} \cos(\theta) \sin(\theta) (-3 + 7 \cos^2(\theta)) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{\pi}} \frac{x}{\sqrt{x^2+y^2}} \frac{z}{\sqrt{x^2+y^2+z^2}} \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}} \left(-3 + 7 \left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right)^2 \right) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{\pi}} \frac{xz}{x^2+y^2+z^2} \left(-3 + 7 \left(\frac{z^2}{x^2+y^2+z^2} \right) \right) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{\pi}} \frac{xz}{x^2+y^2+z^2} \left(7 \left(\frac{z^2}{x^2+y^2+z^2} \right) - 3 \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(Y_{4,-2}(\Omega_k) + (-1)^2 Y_{4,2}(\Omega_k) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{8} e^{-2i\phi} \sqrt{\frac{5}{2\pi}} \sin^2(\theta) (-1 + 7 \cos^2(\theta)) + \frac{3}{8} e^{2i\phi} \sqrt{\frac{5}{2\pi}} \sin^2(\theta) (-1 + 7 \cos^2(\theta)) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{2\pi}} \left(\frac{x^2-y^2}{x^2+y^2} \right) \left(\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}} \right)^2 \left(-1 + 7 \left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right)^2 \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{2\pi}} \left(\frac{x^2-y^2}{x^2+y^2+z^2} \right) \left(-1 + 7 \left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right)^2 \right) \right)$$

$$\begin{aligned}
[l=4, m=3] &= B_{4,3}^+(q) \left(Y_{4,-3}(\Omega_k) + (-1)^3 Y_{4,3}(\Omega_k) \right) \\
[l=4, m=3] &= B_{4,3}^+(q) \left(\frac{3}{8} e^{-3i\phi} \sqrt{\frac{35}{\pi}} \cos(\theta) \sin^3(\theta) - \frac{-3}{8} e^{-3i\phi} \sqrt{\frac{35}{\pi}} \cos(\theta) \sin^3(\theta) \right) \\
[l=4, m=3] &= B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) (e^{-3i\phi}) (\cos(\theta)) (\sin^3(\theta)) \\
[l=4, m=3] &= B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{x(x^2 - 3y^2)}{(x^2 + y^2)^{3/2}} \right) \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \left(\frac{x^2 + y^2}{x^2 + y^2 + z^2} \right)^{3/2} \\
[l=4, m=3] &= B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{zx(x^2 - 3y^2)}{(x^2 + y^2 + z^2)^2} \right) \\
[l=4, m=4] &= B_{4,4}^+(q) \left(Y_{4,-4}(\Omega_k) + (-1)^4 Y_{4,4}(\Omega_k) \right) \\
[l=4, m=4] &= B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{16\sqrt{2\pi}} e^{-4i\phi} \sin^4(\theta) + \frac{3\sqrt{35}}{16\sqrt{2\pi}} e^{-4i\phi} \sin^4(\theta) \right) \\
[l=4, m=4] &= B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) (e^{-4i\phi}) (\sin^4(\theta)) \\
[l=4, m=4] &= B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{x^4 - 6x^2y^2 + y^4}{(x^2 + y^2)^2} \right) \left(\sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}} \right)^4 \\
[l=4, m=4] &= B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{x^4 - 6x^2y^2 + y^4}{(x^2 + y^2 + z^2)^2} \right)
\end{aligned}$$

Change in notation from x, y and z to q's where:

$$x = q_x$$

$$y = q_y$$

$$z = q_z$$

and $q = \sqrt{x^2 + y^2 + z^2}$ and $q^2 = x^2 + y^2 + z^2$

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 - 1 \right) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \frac{q_z^2}{q^2} - 1 \right) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{xz}{(x^2 + y^2 + z^2)} \right) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{q_x q_z}{q^2} \right) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \left(\frac{x^2 - y^2}{x^2 + y^2 + z^2} \right) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{q_x^2 - q_y^2}{q^2} \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \frac{3}{8\sqrt{\pi}} \left(3 - 30 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + 35 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^4 \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \frac{3}{8\sqrt{\pi}} \left(3 - 30 \left(\frac{q_z^2}{q^2} \right) + 35 \left(\frac{q_z^4}{q^4} \right) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{\pi}} \frac{xz}{x^2 + y^2 + z^2} \left(7 \left(\frac{z^2}{x^2 + y^2 + z^2} \right) - 3 \right) \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \frac{3\sqrt{5}}{4\sqrt{\pi}} \left(\frac{q_x q_z}{q^2} \right) \left(7 \left(\frac{q_z^2}{q^2} \right) - 3 \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{2\pi}} \left(\frac{x^2 - y^2}{x^2 + y^2 + z^2} \right) \left(-1 + 7 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{2\pi}} \left(\frac{q_x^2 - q_y^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right) \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{zx(x^2 - 3y^2)}{(x^2 + y^2 + z^2)^2} \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{q_x q_z (q_x^2 - 3q_y^2)}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{x^4 - 6xy + y^4}{(x^2 + y^2 + z^2)^2} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{(q_x^4 - 6q_x q_y - q_y^4)}{q^4} \right)$$

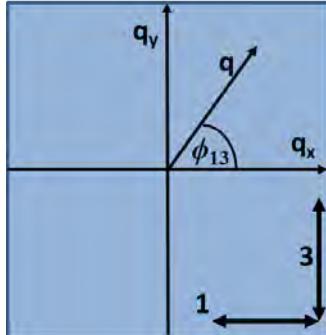
Now define each set of spherical harmonics for each experiment (dependent upon geometry)

Radial (velocity-vorticity) where the neutrons propagate along the gradient direction such that

$$\frac{q_x}{q} = \cos(\phi_{13})$$

$$\frac{q_y}{q} = \sin(\phi_{13})$$

$$q_z = 0$$



$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \frac{q_z^2}{q^2} - 1 \right) \right)$$

$$[l=2, m=0] = -B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{q_x q_z}{q^2} \right) \right)$$

$$[l=2, m=1] = 0$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{q_x^2 - q_y^2}{q^2} \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} (\cos^2(\phi_{13}) - \sin^2(\phi_{13}))$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13})$$

$$[l=4, m=0] = B_{4,0}^+(q) \frac{3}{8\sqrt{\pi}} \left(3 - 30 \left(\frac{q_z^2}{q^2} \right) + 35 \left(\frac{q_z^4}{q^4} \right) \right)$$

$$[l=4, m=0] = B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right)$$

$$[l=4, m=1] = B_{4,1}^+(q) \frac{3\sqrt{5}}{4\sqrt{\pi}} \left(\frac{q_x q_z}{q^2} \right) \left(7 \left(\frac{q_z^2}{q^2} \right) - 3 \right)$$

$$[l=4, m=1] = 0$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{q_x^2 - q_y^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{q_x^2 - q_y^2}{q^2} \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) (\cos^2(\phi_{13}) - \sin^2(\phi_{13}))$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13})$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{q_x q_z (q_x^2 - 3q_y^2)}{q^4} \right)$$

$$[l=4, m=3] = 0$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{q_x^4 - 6q_x^2 q_y^2 + q_y^4}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) (\cos^4(\phi_{13}) - 6\cos^2(\phi_{13})\sin^2(\phi_{13}) + \sin^4(\phi_{13}))$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{3}{8} + \frac{1}{2}\cos(2\phi_{13}) + \frac{1}{8}\cos(4\phi_{13}) - 6\frac{1}{8}(1 - \cos(4\phi_{13})) + \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\phi_{13}) \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13})$$

The structure factor for the neutrons propagate along the gradient direction

$$\begin{aligned} S^{gradient}(q) = & 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) - B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) + \\ & B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13}) + B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13}) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13}) \end{aligned}$$

Therefore we define the W_n 's for the neutrons in the gradient direction as:

$$W_n^{gradient}(q)S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi_{13}) S(q) d\phi_{13}$$

$$W_0^{gradient}(q)S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(0\phi_{13}) S(q) d\phi_{13}$$

$$W_0^{gradient}(q)S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left[1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) - B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) + B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13}) + B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right) \cos(0\phi_{13}) \right] d\phi_{13}$$

$$+ B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13}) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13}) + \dots (l > 4)$$

$$W_0^{gradient}(q)S^{eq}(q) = 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) - B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) + B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right) + \dots (l > 4)$$

$$W_2^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\phi_{13}) S(q) d\phi_{13}$$

$$W_2^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left[1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) - B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) + B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13}) + B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right) \right. \\ \left. + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13}) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13}) + \dots (l > 4) \right] \cos(2\phi_{13}) d\phi_{13}$$

$$W_2^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left[B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13}) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13}) + \dots (l > 4) \right] \cos(2\phi_{13}) d\phi_{13}$$

$$W_2^{\text{gradient}}(q) S^{eq}(q) = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \right) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) + \dots (l > 4)$$

$$W_4^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(4\phi_{13}) S(q) d\phi_{13}$$

$$W_4^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left[1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) - B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) + B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \cos(2\phi_{13}) + B_{4,0}^+(q) \left(\frac{9}{8\sqrt{\pi}} \right) \right. \\ \left. + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \cos(2\phi_{13}) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13}) + \dots (l > 4) \right] \cos(4\phi_{13}) d\phi_{13}$$

$$W_4^{\text{gradient}}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \left[B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) \cos(4\phi_{13}) + \dots (l > 4) \right] \cos(4\phi_{13}) d\phi_{13}$$

$$W_4^{\text{gradient}}(q) S^{eq}(q) = 4 \left(B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) + \dots (l > 4) \right)$$

$$W_6^{gradient}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(6\phi_{13}) S(q) d\phi_{13} = 0$$

$$W_{12}^{gradient}(q) S^{eq}(q) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\phi_{13}) \sin(\phi_{13}) S(q) d\phi_{13} = 0$$

1-2 plane (velocity- velocity gradient)

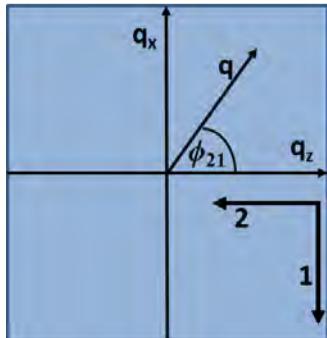
For the 2D detector when the neutrons propagate along the vorticity direction:

$$\frac{q_x}{q} = -\sin(\phi_{21})$$

$$\frac{q_y}{q} = 0$$

$$q_z = -\cos(\phi_{21})$$

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$



$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \frac{q_z^2}{q^2} - 1 \right) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) (3 \cos^2(\phi_{21}) - 1)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) \left(3 \left(\frac{1}{2} (1 + \cos(2\phi_{21})) \right) - 1 \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \right) \left(\frac{3}{2} (1 + \cos(2\phi_{21})) - \frac{2}{2} \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\frac{1}{2} \sqrt{\frac{5}{4\pi}} \right) \left(\frac{1}{2} + \frac{3}{2} \cos(2\phi_{21}) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3 \cos(2\phi_{21}))$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{q_x q_z}{q^2} \right) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21})$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{q_x^2 - q_y^2}{q^2} \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{q_x^2}{q^2} \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \right) \sin^2(\phi_{21})$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{2} \sqrt{\frac{15}{2\pi}} \right) \left(\frac{1}{2} (1 - \cos(2\phi_{21})) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21}))$$

$$\begin{aligned}
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(3 - 30 \left(\frac{q_z^2}{q^2} \right) + 35 \left(\frac{q_z^4}{q^4} \right) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(3 - 30 \cos^2(\phi_{21}) + 35 \cos^4(\phi_{21}) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(3 - 15(1 + \cos(2\phi_{21})) + \frac{35}{8}(3 + 4\cos(2\phi_{21}) + \cos(4\phi_{21})) \right) \\
[l=4, m=0] &\equiv B_{4,0}^+(q) \left(\frac{3\sqrt{3}}{4\sqrt[4]{\pi}} \right) \left(\frac{q_z^2}{q^2} \right)^6 \left(7 \left(\frac{120}{q^2} \right) \cos(2\phi_{21}) + \frac{105}{8} + \frac{140}{8} \cos(2\phi_{21}) + \frac{35}{8} \cos(4\phi_{21}) \right) \\
[l=4, m=0] &\equiv B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt[4]{\pi}} \right) \left(\frac{9}{8} \left(\frac{20}{\sin(\phi_{21})} \cos(2\phi_{21}) + \frac{35}{8} \cos^2(\phi_{21}) \cos(4\phi_{21}) \right) \right) \\
[l=4, m=0] &= B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{64\sqrt{\pi}} \right) \left(\frac{9}{8} + \frac{20}{\sin(\phi_{21})} \cos(\phi_{21}) \cos(\phi_{21}) \left(\frac{1}{2} (1 + \cos(2\phi_{21})) \right) - 3 \right) \\
[l=4, m=1] &= B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) (1 + 7 \cos(2\phi_{21})) \\
[l=4, m=1] &= B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right)
\end{aligned}$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{q_x^2 - q_y^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{q_x^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \sin^2(\phi_{21}) (-1 + 7 \cos^2(\phi_{21}))$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{1}{2} (1 - \cos(2\phi_{21})) \right) \left(-1 + 7 \frac{1}{2} (1 + \cos(2\phi_{21})) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) (1 - \cos(2\phi_{21})) (5 + 7 \cos(2\phi_{21}))$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(5 + 2 \cos(2\phi_{21}) - \frac{7}{2} - \frac{7}{2} \cos(4\phi_{21}) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2 \cos(2\phi_{21}) - \frac{7}{2} \cos(4\phi_{21}) \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{q_x^3 q_z}{q^4} \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \sin^3(\phi_{21}) \cos(\phi_{21})$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2 \sin(2\phi_{21}) + \sin(4\phi_{21}))$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{(q_x^2 - 2q_x q_y - q_y^2)(q_x^2 + 2q_x q_y - q_y^2)}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{q_x^4}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \sin^4(\phi_{21})$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{3}{8} - \frac{1}{2} \cos(2\phi_{21}) + \frac{1}{8} \cos(4\phi_{21}) \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{21}) + \cos(4\phi_{21}))$$

Structure factor for the 2D detector when the neutrons propagate along the vorticity direction:

$$\begin{aligned} S^{vorticity}(q) = & 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3 \cos(2\phi_{21})) + B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21})) \\ & + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 + 20 \cos(2\phi_{21}) + 35 \cos(4\phi_{21})) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right) \\ & + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2 \cos(2\phi_{21}) - \frac{7}{2} \cos(4\phi_{21}) \right) + B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2 \sin(2\phi_{21}) + \sin(4\phi_{21})) \\ & + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{21}) + \cos(4\phi_{21})) + \dots (l > 4) \end{aligned}$$

W_n's for the 2D detector when the neutrons propagate along the vorticity direction:

$$\begin{aligned}
W_n^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi_{21}) S(q) d\phi_{21} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3\cos(2\phi_{21})) + B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21})) \\
&\quad + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 + 20\cos(2\phi_{21}) + 35\cos(4\phi_{21})) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right) \\
W_0^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(0\phi_{21}) d\phi_{21} \\
&\quad + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2\cos(2\phi_{21}) - \frac{7}{2} \cos(4\phi_{21}) \right) + B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2\sin(2\phi_{21}) + \sin(4\phi_{21})) \\
&\quad + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4\cos(2\phi_{21}) + \cos(4\phi_{21})) + \dots (l > 4) \\
W_0^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(0\phi_{21}) d\phi_{21} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) \\
&\quad + B_{4,0}^+(q) \frac{27}{64\sqrt{\pi}} + B_{4,2}^+(q) \left(\frac{9\sqrt{5}}{32\sqrt{2\pi}} \right) + B_{4,4}^+(q) \left(\frac{9\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \\
W_0^{vorticity}(q)S^{eq}(q) &= 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) + B_{4,0}^+(q) \left(\frac{27}{64\sqrt{\pi}} \right) + B_{4,2}^+(q) \left(\frac{9\sqrt{5}}{32\sqrt{2\pi}} \right) + B_{4,4}^+(q) \left(\frac{9\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4)
\end{aligned}$$

$$\begin{aligned}
W_2^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\phi_{21}) S(q) d\phi_{21} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3\cos(2\phi_{21})) + B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21})) \\
&\quad + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 + 20\cos(2\phi_{21}) + 35\cos(4\phi_{21})) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right) \\
W_2^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\phi_{21}) d\phi_{21} \\
&\quad + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2\cos(2\phi_{21}) - \frac{7}{2}\cos(4\phi_{21}) \right) + B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2\sin(2\phi_{21}) + \sin(4\phi_{21})) \\
&\quad + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4\cos(2\phi_{21}) + \cos(4\phi_{21})) + \dots (l > 4) \\
W_2^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} B_{2,0}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{\pi}} \right) \cos(2\phi_{21}) + B_{2,2}^+(q) \left(-\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) \cos(2\phi_{21}) \\
&\quad + B_{4,0}^+(q) \frac{60}{64\sqrt{\pi}} \cos(2\phi_{21}) + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{2\pi}} \right) \cos(2\phi_{21}) + B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{16\sqrt{2\pi}} \right) \cos(2\phi_{21}) + \dots (l > 4) \\
W_2^{vorticity}(q)S^{eq}(q) &= \frac{1}{2} \left(B_{2,0}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{\pi}} \right) - B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) + B_{4,0}^+(q) \left(\frac{60}{64\sqrt{\pi}} \right) + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{2\pi}} \right) - B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{16\sqrt{2\pi}} \right) + \dots (l > 4) \right) \\
W_2^{vorticity}(q)S^{eq}(q) &= B_{2,0}^+(q) \left(\frac{3}{8} \sqrt{\frac{5}{\pi}} \right) - B_{2,2}^+(q) \left(\frac{1}{8} \sqrt{\frac{15}{2\pi}} \right) + B_{4,0}^+(q) \left(\frac{60}{128\sqrt{\pi}} \right) + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) - B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{32\sqrt{2\pi}} \right) + \dots (l > 4)
\end{aligned}$$

$$\begin{aligned}
W_4^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(4\phi_{21}) S(q) d\phi_{21} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3 \cos(2\phi_{21})) + B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21})) \\
&\quad + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 + 20 \cos(2\phi_{21}) + 35 \cos(4\phi_{21})) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right) \\
W_4^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \\
&\quad + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2 \cos(2\phi_{21}) - \frac{7}{2} \cos(4\phi_{21}) \right) + B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2 \sin(2\phi_{21}) + \sin(4\phi_{21})) \\
&\quad + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{21}) + \cos(4\phi_{21})) + \dots (l > 4) \\
W_4^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} B_{4,0}^+(q) \left(\frac{105}{64\sqrt{\pi}} \right) \cos(4\phi_{21}) + B_{4,2}^+(q) \left(\frac{-21\sqrt{5}}{32\sqrt{2\pi}} \right) \cos(4\phi_{21}) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) \cos(4\phi_{21}) + \dots (l > 4) \cos(4\phi_{21}) d\phi_{21} \\
W_4^{vorticity}(q)S^{eq}(q) &= 2 \left(B_{4,0}^+(q) \left(\frac{105}{64\sqrt{\pi}} \right) - B_{4,2}^+(q) \left(\frac{21\sqrt{5}}{16\sqrt{2\pi}} \right) + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \right) \\
W_6^{vorticity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(6\phi_{21}) S(q) d\phi_{21} \\
W_6^{vorticity}(q)S^{eq}(q) &= 0
\end{aligned}$$

$$\begin{aligned}
W_{12}^{vorticity}(q) S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\phi_{21}) \sin(\phi_{21}) S(q) d\phi_{21} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) (1 + 3 \cos(2\phi_{21})) + B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{21})) \\
&\quad + B_{4,0}^+(q) \frac{3}{64\sqrt{\pi}} (17 + 260 \cos(2\phi_{21}) + 35 \cos(4\phi_{21})) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \left(\sin(\phi_{21}) \cos(\phi_{21}) + \frac{7}{4} \sin(4\phi_{21}) \right) \\
W_{12}^{vorticity}(q) S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\phi_{21}) \sin(\phi_{21}) d\phi_{21} \\
&\quad + B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{16\sqrt{2\pi}} \right) \left(\frac{3}{2} + 2 \cos(2\phi_{21}) - \frac{7}{2} \cos(4\phi_{21}) \right) + B_{4,3}^+(q) \left(\frac{3}{32} \sqrt{\frac{35}{\pi}} \right) (2 \sin(2\phi_{21}) + \sin(4\phi_{21})) \\
&\quad + B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{21}) + \cos(4\phi_{21})) + \dots (l > 4) \\
W_{12}^{vorticity}(q) S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \left(B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) \sin(\phi_{21}) \cos(\phi_{21}) + \dots (l > 4) \right) \cos(\phi_{21}) \sin(\phi_{21}) d\phi_{21} \\
W_{12}^{vorticity}(q) S^{eq}(q) &= \frac{1}{8} \left(B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \right) + B_{4,1}^+(q) \left(\frac{3\sqrt{5}}{8\sqrt{\pi}} \right) + \dots (l > 4) \right)
\end{aligned}$$

Tangential (velocity gradient-vorticity)

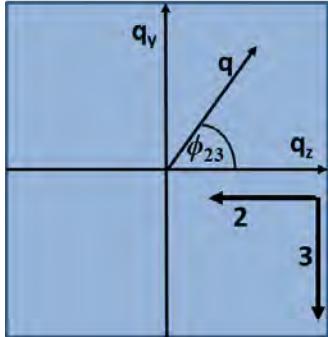
For the 2D detector when the neutrons propagate along the flow direction:

$$\frac{q_x}{q} = 0$$

$$\frac{q_y}{q} = -\sin(\phi_{23})$$

$$q_z = -\cos(\phi_{23})$$

$$[l=0, m=0] = B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right)$$



$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(3 \frac{q_z^2}{q^2} - 1 \right) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(-1 + 3 \cos^2(\phi_{23}) \right) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\sqrt{\frac{5}{4\pi}} \left(-1 + 3 \left(\frac{1}{2} (1 + \cos(2\phi_{23})) \right) \right) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\frac{1}{2} \sqrt{\frac{5}{4\pi}} (1 + 3 \cos(2\phi_{23})) \right)$$

$$[l=2, m=0] = B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} (1 + 3 \cos(2\phi_{23})) \right)$$

$$[l=2, m=1] = B_{2,1}^+(q) \left(\sqrt{\frac{15}{2\pi}} \left(\frac{q_x q_z}{q^2} \right) \right)$$

$$[l=2, m=1] = 0$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{q_x^2 - q_y^2}{q^2} \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \sqrt{\frac{15}{8\pi}} \left(\frac{-q_y^2}{q^2} \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \right) \sin^2(\phi_{23})$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\sqrt{\frac{15}{8\pi}} \right) \left(\frac{1}{2} (1 - \cos(2\phi_{23})) \right)$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{2} \sqrt{\frac{15}{8\pi}} \right) (1 - \cos(2\phi_{23}))$$

$$[l=2, m=2] = B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{23}))$$

$$\begin{aligned}
[l=4, m=0] &= B_{4,0}^+(q) \frac{3}{8\sqrt{\pi}} \left(3 - 30 \left(\frac{q_z^2}{q^2} \right) + 35 \left(\frac{q_z^4}{q^4} \right) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) (3 - 30 \cos^2(\phi_{23}) + 35 \cos^4(\phi_{23})) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(3 - 15(1 + \cos(2\phi_{23})) + \frac{35}{8}(3 + 4 \cos(2\phi_{23}) + \cos(4\phi_{23})) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(-12 - 15 \cos(2\phi_{23}) + \frac{35}{8}(3 + 4 \cos(2\phi_{23}) + \cos(4\phi_{23})) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{8\sqrt{\pi}} \right) \left(-\frac{96}{8} - \frac{120}{8} \cos(2\phi_{23}) + \frac{105}{8} + \frac{140}{8} \cos(2\phi_{23}) + \frac{35}{8} \cos(4\phi_{23}) \right) \\
[l=4, m=0] &= B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 - 20 \cos(2\phi_{23}) - 35 \cos(4\phi_{23})) \\
[l=4, m=1] &= B_{4,1}^+(q) \frac{3\sqrt{5}}{4\sqrt{\pi}} \left(\frac{q_x q_z}{q^2} \right) \left(7 \left(\frac{q_z^2}{q^2} \right) - 3 \right) \\
[l=4, m=1] &= 0
\end{aligned}$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{q_x^2 - q_y^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{-q_y^2}{q^2} \right) \left(-1 + 7 \left(\frac{q_z^2}{q^2} \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \sin^2(\phi_{23}) (-1 + 7 \cos^2(\phi_{23}))$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{4\sqrt{2\pi}} \right) \left(\frac{1}{2} (1 - \cos(2\phi_{23})) \right) \left(-1 + 7 \left(\frac{1}{2} (1 + \cos(2\phi_{23})) \right) \right)$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{8\sqrt{2\pi}} \right) (1 - \cos(2\phi_{23})) (5 + 7 \cos(2\phi_{23}))$$

$$[l=4, m=2] = B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{32\sqrt{2\pi}} \right) \left(5 + 2 \cos(2\phi_{23}) - \frac{7}{2} \cos(4\phi_{23}) \right)$$

$$[l=4, m=3] = B_{4,3}^+(q) \left(\frac{3}{4} \sqrt{\frac{35}{\pi}} \right) \left(\frac{q_x q_z (q_x^2 - 3q_y^2)}{q^4} \right)$$

$$[l=4, m=3] = 0$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{(q_x^2 - 2q_x q_y - q_y^2)(q_x^2 + 2q_x q_y - q_y^2)}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{8\sqrt{2\pi}} \right) \left(\frac{q_y^4}{q^4} \right)$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{8\sqrt{2\pi}} \right) (\sin^4(\phi_{23}))$$

$$[l=4, m=4] = B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4\cos(2\phi_{23}) + \cos(4\phi_{23}))$$

Structure factor for the 2D detector when the neutrons propagate along the flow direction:

$$\begin{aligned} S^{velocity}(q) = & 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} (1 + 3\cos(2\phi_{23})) \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{23})) \\ & + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 - 20\cos(2\phi_{23}) - 35\cos(4\phi_{23})) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{32\sqrt{2\pi}} \right) \left(5 + 2\cos(2\phi_{23}) - \frac{7}{2}\cos(4\phi_{23}) \right) \\ & + B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4\cos(2\phi_{23}) + \cos(4\phi_{23})) + \dots (l > 4) \end{aligned}$$

W_n's for the 2D detector when the neutrons propagate along the flow direction:

$$\begin{aligned}
W_n^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi_{23}) S(q) d\phi_{23} \\
&\quad 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} (1 + 3 \cos(2\phi_{23})) \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) (1 - \cos(2\phi_{23})) \\
W_0^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \left[B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 - 20 \cos(2\phi_{23}) - 35 \cos(4\phi_{23})) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{32\sqrt{2\pi}} \right) \left(5 + 2 \cos(2\phi_{23}) - \frac{7}{2} \cos(4\phi_{23}) \right) \right] \cos(0\phi_{23}) d\phi_{23} \\
&\quad + B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{23}) + \cos(4\phi_{23})) + \dots (l > 4) \\
W_0^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \left[1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) \right. \\
&\quad \left. + B_{4,0}^+(q) \left(\frac{27}{64\sqrt{\pi}} \right) + B_{4,2}^+(q) \left(\frac{-15\sqrt{5}}{32\sqrt{2\pi}} \right) + B_{4,4}^+(q) \left(\frac{-9\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \right] \cos(0\phi_{23}) d\phi_{23} \\
W_0^{velocity}(q)S^{eq}(q) &= 1 + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) + B_{4,0}^+(q) \left(\frac{27}{64\sqrt{\pi}} \right) - B_{4,2}^+(q) \left(\frac{15\sqrt{5}}{32\sqrt{2\pi}} \right) - B_{4,4}^+(q) \left(\frac{9\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4)
\end{aligned}$$

$$\begin{aligned}
W_2^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\phi_{23}) S(q) d\phi_{23} \\
&\quad + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} (1 + 3 \cos(2\phi_{23})) \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} (1 - \cos(2\phi_{23})) \right) \\
W_2^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \\
&\quad + B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 - 20 \cos(2\phi_{23}) - 35 \cos(4\phi_{23})) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{32\sqrt{2\pi}} \right) \left(5 + 2 \cos(2\phi_{23}) - \frac{7}{2} \cos(4\phi_{23}) \right) \cos(2\phi_{23}) d\phi_{23} \\
&\quad + B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{23}) + \cos(4\phi_{23})) + \dots (l > 4) \\
W_2^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \\
&\quad + B_{2,0}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{\pi}} \right) \cos(2\phi_{23}) + B_{2,2}^+(q) \left(\frac{-1}{4} \sqrt{\frac{15}{2\pi}} \right) \cos(2\phi_{23}) \\
&\quad + B_{4,0}^+(q) \left(\frac{-60}{64\sqrt{\pi}} \right) \cos(2\phi_{23}) + B_{4,2}^+(q) \left(\frac{-6\sqrt{5}}{32\sqrt{2\pi}} \right) \cos(2\phi_{23}) + B_{4,4}^+(q) \left(\frac{12\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \\
&\quad \cos(2\phi_{23}) d\phi_{23} \\
W_2^{velocity}(q)S^{eq}(q) &= \frac{1}{2} \left(B_{2,0}^+(q) \left(\frac{3}{4} \sqrt{\frac{5}{\pi}} \right) - B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} \right) - B_{4,0}^+(q) \left(\frac{60}{64\sqrt{\pi}} \right) \cos(2\phi_{23}) - B_{4,2}^+(q) \left(\frac{6\sqrt{5}}{32\sqrt{2\pi}} \right) \cos(2\phi_{23}) + B_{4,4}^+(q) \left(\frac{12\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \right)
\end{aligned}$$

$$\begin{aligned}
W_4^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(4\phi_{23}) S(q) d\phi_{23} \\
&\quad + B_{0,0}^+(q) \left(\frac{1}{\sqrt{\pi}} \right) + B_{2,0}^+(q) \left(\frac{1}{4} \sqrt{\frac{5}{\pi}} (1 + 3 \cos(2\phi_{23})) \right) + B_{2,2}^+(q) \left(\frac{1}{4} \sqrt{\frac{15}{2\pi}} (1 - \cos(2\phi_{23})) \right) \\
W_4^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \left[+ B_{4,0}^+(q) \left(\frac{3}{64\sqrt{\pi}} \right) (9 - 20 \cos(2\phi_{23}) - 35 \cos(4\phi_{23})) + B_{4,2}^+(q) \left(\frac{-3\sqrt{5}}{32\sqrt{2\pi}} \right) \left(5 + 2 \cos(2\phi_{23}) - \frac{7}{2} \cos(4\phi_{23}) \right) \right] \cos(4\phi_{23}) d\phi_{23} \\
W_6^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(6\phi_{23}) S(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) (3 - 4 \cos(2\phi_{23}) + \cos(4\phi_{23})) + \dots (l > 4) \\
W_4^{velocity}(q)S^{eq}(q) &= \frac{1}{2\pi} \int_0^{2\pi} B_{4,0}^+(q) \left(\frac{-105}{64\sqrt{\pi}} \right) \cos(4\phi_{23}) + B_{4,2}^+(q) \left(\frac{21\sqrt{5}}{64\sqrt{2\pi}} \right) \cos(4\phi_{23}) + B_{4,4}^+(q) \left(\frac{-3\sqrt{35}}{64\sqrt{2\pi}} \right) \cos(4\phi_{23}) + \dots (l > 4) \cos(4\phi_{23}) d\phi_{23} \\
W_{12}^{velocity}(q)S^{eq}(q) &= 2 \left(\frac{1}{2\pi} \int_0^{2\pi} B_{4,0}^+(q) \left(\frac{105}{64\sqrt{\pi}} \right) \sin(\phi_{23}) B_{4,2}^+(q) \left(\frac{21\sqrt{5}}{64\sqrt{2\pi}} \right) - B_{4,4}^+(q) \left(\frac{3\sqrt{35}}{64\sqrt{2\pi}} \right) + \dots (l > 4) \right) \\
W_{12}^{velocity}(q)S^{eq}(q) &= 0
\end{aligned}$$

Identities:

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

$$\sin^4(\theta) = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

$$\cos^4(\theta) = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

$$\cos^2(\theta)\sin^2(\theta) = \frac{1}{8}(1 - \cos(4\theta))$$

$$\cos(\theta)\sin^3(\theta) = \frac{1}{4}\sin(2\theta) + \frac{1}{8}\sin(4\theta)$$

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

$$\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))$$

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$\cos(-a) = \cos(a)$$

$$\sin(-a) = -\sin(a)$$

$$\int_0^{2\pi} \sin \varphi d\varphi = -\cos \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2 \varphi d\varphi = \left(\frac{\varphi}{2} - \frac{1}{4} \sin 2\varphi \right) \Big|_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \sin^3 \varphi d\varphi = \left(-\cos \varphi - \frac{1}{3} \cos^3 \varphi \right) \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^4 \varphi d\varphi = \left(\frac{3\varphi}{8} - \frac{1}{4} \sin 2\varphi + -\frac{1}{32} \sin 4\varphi \right) \Big|_0^{2\pi} = \frac{3\pi}{4}$$

$$\int_0^{2\pi} \cos \varphi d\varphi = \sin \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos^2 \varphi d\varphi = \left(\frac{\varphi}{2} + \frac{1}{4} \sin 2\varphi \right) \Big|_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \cos^3 \varphi d\varphi = \left(\sin \varphi - \frac{1}{3} \sin^3 \varphi \right) \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos^4 \varphi d\varphi = \left(\frac{3\varphi}{8} + \frac{1}{4} \sin 2\varphi + \frac{1}{32} \sin 4\varphi \right) \Big|_0^{2\pi} = \frac{3\pi}{4}$$

$$\int_0^{2\pi} \sin \varphi \cos \varphi d\varphi = \frac{1}{2} \sin^2 \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin \varphi \cos^2 \varphi d\varphi = -\frac{1}{3} \cos^3 \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2 \varphi \cos \varphi d\varphi = \frac{1}{3} \sin^3 \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2 \varphi \cos^2 \varphi d\varphi = \left(\frac{\varphi}{8} - \frac{1}{32} \sin 4\varphi \right) \Big|_0^{2\pi} = \frac{\pi}{4}$$

$$\int_0^{2\pi} \sin \varphi \cos^3 \varphi d\varphi = -\frac{1}{4} \cos^4 \varphi \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^3 \varphi \cos \varphi d\varphi = 1 \Big|_0^{2\pi} = 1$$