MEASURE THEORY AND COMPLEX VARIABLES

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Contents

1. Introduction

These notes were taken in University of Delaware's MATH602 (Measure, Integration, and Complex Variables) course, taught by Dr. Mahya Ghandehari in Spring 2021. I typed them based on hand-written notes taken during class each week- the hope was that a typed version would provide a better record in the future and be much more useful. Dr. Ghandehari's lecture notes were self-contained, though we took material from:

- Rudin, *Principles of Mathematical Analysis (3rd Edition)*
- Royden & Fitzpatrick, *Real Analysis (4th Edition)*
- Silverman, *Complex Analysis with Applications*
- Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*

These notes are a work in progress; all mistakes are mine and mine alone (either through mistyping or a misunderstanding of the material). If you have any error corrections, tips, or general comments, please reach out to me at: ghoefer@udel.edu.

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2. Review of the basics

2.1. **Space of continuous functions and review.** We've defined Riemann-Stieltjes integrals previously (in MATH600); if we let $\alpha(x) = x$, we get the definition of the Riemann integral.

Definition 2.1. *Let $f : [a,b] \to \mathbb{R}$ be bounded, so there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Consider a partition $\mathcal{P} := a = x_0 < \cdots < x_n = b$ of $[a,b]$. Define $M_i = \sup(f|_{[x_{i-1},x_i]}), m_i = \inf(f|_{[x_{i-1},x_i]})$. Define*

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad \mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Finally, define

$$\overline{\int_a^b} f dx = \inf_{\mathcal{P} \text{ partition}} \mathcal{U}(\mathcal{P}, f), \quad \underline{\int_a^b} f dx = \sup_{\mathcal{P} \text{ partition}} \mathcal{L}(\mathcal{P}, f)$$

We say f is Riemann integrable, denoted by $\int_a^b f dx$ if $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$.

Remark: When we say f is Riemann integrable, we are always assuming f is a bounded function.

Recall: If (X, d) is a metric space and f_n, f are functions which map $X \to \mathbb{C}$ we say $f_n \to f$ pointwise if for all $x \in X, f_n(x) \to f(x)$ as $n \to \infty$ (and fixed $x \in X$).

Examples:

(i)

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = \begin{cases} n, & 0 < x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

$$f : [0,1] \to \mathbb{R}, \quad f(x) \equiv 0$$

(ii)

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) = \begin{cases} 1, & x \in \mathbb{Q},\ x = \frac{p}{q},\ q \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

Note that in (i), both functions were Riemann integrable, but $\int f_n \not\to \int f$. Additionally, in (ii) note that f_n is Riemann integrable as it has a finite number of discontinuities, but f is not Riemann integrable. These illuminate the issues with pointwise convergence. Most of the time, our convergence of integrals breaks down for one of two reasons: either the integral itself has issues, or our convergence is pointwise and not uniform.

Definition 2.2. *Let (X, d) be a metric space, and let $f_n, f : X \to \mathbb{R}$. We say $f_n \to f$ uniformly if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$.*

Properties of uniform convergence:

(i) If $f_n \to f$ uniformly and f_n is continuous on X for all $n \in \mathbb{N}$, then f must be continuous.

(ii) If $f_n \to f$ uniformly and each f_n is Riemann integrable, then f is as well and $\int f_n \to \int f$.

Putting things in context:
Define a new space $C_{[a,b]} := \{f : [a,b] \to \mathbb{R} : \text{ continuous functions } \}$

- $C_{[a,b]}$ is a metric space
- The norm on $C_{[a,b]}$ is $\|f\|_{\sup} = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|$ for $f \in C_{[a,b]}$

Exercise: Check that $\|\cdot\|_{\sup}$ does indeed have all properties of a norm.

Theorem 2.3. *The metric space $C_{[a,b]}$ is complete.*

Proof. (Idea) Take some Cauchy sequence in $C_{[a,b]}$, and define $f(x) := \lim_{n\to\infty} f_n(x)$ for each $x \in [a,b]$. Show that $f_n \to f$ uniformly, meaning f must be continuous, and therefore $f \in C_{[a,b]}$. □

2.2. **Weierstrass Approximation Theorem and applications.**

Theorem 2.4 (Weierstrass Approximation Theorem (Real Version))**.** *Let $f \in C_{[a,b]}$. There exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n \to f$ uniformly (i.e. $p_n \to f$ in $C_{[a,b]}$). If our function f is real, we may take our sequence of polynomials to have real coefficients.*

Proof. We break the proof into steps.
Step 1: We claim it is enough to show the problem for $C_{[0,1]}$. Define $\phi : [a,b] \to [0,1]$ by $x \mapsto \frac{x-a}{b-a}$, and inverse $\phi^{-1} : [0,1] \to [a,b]$ where $x \mapsto (b-a)x+a$. Define $\Gamma : C_{[0,1]} \to C_{[a,b]}$ where $\Gamma(f) = f(\phi(x))$, and $\Gamma^{-1} : C_{[a,b]} \to C_{[0,1]}$ where $\Gamma^{-1}(f) = f(\phi^{-1}(x))$. It is clear that Γ is an isometry, so $\|\Gamma(f) - \Gamma(g)\|_{\sup} = \sup_{x\in[a,b]} |f(\phi(x)) - g(\phi(x))| = \sup_{y\in[0,1]} |f(y) - g(y)| = \|f-g\|_{\sup}$ (in $C_{[0,1]}$). We also note that $p \in C_{[0,1]}$ is a polynomial if and only $\Gamma(p) \in C_{[a,b]}$ is a polynomial. Therefore, we can translate the problem from $C_{[a,b]}$ to $C_{[0,1]}$ without any trouble, simplifying matters somewhat. Without loss of generality, from here on out assume $a = 0, b = 1$.

Step 2: Let $\epsilon > 0$ be given. We also claim that we may assume $f(0) = f(1) = 0$. We may make the previous statement, as if $f, p \in C_{[0,1]}$ with $\|f - p\|_{\sup} < \epsilon$ if we define $g(x) = f(x) - ((f(1) - f(0))x + f(0)), q(x) = p(x) - ((f(1) - f(0))x + f(0))$ then $g(0) = g(1) = 0$, and $\|g - q\|_{\sup} = \|f - p\|_{\sup} < \epsilon$. This allows us to assume without loss of generality that f is zero at both endpoints.

Step 3: We now want to prove that $(1 - x^2)^n \geq 1 - nx^2$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Define

$$h(x) = (1 - x^2)^n - (1 + nx^2)$$

We see $h'(x) = -2x(1 - x^2)^{n-1} + 2nx$. We see $h(0) = 0$, so we want to show $h'(x) \geq 0$ for all $x \in (0,1]$- this follows immediately from the fact that

$$h'(x) = -2x(1 - x^2)^{n-1} + 2nx = 2x(n - (1 - x^2)^{n-1}) \geq 0$$

for all $x \in [0,1]$.

Step 4: Assume $f \in C_{[0,1]}$ such that $f(0) = f(1) = 0$. Extend f to a continuous function on $\mathbb{R}$, again called f, by setting $f(x) = 0$ for all $x \in \mathbb{R} \setminus [0,1]$. Clearly, f is uniformly continuous (**exercise!**). Define

$$Q_n(x) = c_n(1-x^2)^n, \text{ such that } \int_{-1}^{1} Q_n(x)dx = 1.$$

Since $(1-x^2)^n \geq 1 - nx^2$ on $[0,1]$, we have

$$\int_{-1}^{1} (1-x^2)^n dx = 2\int_{0}^{1} (1-x^2)^n dx \geq 2\int_{0}^{\frac{1}{\sqrt{n}}} 1 - nx^2 dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

This implies $c_n < \sqrt{n}$ for $n \in \mathbb{N}$. Take $0 < \delta < 1$. For all $x \in [-1,-\delta] \cup [\delta, 1]$ we have $c_n(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n$. Let

$$p_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$

$$= \int_{0}^{1} f(u)Q_n(u-x)du \text{ (as } f(y) = 0 \text{ if } y < 0, y > 1).$$

It is not hard to see that $p_n(x)$ is a polynomial for each $n \in \mathbb{N}$; we also note that p_n has degree at most $2n$. Let $\epsilon > 0$ be given, and let $M = \|f\|_{\sup}$. As f is uniformly continuous, there exists some $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in [0,1]$. Let $x \in [0,1]$. We see

$$|f(x) - p_n(x)| = \left| f(x)\int_{-1}^{1} Q_n(t)dt - \int_{-1}^{1} f(x+t)Q_n(t)dt \right| = \left| \int_{-1}^{1} f(x)Q_n(t)dt - \int_{-1}^{1} f(x+t)Q_n(t)dt \right|$$

$$= \left| \int_{-1}^{1} (f(x) - f(x+t))Q_n(t)dt \right| \leq \int_{-1}^{1} |f(x) - f(x+t)|Q_n(t)dt$$

$$= \int_{-1}^{-\delta} |f(x) - f(x+t)|Q_n(t)dt + \int_{-\delta}^{\delta} |f(x) - f(x+t)|Q_n(t)dt + \int_{\delta}^{1} |f(x) - f(x+t)|Q_n(t)dt.$$

We note

$$\int_{-\delta}^{\delta} |f(x) - f(x+t)|Q_n(t)dt \leq \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)dt$$

$$\leq \frac{\epsilon}{2}\int_{-1}^{1} Q_n(t)dt = \frac{\epsilon}{2}.$$

We also note that

$$\int_{-1}^{-\delta} |f(x) - f(x+t)|Q_n(t)dt \leq 2M(\sqrt{n}(1-\delta^2)^n), \text{ and}$$

$$\int_{\delta}^{1} |f(x) - f(x+t)|Q_n(t)dt \leq 2M(\sqrt{n}(1-\delta^2)^n).$$

Then $\|f-p_n\|_{\sup} \le 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2}$, as $x \in [0,1]$ was arbitrary. We can then choose N large enough so that $4M\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}$ for all $n \ge N$. Putting these together, we see

$$\|f-p_n\|_{\sup} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \ge N$, which completes the proof. □

Application: $C_{[a,b]}$ is separable.

Proof. Let $n \in \mathbb{N}$. Let $R_n = \{$ polynomials of degree n with coefficients in $\mathbb{Q}\}$. It is clear R_n is countable for each $n \in \mathbb{N}$. Let $R = \bigcup_{n\in\mathbb{N}} R_n$. As R is a countable union of countable sets, it is countable. Suppose $f \in C_{[a,b]}$, and let $\epsilon > 0$ be given. By the Weierstrass Approximation Theorem, there exists a polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with $a_0, \ldots, a_k \in \mathbb{R}$ such that $\|f-P\|_{\sup} < \frac{\epsilon}{2}$. Let $b_0, \ldots, b_k \in \mathbb{Q}$ such that $|a_i - b_i| < \frac{\epsilon}{2k \sup_{x\in[a,b]} |x^i|}$. Then if $Q(x) = b_0 + b_1x + \ldots + b_kx^k$, we have

$$\|Q(x) - P(x)\|_{\sup} \le \sum_{i=1}^{k} |a_i - b_i| \sup_{x\in[a,b]} |x^i| < \frac{\epsilon}{2}.$$

This shows $\|f-Q\|_{\sup} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This implies R is dense in $C_{[a,b]}$, and so $C_{[a,b]}$ has a countably dense subset. □

Theorem 2.5 (Stone-Weierstrass Theorem (real case))**.** *Let K be a compact subset of a metric space (X,d). Define $C(K) := \{f : K \to \mathbb{R}$ continuous$\}$, and let the norm on $C(K)$ be $\|f\|_{sup} := \sup_{k\in K} |f(x)|$. Let $\mathcal{A}$ be a subset of $C(K)$ that satifies the following properties:*

(i) $\mathcal{A}$ is an algebra
(ii) $\mathcal{A}$ separates points: if $x \ne y$ and $x, y \in K$ there exists some $f \in \mathcal{A}$ such that $f(x) \ne f(y)$
(iii) $\mathcal{A}$ does not vanish at any point in K: for all $x \in K$, there exists an $f \in \mathcal{A}$ such that $f(x) \ne 0$

Then $\mathcal{A}$ is $\|\cdot\|_{sup}$-dense in $C(K)$.

Application: Let X, Y be compact metric spaces, and let $f : X \times Y \to \mathbb{R}$ be continuous. Then for all $\epsilon > 0$ there exists an $n > 0$ and functions $f_1, \ldots, f_n : X \to \mathbb{R}$ and $g_1, \ldots, g_n : Y \to \mathbb{R}$ which are continuous such that

$$\|f - \sum_{i=1}^{n} f_i \otimes g_i\|_{\sup} < \epsilon.$$

Definition 2.6. *For $f_i : X \to \mathbb{R}$, $g_i : Y \to \mathbb{R}$ we define $f_i \otimes g_i : X \times Y \to \mathbb{R}$ by $(f_i \otimes g_i)(x,y) = f_i(x)g_i(y)$ for $i = 1, \ldots, n$.*

Proof. (Idea) Apply the Stone-Weierstrass theorem to the set $\mathcal{A} = \text{span}\{h \otimes g : h \in C(X), g \in C(Y)\}$, where $\mathcal{A} \subseteq C(X \times Y)$. □

Application: Let $\mathbb{T} = [0,1]/(0 \equiv 1) \to \mathbb{R}/\mathbb{Z}$ (i.e. $\mathbb{T}$ is the torus). Define $C_{\mathbb{C}}(\mathbb{T}) := \{f : [0,1] \to \mathbb{C}$ continuous, and $f(0) = f(1)\}$ and take $\mathcal{A} = \text{span}\{\chi_n : n \in \mathbb{Z}\}$ where $\chi_n : [0,1] \to \mathbb{C}$ such that $\chi_n(t) = e^{2\pi i n t}$. The Stone-Weierstrass theorem tells us $\mathcal{A}$ is dense in $C_{\mathbb{C}}(\mathbb{T})$.

Remark: $\mathcal{A} \subseteq C_{\mathbb{C}}(K)$ is called self-adjoint if $\mathcal{A} = \overline{\mathcal{A}}$.

3. The Lebesgue measure

3.1. Lebesgue measure.

Definition 3.1. *$\mathcal{P}(X)$ is the set of all subsets of set X, and is called the power set of X.*

As a little bit of background, the idea of measure arose to generalize the concept of size or length to arbitrary spaces (when applicable). Measure, in turn, will allow us to define Lebesgue integration, which developed due to the failure of Riemann integration to behave in certain ways for a lot of functions. We define a measure to be a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ which satisfies the following properties:

(i) $m(\text{interval}) = \text{length of that interval}$

(ii) If $\{E_i\}_{i=1}^{\infty}$ is a collection of pairwise disjoint subsets of $\mathbb{R}$, we have $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$ (i.e. m is countably additive)

(iii) m is translation invariant: $m(x + E) = m(E)$

(iv) If $A \subseteq B$, then $m(A) \leq m(B)$

Notation: $x + E = \{x + y : y \in E\}$ where $E \subseteq \mathbb{R}$.

Proposition 3.2. *Such a function m as described above does not exist (rats).*

Proof. Long- see handwritten notes if you want the construction. □

The following is the definition for the Lebesgue outer measure.

Definition 3.3. *Let $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ such that*

$$\lambda^*(E) = \inf\left\{\sum_{i=1}^{\infty} \ell(I_i) : E \subseteq \bigcup_{i=1}^{\infty} I_i\right\}.$$

Note: we must have a countable union of open intervals for the definition of the outer measure (although technically it could work with a countable union of closed intervals, as we can approach open intervals using closed ones and vice versa).

Examples:

(i) $\lambda^*(\{x\}) = 0$

(ii) $\lambda^*(\mathbb{Z}) = 0$ (**exercise!**)

Proof. Let $\epsilon > 0$ be given. Take $I_n = (n - \frac{\epsilon}{2^{|n|}}, n + \frac{\epsilon}{2^{|n|}})$ for every $n \in \mathbb{Z}$. Then

$$\lambda^*(\mathbb{Z}) \leq \sum_{n \in \mathbb{Z}} \ell(I_n) = \sum_{n \in \mathbb{Z}} \frac{\epsilon}{2^{|n|-1}} = 4\epsilon.$$

As $\epsilon > 0$ was arbitrary, this implies $\lambda^*(\mathbb{Z}) = 0$. □

Properties of outer measure

(i) $\lambda^*(E) \geq 0$

(ii) $\lambda^*(\emptyset) = 0, \lambda^*(\{x\}) = 0$

(iii) λ^* is increasing: if $A \subseteq B$, then $\lambda^*(A) \leq \lambda^*(B)$

Proof. (Rough sketch) Find a covering of B with a countable union of intervals- as $A \subseteq B$, this is a covering of A as well, with the infimum for the A cover less than or equal to the infimum of the B cover just by definition. This implies $\lambda^*(A) \leq \lambda^*(B)$. □

(iv) λ^* is translation invariant: for all $E \subseteq \mathbb{R}, x \in \mathbb{R}$, then $\lambda^*(x + E) = \lambda^*(E)$

(v) $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(E_i)$ (σ sub-additivity of λ^*. If we drop σ, we have sub-additivity which only works for finite sums).

(vi) $\lambda^*(\text{interval}) = \text{length of the interval}$

Proof. (i), (ii), (iii), (iv)- **exercise!**

(v) If $\sum_{n=1}^{\infty} \lambda^*(E_n) = \infty$, then we are done- so assume $\sum_{n=1}^{\infty} \lambda^*(E_n) < \infty$. Let $\epsilon > 0$ be arbitrary. There exists $C_n = \{I_i^n : i \in \mathbb{N}\}$ an open cover of intervals for E_n, for each $n \in \mathbb{N}$, such that

$$\sum_{i=1}^{n} \ell(I_i^n) < \lambda^*(E_n) + \frac{\epsilon}{2^n}.$$

Consider $C = \cup C_n = \{I_i^n : i, n \in \mathbb{N}\}$. We see that C is countable, as a countable union of countable sets. We also note that C is then a countable open covering of $\bigcup_{n=1}^{\infty} E_n$. Therefore,

$$\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \ell(I_i^n) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_i^n) \leq \sum_{n=1}^{\infty}\left(\lambda^*(E_n) + \frac{\epsilon}{2^n}\right)$$
$$= \sum_{n=1}^{\infty} \lambda^*(E_n) + \epsilon.$$

Then as $\epsilon > 0$ was arbitrary, this shows $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$.

(vi) We'll prove (vi) for a closed interval $[a, b]$. As $[a, b] \subseteq (a - \epsilon, b + \epsilon)$, then $\lambda^*([a, b]) \leq b - a + 2\epsilon$ for all $\epsilon > 0$; if we let $\epsilon \to 0$, this shows $\lambda^*([a, b]) \leq b - a$. We next want to show that for every countable open covering of $[a, b]$ with intervals, we have $\sum_{i=1}^{\infty} \ell(I_i) \geq b - a$. Suppose $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$, where I_i is an open interval for each $i \in \mathbb{N}$. As $[a, b]$ is compact, there exists a finite subcover $[a, b] \subseteq I_{i_1} \cup \cdots \cup I_{i_n}$. We know that $a \in I_{i_1} \cup \cdots \cup I_{i_n}$, and so $a \in I_{i_{n_1}}$ for some interval in the finite subcover. If $b \in I_{i_{n_1}}$, then $[a, b] \subseteq I_{i_{n_1}}$ and so

$$b - a \leq \ell(I_{i_{n_1}}) \leq \sum_{i=1}^{\infty} \ell(I_i).$$

If not, let $I_{i_{n_1}} = (a_1, b_1)$. Choose $I_{i_{n_2}}$ such that $b_1 \in I_{i_{n_2}}$; continuing this process (which ends after a finite number of steps), we obtain a set of overlapping intervals $I_{i_{n_1}}, I_{i_{n_2}}, \ldots, I_{i_{n_k}}$ which together contain $[a, b]$. Then

$$b - a \leq \ell(I_{i_{n_1}}) + \ldots + \ell(I_{i_{n_k}}) \leq \sum_{i=1}^{\infty} \ell(I_i).$$

This shows $b - a \leq \sum_{i=1}^{\infty} \ell(I_i)$, so $b - a \leq \lambda^*([a, b])$. This completes the proof. □

Proposition 3.4. *Using the fact above, we can show* $\lambda^*((a, b)) = b - a$.

Proof. It is clear that $\lambda^*((a, b)) \leq b - a$, as (a, b) is an open cover of (a, b). Note, for $\epsilon > 0$ we have $[a + \epsilon, b - \epsilon] \subseteq (a, b)$. As λ^* is increasing, $\lambda^*([a + \epsilon, b - \epsilon]) \leq \lambda^*((a, b))$ with $b - a + 2\epsilon \leq \lambda^*([a + \epsilon, b - \epsilon])$. Sending $\epsilon \to 0$, we have $b - a \leq \lambda^*((a, b))$, and therefore $\lambda^*((a, b)) = b - a$. □

Note: The same result can be shown for other types of intervals.

Corollary 3.5. *If $A \subseteq \mathbb{R}$ is countable, then $\lambda^*(A) = 0$.*

Proof. (Idea) Let $A = \{a_1, \dots, a_k, \dots\}$. Then
$$\lambda^*(A) = \lambda^*\left(\bigcup_{i=1}^{\infty} a_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(\{a_i\}) = \sum_{i=1}^{\infty} 0 = 0.$$
□

Example: Let $A = \mathbb{Q} \cap [0,1]$, and $B = \mathbb{Q}^C \cap [0,1]$. We see $\lambda^*(A) = 0$ as A is countable, and $\lambda^*(B) = 1$ as $B \subseteq [0,1]$ with
$$\lambda^*(B) \leq \lambda^*([0,1]) = 1;$$
for the other side of the inequality, we see
$$[0,1] = A \cup B$$
$$\Rightarrow 1 = \lambda^*([0,1]) \leq \lambda^*(A) + \lambda^*(B) = \lambda^*(B).$$
Recap: We defined $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ where λ^* is σ sub-additive, translation invariant, increasing, and generalizes the notion of length; *but*, λ^* is not additive.

Definition 3.6. *We say a subset $A \subseteq \mathbb{R}$ is Lebesgue measurable if for all $E \subseteq \mathbb{R}$ we have $\lambda^*(E) = \lambda^*(A \cap E) + \lambda^*(A^C \cap E)$. The set of of all Lebesgue measurable subsets of $\mathbb{R}$ is denoted by $\mathcal{L}(\mathbb{R})$, with $\mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$.*

Theorem 3.7 (Properties of Lebesgue measureable sets)**.**
(i) $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$
(ii) *If $E \in \mathcal{L}(\mathbb{R})$, then $E^C = \mathbb{R} \setminus E \in \mathcal{L}(\mathbb{R})$*
(iii) *If $E_1, E_2, \dots \in \mathcal{L}(\mathbb{R})$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R})$ (i.e. closed under countable union)*

Notation: Let X be a set. Let $\mathcal{M} \subseteq \mathcal{P}(X)$. We say $\mathcal{M}$ is an algebra of subsets if
(i) $\emptyset, X \in \mathcal{M}$
(ii) If $A \in \mathcal{M}$, then $X \setminus A \in \mathcal{M}$
(iii) If $A_1, A_2, \dots, A_n \in \mathcal{M}$ then $A_1 \cup \dots \cup A_n \in \mathcal{M}$

$\mathcal{M}$ is called a σ-algebra if it satisfies (i), (ii), and (iii)' (where we replace finite with countable number of sets in condition (iii)).

Remark: $\mathcal{L}(\mathbb{R})$ is a σ-algebra.

Definition 3.8. *Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra in $\mathbb{R}$ containing all open sets.*

Remark: For any $A, E \in \mathcal{P}(\mathbb{R})$, $\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \cap E^C)$.

In showing that $\mathcal{L}(\mathbb{R})$ is a σ-algebra, we will prove (iii)'. We'll break the proof down into steps:

Proof.
Step 1: We'll first prove that if $E_1, E_2 \in \mathcal{L}(\mathbb{R})$, then $E_1 \cup E_2 \in \mathcal{L}(\mathbb{R})$. Moreover, if $E_1 \cap E_2 = \emptyset$, then $\lambda^*(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2)$. To that end, let $A \in \mathcal{P}(\mathbb{R})$ be arbitrary. We want to show
$$\lambda^*(A) = \lambda^*(A \cap (E_1 \cup E_2)) + \lambda^*(A \cap (E_1 \cup E_2)^C).$$
Note that $A = (A \cap (E_1 \cup E_2)) \cup (A \cap (E_1 \cup E_2)^C)$. Then
$$\begin{aligned}\lambda^*(A) &= \lambda^*\Big((A \cap (E_1 \cup E_2)) \cup (A \cap (E_1 \cup E_2)^C)\Big) \\ &\leq \lambda^*(A \cap (E_1 \cup E_2)) + \lambda^*(A \cap (E_1 \cup E_2)^C),\end{aligned}$$

as λ^* is sub-additive. We see that

$$\lambda^*(A) = \lambda^*(A \cap E_1) + \lambda^*(A \cap E_1^C),$$

as $E_1 \in \mathcal{L}(\mathbb{R})$. Then

$$\begin{aligned}\lambda^*(A) &= \lambda^*(A \cap E_1 \cap E_2) + \lambda^*(A \cap E_1 \cap E_2^C) + \lambda^*(A \cap E_1^C) \\ &\geq \lambda^*(A \cap E_1 \cap E_2) + \lambda^*((A \cap E_1 \cap E_2^C) \cup (A \cap E_1^C)) \\ &\geq \lambda^*(A \cap E_1 \cap E_2) + \lambda^*(A \cap (E_1 \cap E_2)^C).\end{aligned}$$

This implies

$$\lambda^*(A) \geq \lambda^*(A \cap E_1 \cap E_2) + \lambda^*(A \cap (E_1 \cap E_2)^C),$$

which means

$$\lambda^*(A) = \lambda^*(A \cap E_1 \cap E_2) + \lambda^*(A \cap (E_1 \cap E_2)^C).$$

Therefore, $E_1 \cap E_2 \in \mathcal{L}(\mathbb{R})$. If $E_1, E_2 \in \mathcal{L}(\mathbb{R})$, then $E_1^C, E_2^C \in \mathcal{L}(\mathbb{R})$ as well and therefore $E_1^C \cap E_2^C \in \mathcal{L}(\mathbb{R})$. Then $(E_1^C \cap E_2^C)^C = E_1 \cup E_2 \in \mathcal{L}(\mathbb{R})$.

Next, if we suppose $E_1 \cap E_2 = \emptyset$, let $A = E_1 \cup E_2$. Then

$$\begin{aligned}\lambda^*(A) = \lambda^*(E_1 \cup E_2) &= \lambda^*((E_1 \cup E_2) \cap E_1) + \lambda^*((E_1 \cup E_2) \cap E_1^C) \\ &= \lambda^*(E_1) + \lambda^*(E_2).\end{aligned}$$

It is easy to see we can use induction to extend our claim to a finite union of Lebesgue measurable sets.

Step 2: We know if $E_1, \ldots, E_n \in \mathcal{L}(\mathbb{R})$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{L}(\mathbb{R})$. Moreover, if $E_1, \ldots, E_n$ are pairwise disjoint, then for all $A \in \mathcal{P}(\mathbb{R})$ we have

$$\lambda^*(A \cap \left(\bigcup_{i=1}^{n} E_i\right)) = \sum_{i=1}^{n} \lambda^*(A \cap E_i).$$

Step 3: Let $E_1, E_2, \ldots$ be a countable collection of sets in $\mathcal{L}(\mathbb{R})$. Fix $n \in \mathbb{N}$. Define $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_3 = E_3 \setminus (E_1 \cup E_2)$, etc. We note $F_i \cap F_j = \emptyset$ for $i \neq j$, and $F_i \in \mathcal{L}(\mathbb{R})$ for $1 \leq i \leq n$. Then

$$\begin{aligned}\lambda^*(A) &= \lambda^*\left(A \cap \left(\bigcup_{i=1}^{n} F_i\right)\right) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{n} F_i\right)^C\right) \\ &= \sum_{i=1}^{n} \lambda^*(A \cap F_i) + \lambda^*(A \cap (\cup_{i=1}^{n} F_i)^C) \\ &\geq \sum_{i=1}^{n} \lambda^*(A \cap F_i) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)^C\right),\end{aligned}$$

as λ^* is increasing. So for all $n \in \mathbb{N}$, we have

$$\lambda^*(A) \geq \sum_{i=1}^{n} \lambda^*(A \cap F_i) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)^C\right).$$

Letting $n \to \infty$, we then have

$$\lambda^*(A) \geq \sum_{i=1}^{\infty} \lambda^*(A \cap F_i) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)^C\right)$$
$$\geq \lambda^*\left(\bigcup_{i=1}^{\infty}(A \cap F_i)\right) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)^C\right)$$
$$\Rightarrow \lambda^*(A) \geq \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right) + \lambda^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)^C\right),$$

for all $A \in \mathcal{L}(\mathbb{R})$. Then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{L}(\mathbb{R})$, and so $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R})$. □

Remark: If $E_1, E_2, \ldots \in \mathcal{L}(\mathbb{R})$ and are all pairwise disjoint, then $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda^*(E_i)$.

Proof. **Exercise!** □

Theorem 3.9. *The collection of Lebesgue measurable sets is a σ-algebra. Additionally, $\mathcal{L}(\mathbb{R})$ contains the Borel σ-algebra.*

Notation: Restriction of λ^* to $\mathcal{L}(\mathbb{R})$ is called the Lebesgue measure on $\mathbb{R}$, and is denoted by λ. So $\lambda : \mathcal{L}(\mathbb{R}) \to [0, \infty]$ where $\lambda(E) = \lambda^*(E)$ for $E \in \mathcal{L}(\mathbb{R})$.

Theorem 3.10 (Lebesgue measure on $\mathbb{R}$)**.**

(i) If $E_1, E_2, \ldots, \in \mathcal{L}(\mathbb{R})$ are pairwise disjoint then $\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i)$ (this is called σ-additivity of λ)
(ii) λ is σ sub-additive
(iii) $\lambda(x + E) = \lambda(E)$ for all $x \in \mathbb{R}$ and $E \in \mathcal{L}(\mathbb{R})$
(iv) $\lambda(\emptyset) = 0$, λ(interval) = length of the interval

More properties of λ

(i) If $A, B \in \mathcal{L}(\mathbb{R})$ and $A \subseteq B$ we have $\lambda(B \setminus A) = \lambda(B) - \lambda(A)$
(ii) If K is any compact set, $\lambda(K) < \infty$, and if O is an open set then $\lambda(O) > 0$ (here we are assuming both $K, O \in \mathcal{L}(\mathbb{R})$)
(iii) Upward continuity of measures: if $A_1, A_2, \ldots \in \mathcal{L}(\mathbb{R})$ and $A_1 \subseteq A_2 \subseteq \ldots$ then $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n)$
(iv) Downwards continuity of measures: if $A_1 \supseteq A_2 \supseteq \ldots$ where each A_i is Lebesgue measurable and $\lambda(A_1) < \infty$, then $\lambda\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n)$

Proof. (i) As $A \subseteq B$, we may write $B = A \cup (B \setminus A)$, where $A \cap (B \setminus A) = \emptyset$. Then

$$\lambda(B) = \lambda(A) + \lambda(B \setminus A)$$
$$\Rightarrow \lambda(B) - \lambda(A) = \lambda(B \setminus A).$$

□

Proof. (ii) As K is compact, K is closed, and therefore belongs to the Borel algebra on $\mathbb{R}$. This means K is Lebesgue measurable. Furthermore, as K is compact, it is bounded, and so there exists an $M > 0$ such that $K \subseteq [-M, M]$ with

$$\lambda(K) \leq \lambda([-M, M]) = 2M < \infty.$$

If we suppose $O \subseteq \mathbb{R}$ is a non-empty open set, take $x_0 \in O$; there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq O$ with

$$\lambda((x_0 - \delta, x_0 + \delta)) \leq \lambda(O).$$

As $\lambda((x_0 - \delta, x_0 + \delta)) = 2\delta > 0$, this means $\lambda(O) > 2\delta > 0$. Therefore, as this holds for any $x_0 \in O$, and as O was an arbitrary open set, this shows any open set has positive measure. □

Proof. Suppose $A_1 \subseteq A_2 \subseteq \ldots$ is a increasing chain of Lebesgue measurable sets. Define $B_1 = A_1, B_2 = A_2 \setminus A_1, \ldots, B_n = A_n \setminus A_{n-1}, \ldots$ for each $n \in \mathbb{N}$ where $n \geq 2$. It is clear that $B_1, B_2, \ldots$ are all Lebesgue measurable and pairwise disjoint. We also note that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n, \text{ and } A_n = \bigcup_{k=1}^{n} B_k$$

for each $n \in \mathbb{N}$. We then have

$$\lambda(A_n) = \lambda\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} \lambda(B_k).$$

Taking limits, we see

$$\begin{aligned}\lim_{n\to\infty} \lambda(A_n) &= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(B_k) = \sum_{k=1}^{\infty} \lambda(B_k) \\ &= \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right).\end{aligned}$$

□

Proof. Suppose $\lambda(A_1) < \infty$, and $A_1 \supseteq A_2 \supseteq \ldots$ is a decreasing chain of Lebesgue measurable sets. Define $B_n = A_1 \setminus A_n$ for each $n \in \mathbb{N}$; then $B_1 \subseteq B_2 \subseteq \ldots$ where each set is Lebesgue measurable. By (iii), we see

$$\lim_{n\to\infty} \lambda(B_n) = \lambda\left(\bigcup_{n=1}^{\infty} B_n\right),$$

and so

$$\lim_{n\to\infty} (\lambda(A_1) - \lambda(A_n)) = \lambda\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \lambda(A_1) - \lambda\left(\bigcap_{n=1}^{\infty} A_n\right).$$

As $\lambda(A_1) < \infty$, this proves

$$\lim_{n\to\infty} \lambda(A_n) = \lambda\left(\bigcap_{n=1}^{\infty} A_n\right).$$

□

Remark: The condition that $\lambda(A_1) < \infty$ in statement (iv) is necessary: if we let $A_1 = (1, \infty), A_2 = (2, \infty), \ldots, A_n = (n, \infty), \ldots$ we'd have a decreasing chain of measurable sets with $\lim_{n\to\infty} \lambda(A_n) = \infty$, but $\lambda\left(\bigcap_{n=1}^{\infty} A_n\right) = \lambda(\emptyset) = 0$.

IMPORTANT EXAMPLES OF MEASURABLE SETS:

(i) Cantor set

Let $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \ldots$, and set $C = \bigcap_{n=1}^{\infty} C_n$. We note C is Lebesgue measurable, as it is a countable intersection of Lebesgue measurable sets. We also note $\lambda(C_0) = 1, \lambda(C_1) = \frac{2}{3}, \ldots, \lambda(C_n) = (\frac{2}{3})^n$ for $n \in \mathbb{N}$. Then by previous properties of the Lebesgue measure, we have

$$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

(ii) Let $E \subseteq \mathbb{R}$ such that $\lambda^*(E) = 0$. Such a set is called a *null set*. Then $E \in \mathcal{L}(\mathbb{R})$.

Proof. Take any $A \subseteq \mathbb{R}$. We note that as $A \cap E \subseteq E$, then

$$\lambda^*(A \cap E) \leq \lambda^*(E) = 0$$
$$\Rightarrow \lambda^*(A \cap E) = 0.$$

We also note that as $A \cap E^C \subseteq A$, then $\lambda^*(A \cap E^C) \leq \lambda^*(A)$. This means

$$\lambda^*(A) \geq \lambda^*(A \cap E) + \lambda^*(A \cap E^C),$$

which implies E is Lebesgue measurable. □

3.2. **Characterization of Riemann integrable functions.**

Theorem 3.11. *Let $f : [a,b] \Rightarrow \mathbb{R}$ be bounded. Then f is Riemann integrable over $[a,b]$ if and only if $\lambda(\{$discontinuities of $f\}) = 0$.*

Proof. We'll start with some preliminary observations. Let $E = \{\text{discontinuities of } f\}$. By definition, f is continuous at z if and only if there exists a $\delta > 0$ such that for all $x \in [z-\delta, z+\delta]$ we have $|f(x) - f(z)| < \epsilon$. An equivalent condition is that f is continuous at z if there exists a $\delta > 0$ such that for all $x, y \in [z-\delta, z+\delta]$ we have $|f(x) - f(y)| < \epsilon$ (**exercise!**). Then f is not continuous at z if there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist $x, y \in [z-\delta, z+\delta]$ such that $|f(x) - f(y)| \geq \epsilon$. For each $k \in \mathbb{N}$, define

$$E_k = \left\{ z \in [a,b] : \text{ for all } \delta > 0, \text{ there exist } x, y \in [z-\delta, z+\delta] \text{ where } |f(x) - f(y)| \geq \frac{1}{k} \right\}.$$

It is clear that $E = \bigcup_{k=1}^{\infty} E_k$. It can be shown (**exercise!**) that E_k is closed for each $k \in \mathbb{N}$. This means E_k is Lebesgue measurable, and therefore E is Lebesgue measurable (as a Borel set).

First, suppose that $\lambda(E) = 0$. We want to show that f is Riemann integrable over $[a,b]$. To that end, let $\epsilon > 0$ be given, and pick a cover $\{I_n\}_{n\in\mathbb{N}}$ of open intervals for E such that $\sum_{n=1} \ell(I_n) < \epsilon$. Also, pick $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. We see

$$E_k \subseteq E \subseteq \bigcup_{n\in\mathbb{N}} I_n,$$

and so $\{I_n\}_{n\in\mathbb{N}}$ is an open cover of E_k for each $k \in \mathbb{N}$. As E_k is compact (as a closed subset of $[a,b]$), there exists a finite subcover $I_1, I_2, \ldots, I_N$ of E_k. Look at $X = [a,b] \backslash (I_1 \cup \ldots \cup I_N)$; as X is closed, it is compact. For any $x \in X$, we know $x \notin E_k$ (by construction). This means there exists a $\delta_x > 0$ such that for any $z, y \in [x-\delta_x, x+\delta_x]$ we have $|f(z) - f(y)| < \frac{1}{k}$. Consider the open cover $\{(x-\delta_x, x+\delta_x)\}_{x\in X}$ of X; as X is compact, this contains a finite subcover- call it $(x_1 - \delta_{x_1}, x_1 + \delta_{x_1}), \ldots, (x_s - \delta_{x_s}, x_s + \delta_{x_s})$. Let $\mathcal{P}$ be the partition consisting of the endpoints of our intervals in the finite subcover of X, along with the endpoints of the intervals $I_1, \ldots, I_N$. Then

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) = \sum_{k=1}^{s} (\sup f|_{I_k} - \inf f|_{I_k}) \ell(I_k) + \sum_{\ell=1}^{N} (\sup f|_{I_\ell} - \inf f|_{I_\ell}) \ell(I_\ell)$$
$$\leq \epsilon(b-a) + 2\|f\|_{\sup}\epsilon = \epsilon((b-a) + 2\|f\|_{\sup}).$$

As a, b and $\|f\|_{\sup}$ are independent of ϵ, we can make the product arbitrarily small. This suggests

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon,$$

and as $\epsilon > 0$ was arbitrary this shows f is Riemann integrable on $[a, b]$.

(Sketch) Next, suppose f is Riemann integrable on $[a, b]$; to show $\lambda^*(E) = 0$, it is enough to show $\lambda^*(E_k) = 0$ for each $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and let $\epsilon > 0$ be given. By Riemann integrability of f on $[a, b]$ there exists a partition $\mathcal{P} : a = x_0 < \cdots < x_n = b$ such that

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \frac{\epsilon}{k}.$$

Let $J = \{i : (x_{i-1}, x_i) \cap E_k \neq \emptyset\}$. Then

$$\begin{aligned}\sum_{i \in J} (x_{i-1}, x_i) &\leq \sum_{i \in J} (\sup f|_{I_i} - \inf f|_{I_i}) \ell(I_i) \\ &= \mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) \leq \frac{\epsilon}{k}.\end{aligned}$$

This means $E_k \subseteq \bigcup_{i \in J} I_i \cup \{x_j : j = 0, \ldots, m\}$ and therefore

$$\lambda^*(E_k) \leq \sum_{i \in J} \lambda^*(I_i) \leq \epsilon.$$

Then as $\epsilon > 0$ was arbitrary, then $\lambda^*(E_k) < \epsilon$ for all such ϵ. This implies $\lambda^*(E_k) = 0$ for each $k \in \mathbb{N}$. As $E = \bigcup_{k=1}^{\infty} E_k$, this means $\lambda^*(E) = 0$. □

3.3. **Measurable functions.**

Definition 3.12. *A function $f : E \to \mathbb{R}$ is said to be measurable if $E \in \mathcal{L}(\mathbb{R})$ and $f^{-1}((\alpha, \infty)) = \{x \in E : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.*

Examples:

(i) If $f : E \to \mathbb{R}$ is continuous, then f is measurable; as (α, ∞) is open for all $\alpha \in \mathbb{R}$ and $f^{-1}((\alpha, \infty))$ is open, it must therefore be measurable as well.

(ii) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable subset. Define the characteristic function of E as

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$

Then χ_E is a measurable function. If we let $\alpha \in \mathbb{R}$ be given, then
- $\chi_E^{-1}((\alpha, \infty)) = \emptyset$ if $\alpha > 1$,
- $\chi_E^{-1}((\alpha, \infty)) = E$ if $0 < \alpha \leq 1$,
- $\chi_E^{-1}((\alpha, \infty)) = \mathbb{R}$ if $\alpha \leq 0$.

As $\emptyset, E, \mathbb{R}$ are all measurable, this means $\chi_E^{-1}((\alpha, \infty))$ is measurable for all $\alpha \in \mathbb{R}$.

(iii)

Definition 3.13. *Let $a_1, \ldots, a_n \in \mathbb{R}$, and let $E_1 \ldots, E_n \in \mathcal{L}(\mathbb{R})$. Define $f = \sum_{i=1}^{n} a_i \chi_{E_i}$- such a function f is called a simple function. They can be thought of as a generalization of step functions, which are linear combinations of constant intervals.*

Proposition 3.14. *Every simple function is a measurable function.*

Proof. **Exercise!** (hint: Suppose f is as above. First show that there exist $F_1, \dots, F_k \in \mathcal{L}(\mathbb{R})$ and there exist $b_1, \dots, b_k$ such that $f = \sum_{i=1}^{k} b_i \chi_{F_i}$ where $F_i \cap F_j = \emptyset$ and $b_i \neq b_j$ for all $i \neq j$). □

Proposition 3.15. *(Equivalent conditions for measurable functions)*
Suppose $E \in \mathcal{L}(\mathbb{R})$, and let $f : E \to \mathbb{R}$. Then f is a measurable function if one of the following equivalent conditions holds:

(i) $f^{-1}((\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$ *for all* $\alpha \in \mathbb{R}$;
(ii) $f^{-1}((-\infty, \alpha)) \in \mathcal{L}(\mathbb{R})$ *for all* $\alpha \in \mathbb{R}$;
(iii) $f^{-1}([\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$ *for all* $\alpha \in \mathbb{R}$;
(iv) $f^{-1}((-\infty, \alpha]) \in \mathcal{L}(\mathbb{R})$ *for all* $\alpha \in \mathbb{R}$.

Proof.
(i) ⇒ (iii)
We note

$$f^{-1}([\alpha, \infty)) = \{x \in E : f(x) \geq \alpha\} = \bigcap_{n \in \mathbb{N}} \{x \in E : f(x) > \alpha - \frac{1}{n}\}$$
$$= \bigcap_{n=1}^{\infty} f^{-1}((\alpha - \frac{1}{n}, \infty)),$$

for all $\alpha \in \mathbb{R}$. As each $f^{-1}((\alpha - \frac{1}{n}, \infty))$ is Lebesgue measurable (by assumption (i)), and as $\mathcal{L}(\mathbb{R})$ is a σ-algebra, our countable intersection of measurable sets is also Lebesgue measurable. This completes the proof of (iii).

(i) ⇒ (iv)
We note that

$$f^{-1}((-\infty, \alpha]) = \{x \in E : f(x) \leq \alpha\} = E \setminus \{x \in E : f(x) > \alpha\}$$
$$= E \setminus f^{-1}((\alpha, \infty)),$$

for each $\alpha \in \mathbb{R}$. As $f^{-1}((\alpha, \infty))$ is Lebesgue measurable, it's complement is measurable as well- so $f^{-1}((-\infty, \alpha])$ is Lebesgue measurable.

For the rest: **exercise!** □

Proposition 3.16. *Suppose E is measurable and $f : E \to \mathbb{R}$. Then f is measurable if and only if $f^{-1}(G) \in \mathcal{L}(\mathbb{R})$ for every open set $G \subseteq \mathbb{R}$.*

Proof.
(⇐) Trivial.
(⇒) Suppose f is measurable, and let $G \subseteq \mathbb{R}$ be an arbitrary open set. As $G \subseteq \mathbb{R}$ is open, we can write $G = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}_{n \in \mathbb{N}}$ are pairwise disjoint open intervals. We have

$$f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n).$$

If $I_n = (a_n, \infty)$ or $I_n = (-\infty, b_n)$ for some $a_n, b_n \in \mathbb{R}$, then $f^{-1}(I_n) \in \mathcal{L}(\mathbb{R})$ by our previous proposition. If $I_n = (a_n, b_n)$, then

$$f^{-1}(I_n) = f^{-1}((a_n, \infty) \cap (-\infty, b_n)) = f^{-1}((a_n, \infty)) \cap f^{-1}((-\infty, b_n))$$

is Lebesgue measurable as well. Therefore, $f^{-1}(G)$ is the countable union of Lebesgue measurable sets, and so $f^{-1}(G) \in \mathcal{L}(\mathbb{R})$. □

Corollary 3.17. *For function $f : E \to \mathbb{R}$, then f is Lebesgue measurable $\iff$ $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$.*

Remark: Suppose $f : E \to \mathbb{R}$ is Lebesgue measurable. Extend f to all of $\mathbb{R}$ as follows:

$$\tilde{f} : \mathbb{R} \to \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & x \in E, \\ 0, & x \in \mathbb{R} \setminus E \end{cases}$$

Then $\tilde{f}$ is a measurable function $\iff$ for all $\alpha \in \mathbb{R}$, $\tilde{f}^{-1}((\alpha,\infty)) \in \mathcal{L}(\mathbb{R})$; this is because if $\alpha > 0$, then $f^{-1}((\alpha,\infty)) = \tilde{f}^{-1}((\alpha,\infty))$; if $\alpha \leq 0$, then $\tilde{f}^{-1}((\alpha,\infty)) = (\mathbb{R} \setminus E) \cup f^{-1}((\alpha,\infty))$. For the rest of the proof- **exercise!**

This allows us to talk mostly about measurable $f : \mathbb{R} \to \mathbb{R}$ from here on out.

Theorem 3.18. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable functions, and let $c \in \mathbb{R}$. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is continuous. Then*

(i) *$cf : \mathbb{R} \to \mathbb{R}$ is measurable;*
(ii) *$f + g : \mathbb{R} \to \mathbb{R}$ is measurable;*
(iii) *$fg : \mathbb{R} \to \mathbb{R}$ is measurable;*
(iv) *$\phi \circ f : \mathbb{R} \to \mathbb{R}$ is measurable;*
(v) *$f^+, f^-, |f|$ are measurable, where*
- *$f^+ : \mathbb{R} \to \mathbb{R}$ with $f^+(x) = \max\{f(x), 0\}$*
- *$f^- : \mathbb{R} \to \mathbb{R}$ with $f^-(x) = \max\{-f(x), 0\}$*
- *$|f| : \mathbb{R} \to \mathbb{R}$ with $|f|(x) = |f(x)|$*

Note: $f = f^+ - f^-$, and $|f| = f^+ + f^-$. We also note that by the previous theorem, measurable functions form an algebra (along with having some extra nice properties).

Proof.
(i) Easy- essentially just by writing it out.

(ii) Suppose f, g are measurable functions; we want to show that $f + g$ is measurable- i.e., for all $\alpha \in \mathbb{R}$, the set $(f+g)^{-1}((\alpha,\infty))$ is a measurable set. To that end, let $\alpha \in \mathbb{R}$ be arbitrary; we note that

$$\begin{aligned} x \in (f+g)^{-1}((\alpha,\infty)) &\iff (f+g)(x) > \alpha \iff f(x) > \alpha - g(x) \\ &\iff \text{there exists an } r \in \mathbb{Q} \text{ such that } f(x) > r > \alpha - g(x) \\ &\iff \text{there exists an } r \in \mathbb{Q} \text{ such that } f(x) > r \text{ and } g(x) > \alpha - r \\ &\iff \text{there exists an } r \in \mathbb{Q} \text{ such that } x \in f^{-1}((r,\infty)) \text{ and } x \in g^{-1}((\alpha - r,\infty)). \end{aligned}$$

Then

$$(f+g)^{-1}((\alpha,\infty)) = \bigcup_{r \in \mathbb{Q}} f^{-1}((r,\infty)) \cap g^{-1}((\alpha - r,\infty)).$$

As f is measurable, $f^{-1}((r,\infty))$ is measurable for each such $r \in \mathbb{Q}$; similarly, as g is measurable then $g^{-1}((\alpha - r,\infty))$ is measurable for each such $r \in \mathbb{Q}$. Then $(f+g)^{-1}((\alpha,\infty))$ is a countable intersection of measurable sets, and therefore is measurable. As $\alpha \in \mathbb{R}$ was arbitrary, this holds for all $\alpha \in \mathbb{R}$- this proves $f + g$ is measurable.

(iii) Suppose f, g are measurable- we want to show fg is measurable; to do so, we'll show that f^2 is measurable first. Let $\alpha \in \mathbb{R}$- we want to prove $(f^2)^{-1}((\alpha,\infty))$ is measurable. We have two cases:

- If $\alpha \geq 0$, then
$$x \in (f^2)^{-1}((\alpha,\infty)) \iff f^2(x) > \alpha \iff f(x) > \sqrt{\alpha} \text{ or } f(x) < -\sqrt{\alpha}$$
$$\iff x \in f^{-1}((\sqrt{\alpha},\infty)) \cup f^{-1}((-\infty,-\sqrt{\alpha})).$$
So $(f^2)^{-1}((\alpha,\infty)) = f^{-1}((\sqrt{\alpha},\infty)) \cup f^{-1}((-\infty,-\sqrt{\alpha}))$; as f is measurable, both $f^{-1}((\sqrt{\alpha},\infty))$ and $f^{-1}((-\infty,-\sqrt{\alpha}))$ are measurable sets. Then as $(f^2)^{-1}((\alpha,\infty))$ is a countable union of measurable sets, in this case f^2 is measurable.
- If $\alpha < 0$, then $(f^2)^{-1}((\alpha,\infty)) = \mathbb{R}$, which is certainly measurable. Therefore, $(f^2)^{-1}((\alpha,\infty))$ is measurable.

So f^2 is measurable for all $\alpha \in \mathbb{R}$. Now, as f, g are measurable, by (ii) we know $f + g$ is measurable as well. Then
$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$$
is measurable, by (i) and (ii).

(iv) Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, and $f : \mathbb{R} \to \mathbb{R}$ is measurable; let $\alpha \in \mathbb{R}$ be given. We want to show that $(\phi \circ f)^{-1}((\alpha,\infty))$ is a measurable set. We note that
$$x \in (\phi \circ f)^{-1}((\alpha,\infty)) \iff (\phi \circ f)(x) > \alpha \iff \phi(f(x)) > \alpha$$
$$\iff f(x) \in \phi^{-1}((\alpha,\infty)) \iff x \in f^{-1}(\phi^{-1}((\alpha,\infty))).$$
So $(\phi \circ f)^{-1}((\alpha,\infty)) = f^{-1}(\phi^{-1}((\alpha,\infty)))$. As (α,∞) is open and ϕ is continuous, then $\phi^{-1}((\alpha,\infty))$ is open- hence Borel, hence measurable. Then as f is measurable, $f^{-1}(\phi^{1}((\alpha,\infty)))$ is measurable. As $\alpha \in \mathbb{R}$ was given, this holds for all $\alpha \in \mathbb{R}$, and so $\phi \circ f$ is measurable.

(v) Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = |x|$. Then $|f| = \phi \circ f$; as ϕ is continuous, and f is measurable, by (iv) we know $|f|$ is measurable. Furthermore, as
$$f^+ = \frac{f + |f|}{2}, \quad f^- = \frac{|f| - f}{2}$$
are linear combinations of measurable functions, then by (i)-(iv) we see f^+ and f^- are measurable as well. This completes the proof. □

Remark: If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $f(x) \neq 0$ for all $x \in \mathbb{R}$, then $1/f$ is measurable.

Definition 3.19. *A function $f : \mathbb{R} \to [-\infty,\infty]$ is called an extended real-valued function. We say $f : \mathbb{R} \to [-\infty,\infty]$ is a measurable function if $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$ and $f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}) \in \mathcal{L}(\mathbb{R})$.*

Proposition 3.20. *Let $f_n : \mathbb{R} \to [-\infty,\infty]$ be measurable for all $n \in \mathbb{N}$. Then the following functions are measurable:*

(i) $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$ *for all* $x \in \mathbb{R}$

(ii) $g(x) = \inf_{n\in\mathbb{N}} f_n(x)$ *for all* $x \in \mathbb{R}$

(iii) $g(x) = \lim_{n\to\infty} \sup_{k\geq n} f_k(x)$

(iv) $g(x) = \lim_{n\to\infty} \inf_{k\geq n} f_k(x)$

(v) $g(x) = \lim_{n\to\infty} f_n(x)$

Proof. (i) Let $\alpha \in \mathbb{R}$, and $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$ for all $x \in \mathbb{R}$. We note that

$$\begin{aligned} x \in g^{-1}((\alpha,\infty)) &\iff g(x) > \alpha \iff \sup_{n\to\infty} f_n(x) > \alpha \\ &\iff \text{ there exists an } n \in \mathbb{N} \text{ such that } f_n(x) > \alpha \\ &\iff \text{ there exists an } n \in \mathbb{N} \text{ such that } x \in f^{-1}((\alpha,\infty)) \\ &\iff x \in \bigcup_{n\in\mathbb{N}} f_n^{-1}((\alpha,\infty)). \end{aligned}$$

As f_n is measurable for each $n \in \mathbb{N}$, $g^{-1}((\alpha,\infty))$ is a countable union of measurable sets- hence it is measurable. Next, we want to show that $g^{-1}(\{\infty\})$ is measurable. We note that

$$\begin{aligned} x \in g^{-1}(\{\infty\}) &\iff g(x) = \infty \iff \sup_{n\in\mathbb{N}} f_n(x) = \infty \\ &\iff \text{ for all } M \in \mathbb{N}, \text{ there exists an } n \in \mathbb{N} \text{ such that } f_n(x) > M \\ &\iff \text{ for all } M \in \mathbb{N}, \text{ there exists an } n \in \mathbb{N} \text{ such that } x \in f_n^{-1}((M,\infty)) \\ &\iff x \in \bigcap_{M\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} f_n^{-1}((M,\infty)). \end{aligned}$$

Again, as f is measurable, $g^{-1}(\{\infty\})$ is a countable intersection of measurable sets- hence, it is measurable. This shows $g^{-1}(\{\infty\})$ is measurable; then as $g^{-1}(\{-\infty\})$ can never occur by definition, we conclude that $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is measurable.

(ii) **Exercise!** Very similar to (i).

(iii) Let $g(x) = \lim_{n\to\infty} \sup f_n(x)$. For each $k \in \mathbb{N}$, define $h_k(x) = \sup_{n\geq k} f_n(x)$. Then clearly $g(x) = \inf_{k\in\mathbb{N}} h_k(x)$. We note that h_k is measurable for each $k \in \mathbb{N}$ by (i), and so by (ii) g is measurable.

(iv) **Exercise!** Very similar to (iii).

(v) Suppose f_n is measurable; in addition, suppose for all $x \in \mathbb{R}$ that $\lim_{n\to\infty} f_n(x)$ exists. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}$. As the limit exists, the limit is equal to the lim sup and the lim inf- so $f(x) = \lim_{n\to\infty} \sup f_n(x)$, and as the latter is measurable (by (iii)) we see $\lim_{n\to\infty} f_n(x)$ is measurable. □

The following lemma can be fairly useful in many different contexts:

Lemma 3.21 (Simple Approximation Lemma)**.** *Let f be a measurable real-valued function on E. Assume f is bounded on E. Then for each $\epsilon > 0$ there exist simple functions ϕ_ϵ and ψ_ϵ on E such that*

$$\phi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon \text{ on } E.$$

Proof. Let $\epsilon > 0$ be given. Let (c,d) be an open, bounded interval which contains $f(E)$ (we know such an interval exists, as f is a bounded function on E); furthermore, let

$$c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$$

be a partition of $[c,d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$. Define

$$I_k = [y_{k-1}, y_k) \text{ and } E_k = f^{-1}(I_k)$$

for $1 \leq k \leq n$. As I_k is an interval for each value of k, and f is measurable, then E_k is measurable for $1 \leq k \leq n$. Define simple functions $\phi_\epsilon, \psi_\epsilon$ on E where

$$\phi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k},$$

and

$$\psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

Let $x \in E$. As $f(E) \subseteq (c, d)$, there exists a unique k with $1 \leq k \leq n$ such that $y_{k-1} \leq f(x) < y_k$; therefore,

$$\phi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x).$$

We also note that as our partition of $[c, d]$ was chosen so that $y_k - y_{k-1} < \epsilon$, then

$$\psi_\epsilon(x) - \phi_\epsilon(x) = y_k - y_{k-1} < \epsilon.$$

As $\epsilon > 0$, $x \in E$ were arbitrary, we conclude the proof. □

Theorem 3.22 (Characterization of measurable positive-valued functions)**.** *The function $f : \mathbb{R} \to [0, \infty)$ is measurable $\iff$ there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of simple functions such that $\phi_n \leq \phi_{n+1}$ pointwise for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \phi_n = f$.*

Note: Kind of like an analogue to the Stone-Weierstrass Theorem but for measurable functions.

Proof.
($\Leftarrow$) Suppose $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions where $\phi_n \leq \phi_{n+1}$ and $\phi_n \to f$ pointwise. As each simple function ϕ_n is measurable for $n \in \mathbb{N}$, by (v) of the previous proposition we see that $\lim\limits_{n\to\infty} \phi_n = f$ is measurable as well.

($\Rightarrow$) Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function, and fix $n \in \mathbb{N}$ for the moment. Define $E_n := f^{-1}([0, n))$ and $E_{n,k} := f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))$ where $k = 0, \ldots, 2^n n - 1$. We see $E_{n,k}$ is measurable, as f is measurable. We also note that

$$E_n = \bigcup_{k=0}^{2^n n-1} E_{n,k}.$$

For each $n \in \mathbb{N}$, set

$$\phi_n(x) = \begin{cases} \frac{k}{2^n}, & x \in E_{n,k} \\ n, & x \notin E_n \end{cases}.$$

We see

$$\phi_n = \sum_{k=0}^{2^n n-1} \frac{k}{2^n} \chi_{E_{n,k}} + n\chi_{\mathbb{R}\setminus E_n},$$

and so ϕ_n is a simple function. We also note (**exercise!**) that

(i) $\phi_n(x) \leq \phi_{n+1}(x) \leq f(x)$ for all $x \in R$
(ii) $\phi_n \to f$ pointwise

These together finish the other implication. □

Remark:

(i) Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is (by definition) the smallest σ-algebra containing all open subsets of $\mathbb{R}$; i.e., if $\mathcal{F}$ is another σ-algebra containing all open subsets of $\mathbb{R}$, then $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$. An important thing to note is that to construct Borel sets, *we need transfinite induction.* This means: *not all Borel sets can be obtained through countable unions or intersections of open sets.*

(ii) Let $E \in \mathcal{L}(\mathbb{R})$ and $\lambda(E) < \infty$. Then there exists a set $O \in \mathcal{B}(\mathbb{R})$ such that $E \subseteq O$ and $\lambda(O \setminus E) = 0$. Therefore, $E = O \setminus (O \setminus E)$; so we can construct *any* Lebesgue measurable set out of Borel sets and sets with measure 0. Note that if $E \in \mathcal{L}(\mathbb{R})$ and $\lambda(E) = \infty$, we can define $E_n = E \cap [n, n+1)$ for all $n \in \mathbb{Z}$ where E_n is measurable with $\lambda(E_n) \leq 1$. Then there exists an open set O_n such that $E_n \subseteq O_n$ with $\lambda(O_n \setminus E_n) = 0$. Let $O = \bigcup\limits_{n\in\mathbb{Z}} O_n \in \mathcal{B}(\mathbb{R})$. Then $E \subseteq O$, and

$$\lambda(O \setminus E) \leq \lambda\Bigg(\bigcup_{n\in\mathbb{Z}} (O_n \setminus E_n)\Bigg) \leq \sum_{n\in\mathbb{Z}} \lambda(O_n \setminus E_n) = 0.$$

Question: Where in the explanation of (ii) do we use the measurability of E? (Tentative answer: I think it's to ensure that E_n is measurable for each $n \in \mathbb{Z}$, as it may not be if E isn't).

(iii) The cardinality of $\mathcal{L}(\mathbb{R})$ is strictly larger than the cardinality of $\mathcal{B}(\mathbb{R})$.

4. Lebesgue integration

4.1. The Lebesgue integral for simple functions and positive valued functions.

Definition 4.1 (Lebesgue integral of simple functions over E)**.** *Suppose* $\phi = \sum\limits_{k=1}^{n} a_k \chi_{E_k}$ *where* $a_k \in \mathbb{R}$, $E_k \in \mathcal{L}(\mathbb{R})$ *for* $1 \leq k \leq n$. *We define*

$$\int\limits_E \phi \, d\lambda := \sum_{k=1}^{n} a_k \lambda(E_k \cap E).$$

Convention: $0 \cdot \infty = 0$.

Example: Let $f = \chi_{\mathbb{Q}\cap[0,1]}$; then

$$\int\limits_E f \, d\lambda = \lambda(E \cap \mathbb{Q} \cap [0,1]) \leq \lambda(\mathbb{Q}) = 0,$$

as $\mathbb{Q}$ is countable.

Definition 4.2. *Let* ϕ *be a simple function. A representation* $\phi = \sum\limits_{k=1}^{n} a_k \chi_{E_k}$ *is called a canonical representation if the* E_k*'s are all pairwise disjoint, and* $a_i \neq a_j$ *for* $i \neq j$.

Remark: Let ϕ be a simple function with canonical representation. Then $\operatorname{rng}(\phi) = \{a_1, a_2, \ldots, a_n\}$; moreover, $E_k = \phi^{-1}(\{a_k\})$ for each k. This means the canonical representation is unique.

Example: Let $E_1 = (0,2), E_2 = [1,3)$ with $\phi = \chi_{E_1} + 4\chi_{E_2}$. The canonical representation of ϕ is given by

$$F_1 = (0,1), \quad F_2 = [1,2), \quad F_3 = [2,3)$$
$$\phi = \chi_{F_1} + 5\chi_{F_2} + 4\chi_{F_3}$$

Lemma 4.3. *Let ϕ be a simple function with canonical representation $\sum_{k=1}^{n} a_k \chi_{E_k}$. Then*

$$\int_E \phi \, d\lambda = \sum_{k=1}^{n} a_k \lambda(E_k \cap E).$$

The integral defined in (I), using non-canonical forms, is well-defined.

Proof. We skip it! ☐

Definition 4.4 (Lebesgue integral for a positive valued function)**.** *Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function, and let $E \in \mathcal{L}(\mathbb{R})$. Define*

$$\int_E f \, d\lambda = \sup_{\substack{0 \le \phi \le f, \\ \phi \text{ simple}}} \int_E \phi \, d\lambda.$$

Remark: We cannot always cover $f : \mathbb{R} \to [0, \infty)$ by simple functions; that is, sometimes there does not exist a simple function ϕ such that $f \le \phi$ pointwise. This is why we use ϕ such that $0 \le \phi \le f$, as these always exist (by a previous theorem).

Example: Let $f(x) = \sum_{n=1}^{\infty} n \chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}}]}$.

Proposition 4.5. *Let $A, B \in \mathcal{L}(\mathbb{R})$, and $f, g : \mathbb{R} \to [0, \infty)$ be measurable. Then*

(i) If $f \le g$ on A, then $\int_A f \, d\lambda \le \int_A g \, d\lambda$

(ii) If $\emptyset \ne B \subseteq A$ is measurable, then $\int_B f \, d\lambda = \int_A f \chi_B \, d\lambda$

(iii) If ϕ is a simple function, then Definition 1.1 and 1.4 coincide.

Proof. Exercise! ☐

4.2. **Integration and limits of functions.**

Theorem 4.6 (Monotone Convergence Theorem)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a collection of measurable functions, where $f_n : \mathbb{R} \to [0, \infty]$ for all $n \in \mathbb{N}$. Suppose $f_1 \le f_2 \le \cdots$ pointwise, and let $f = \lim_{n\to\infty} f_n(x)$ pointwise. Then for all $A \in \mathcal{L}(\mathbb{R})$, we have*

$$\int_A f \, d\lambda = \lim_{n\to\infty} \int_A f_n \, d\lambda$$

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be as described above. By a previous proposition, we know

$$\int_A f_1 \, d\lambda \le \int_A f_2 \, d\lambda \le \cdots$$

for all $A \in \mathcal{L}(\mathbb{R})$. We have

$$\lim_{n\to\infty} \int_A f_n \, d\lambda = \sup_{n \in \mathbb{N}} \int_A f_n \, d\lambda.$$

We also know that f is measurable, as it is the pointwise limit of a sequence of measurable functions, and $f_n \le f$ pointwise for all $n \in \mathbb{N}$. Then

$$\int_A f_n \, d\lambda \le \int_A f \, d\lambda$$

for all $n \in \mathbb{N}$, and so

$$\lim_{n\to\infty} \int\limits_A f_n \, d\lambda \leq \int\limits_A f \, d\lambda.$$

Next, let $0 \leq \phi \leq f$ be a simple function, and let $0 < \delta < 1$. Define

$$A_n = \{x \in A : f_n(x) \geq \delta\phi(x)\}.$$

Then we have

(i) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$
(ii) $\bigcup\limits_{n\in\mathbb{N}} A_n = A$,

as $f_n(x) \to f(x)$ as $n \to \infty$ and $\delta\phi(x) < \phi(x) \leq f(x)$. Suppose $\phi = \sum\limits_{k=1}^{N} a_k \chi_{E_k}$; then

$$\int\limits_A f_n \, d\lambda \geq \int\limits_A f_n \chi_{A_n} \, d\lambda = \int\limits_{A_n} f_n \, d\lambda \geq \int\limits_{A_n} \delta\phi$$
$$= \sum_{k=1}^{N} \delta a_k \lambda(E_k \cap A_n).$$

So $\int\limits_A f_n \, d\lambda \geq \sum\limits_{k=1}^{N} \delta a_k \lambda(E_k \cap A_n)$. If we take the limit on the left, we see

$$\lim_{n\to\infty} \int\limits_A f_n \, d\lambda \geq \sum_{k=1}^{N} \delta a_k \left(\lim_{n\to\infty} \lambda(E_k \cap A_n) \right) = \sum_{k=1}^{N} \delta a_k \lambda(E_k \cap A).$$

(This holds as a result of the upward continuity of the Lebesgue measure.) We note the latter sum is equal to

$$\sum_{k=1}^{N} \delta a_k \lambda(E_k \cap A) = \delta \int\limits_A \phi \, d\lambda;$$

so

$$\lim_{n\to\infty} \int\limits_A f_n \, d\lambda \geq \delta \int\limits_A \phi \, d\lambda.$$

If we let $\delta \to 1$, we get the limit of the integrals being greater than or equal to the Lebesgue integral over A of ϕ. As we can approximate f with increasing simple functions ϕ, this proves

$$\lim_{n\to\infty} \int\limits_A f_n \, d\lambda \geq \int\limits_A f \, d\lambda,$$

which completes the proof. □

Application: Let $f : \mathbb{R} \to [0, \infty)$ be measurable. Then there exists a sequence of simple functions $\{\phi_n\}_{n\in\mathbb{N}}$ such that $\phi_1 \leq \phi_2 \leq \ldots$ pointwise and $\phi_n \to f$ pointwise, as $n \to \infty$. Then by the Monotone Convergence Theorem, we see

$$\int\limits_A f \, d\lambda = \lim_{n\to\infty} \int\limits_A \phi_n \, d\lambda.$$

Note: The above application gives an equivalent definition for the Lebesgue integral of positive valued functions- this is oftentimes much more useful than the supremum definition.

Note: If we have a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of simple functions where $\phi_1 \leq \phi_2 \leq \ldots$ pointwise and $\phi_n \to f$ pointwise, we denote it by $\phi_n \nearrow f$.

Application of MCT: Let $\emptyset \neq A \in \mathcal{L}(\mathbb{R})$, $c \geq 0$; also, let $f, g : \mathbb{R} \to [0, \infty]$ be measurable. Then

(i) $\int_A cf \, d\lambda = c \int_A f \, d\lambda$;

(ii) $\int_A f + g \, d\lambda = \int_A f \, d\lambda + \int_A g \, d\lambda$;

(iii) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions from $\mathbb{R}$ into $[0, \infty]$, then

$$\int_A \sum_{n=1}^{\infty} f_n \, d\lambda = \sum_{n=1}^{\infty} \int_A f_n \, d\lambda.$$

Proof. We note that (i) and (ii) are easy for simple functions (they follow from the definition of the Lebesgue integral of simple functions). Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ be (pointwise) increasing sequences of simple functions such that $\phi_n \nearrow f$ and $\psi_n \nearrow g$ pointwise. Then $\phi_n + \psi_n \nearrow f + g$ pointwise, where $\{\phi_n + \psi_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions. By the Monotone Convergence Theorem, we know

$$\begin{aligned}\int_A f + g \, d\lambda &= \lim_{n\to\infty} \int_A \phi_n + \psi_n \, d\lambda = \lim_{n\to\infty} \left(\int_A \phi_n \, d\lambda + \int_A \psi_n \, d\lambda \right) \\ &= \lim_{n\to\infty} \int_A \phi_n \, d\lambda + \lim_{n\to\infty} \int_A \psi_n \, d\lambda = \int_A f \, d\lambda + \int_A g \, d\lambda.\end{aligned}$$

Similarly, $c\phi_n \nearrow cf$, and so by the Monotone Convergence Theorem we have

$$\begin{aligned}\int_A cf \, d\lambda &= \lim_{n\to\infty} \int_A c\phi_n \, d\lambda \\ &= \lim_{n\to\infty} c \int_A \phi_n \, d\lambda = c \int_A f \, d\lambda.\end{aligned}$$

For (iii), for all $n \in \mathbb{N}$ define $g_n : \mathbb{R} \to [0, \infty]$ such that $g_n(x) = \sum_{k=1}^{n} f_k(x)$. Then $g_n \nearrow \sum_{k=1}^{\infty} f_k$ (pointwise). By the Monotone Convergence Theorem, we see

$$\lim_{n\to\infty} \int_A g_n \, d\lambda = \int_A \sum_{k=1}^{\infty} f_k \, d\lambda.$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_A f_k \, d\lambda = \int_A \sum_{k=1}^{\infty} f_k \, d\lambda \Rightarrow \sum_{k=1}^{\infty} \int_A f_k \, d\lambda = \int_A \sum_{k=1}^{\infty} f_k \, d\lambda.$$

□

Lemma 4.7 (Fatou's Lemma)**.** *Let $f_n : \mathbb{R} \to [0, \infty]$ be measurable for each $n \in \mathbb{N}$, and $A \in \mathcal{L}(\mathbb{R})$. Then*

$$\int_A \liminf_{n\to\infty} f_n \, d\lambda \leq \liminf_{n\to\infty} \left(\int_A f_n \, d\lambda \right).$$

Proof. For all $n \in \mathbb{N}$, define $g_n = \inf_{k \geq n} f_k$. Note that $0 \leq g_1 \leq g_2 \leq \dots$ and $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$. By the Monotone Convergence Theorem, we see

$$\textbf{(*)} \quad \lim_{n\to\infty} \int_A g_n \, d\lambda = \int_A \liminf_{n\to\infty} f_n \, d\lambda.$$

So

$$\int_A g_n \, d\lambda \leq \int_A f_k \, d\lambda$$

for all $k \geq n$, meaning

$$\textbf{(**)} \quad \int_A g_n \, d\lambda \leq \inf_{k\geq n} \int_A f_k \, d\lambda \leq \lim_{n\to\infty} \left(\inf_{k\geq n} \int_A f_k \, d\lambda \right) = \liminf_{n\to\infty} \int_A f_n \, d\lambda.$$

Then by **(*)** and **(**)**, we are done. □

Example: Let $f_n : \mathbb{R} \to [0, \infty]$ where

$$f_n(x) = \begin{cases} n, & x \in (0, \frac{1}{n}), \\ 0, & x \notin (0, \frac{1}{n}). \end{cases}$$

Then $f_n \to f$ pointwise, where $f \equiv 0$ everywhere. For all $n \in \mathbb{N}$, we have $\int_{\mathbb{R}} f_n \, d\lambda = 1$, and $\int_{\mathbb{R}} f \, d\lambda = 0$. This matches the results of Fatou's Lemma.

4.3. **The general Lebesgue integral. Recall:** Let $f : \mathbb{R} \to [-\infty, \infty]$. We recall $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. We know that f^+, f^- are measurable, along with $|f| = f^+ + f^-$.

Question: If $|f|$ is measurable, does f need to be measurable?

Answer: No- note that if E is not measurable, then χ_E is a non-measurable function. If we change χ_E to

$$f(x) = \begin{cases} 1, & x \in E, \\ -1, & x \in E^C \end{cases}$$

then f is not measurable as $f^{-1}((0, \infty)) = E \notin \mathcal{L}(\mathbb{R})$. However, $|f| = 1$, which is certainly measurable.

Definition 4.8. *We say $f : \mathbb{R} \to [-\infty, \infty]$ is Lebesgue integrable if*

(i) f is measurable,
(ii) $\int_{\mathbb{R}} f^+ \, d\lambda < \infty$, and $\int_{\mathbb{R}} f^- \, d\lambda < \infty$.

Then $\int f \, d\lambda = \int f^+ \, d\lambda - \int f^- \, d\lambda$. Moreover, if $A \in \mathcal{L}(\mathbb{R})$ where $A \neq \emptyset$ we say f is integrable over A if $f\chi_A$ is integrable.

Definition 4.9. *Let $f : \mathbb{R} \to \mathbb{C}$. We say f is integrable if both the real and imaginary parts of f are Lebesgue integrable. We define*

$$\int f \, d\lambda = \int Ref \, d\lambda + i \int Imf \, d\lambda.$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \frac{1}{x^2}\chi_{[1,\infty)}$. We see f is Lebesgue integrable, with $\int f(x) \, d\lambda = 1$.

Remark:

(i) If $f : \mathbb{R} \to [-\infty, \infty]$ is integrable, then $\lambda(f^{-1}(\{-\infty, +\infty\})) = 0$.

Proof. We prove $\lambda(f^{-1}(\{+\infty\})) = 0$; the other case is similar. As f is integrable, then f^+, f^- are measurable with $\int f^+ \, d\lambda < \infty$, and $\int f^- \, d\lambda < \infty$. Let $E = f^{-1}(\{+\infty\})$. Then $n\chi_E \leq f^+$ for all $n \in \mathbb{N}$, and therefore

$$\int n\chi_E \, d\lambda \leq \int f^+ \, d\lambda$$

for all $n \in \mathbb{N}$. As $n \to \infty$, we see $n\chi_E \to \infty$ if χ_E is positive. However, as $\int n\chi_E \, d\lambda$ is bounded above, this forces

$$n\lambda(E) \leq \int f^+ \, d\lambda < \infty.$$

Therefore, $\lambda(E) = 0$. □

(ii)

Definition 4.10. *Let $f, g : \mathbb{R} \to [-\infty, \infty]$. We say $f = g$ almost everywhere (a.e.) if*

$$\lambda(\{x \in \mathbb{R} : f(x) \neq g(x)\}) = 0.$$

Fact: Suppose $f, g : \mathbb{R} \to [\infty, \infty]$, where f is measurable and $f = g$ almost everywhere. Then g is measurable.

Proof. Let $N = \{x \in \mathbb{R} : f(x) \neq g(x)\}$. Then $\lambda(N) = 0$. Take $A \in \mathcal{B}(\mathbb{R})$- then

$$g^{-1}(A) = \left(g^{-1}(A) \cap N\right) \cup \left(g^{-1}(A) \cap N^C\right).$$

As $g^{-1}(A) \cap N \subseteq N$, and $\lambda(N) = 0$, this means $g^{-1}(A) \cap N$ is measurable. Furthermore, as $g^{-1}(A) \cap N^C = f^{-1}(A) \cap N^C$ and f is measurable, then $f^{-1}(A) \cap N^C$ is measurable. This shows $g^{-1}(A)$ is measurable, and so g is a measurable function. □

Fact: Suppose $f, g : \mathbb{R} \to [-\infty, \infty]$, where f is integrable and $f = g$ a.e. Then g is integrable; moreover, $\int f \, d\lambda = \int g \, d\lambda$.

Proof. **Exercise!!** □

(iii) Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable. Define $g : \mathbb{R} \to (-\infty, \infty)$ where

$$g(x) = \begin{cases} f(x), & f(x) \neq \pm\infty, \\ 0, & \text{otherwise} . \end{cases}$$

Then $f = g$ a.e. and so g is integrable as well.

Definition 4.11 (Convergence almost everywhere)**.** *Let $f_n, f : \mathbb{R} \to \mathbb{R}$ and suppose $\{f_n\}_{n \in \mathbb{N}}$ are measurable. We say $f_n \to f$ a.e. (or $\lim_{n\to\infty} f_n = f$ a.e.) if there exists an $N \subseteq \mathbb{R}$ such that*

(i) $\lambda(N) = 0$,

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in N^C$.

<u>Exercise:</u> Show that if $f_n \to f$ a.e. and each f_n is measurable, then f is measurable as well.

Proposition 4.12 (Properties of Lebesgue integration)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be integrable and $c \in \mathbb{R}$. Then*

(i) cf *is integrable, with* $\int cf \, d\lambda = c \int f \, d\lambda$
(ii) $f + g$ *is integrable, with* $\int f + g \, d\lambda = \int f \, d\lambda + \int g \, d\lambda$
(iii) $|f|$ *is integrable, and* $\left| \int f \, d\lambda \right| \leq \int |f| \, d\lambda$

Proof.
(i) Exercise!

(ii) As f, g are measurable, then they are both measurable. Therefore, $f + g$ is measurable. We see

$$(f+g)^+ \leq f^+ + g^+ \Rightarrow \int (f+g)^+ \, d\lambda \leq \int f^+ \, d\lambda + \int g^+ \, d\lambda < \infty.$$

Similarly, we have

$$(f+g)^- \leq f^- + g^- \Rightarrow \int (f+g)^- \, d\lambda < \infty.$$

So $f + g$ is integrable. We note

$$\begin{aligned} (f+g)^+ - (f+g)^- &= f + g = (f^+ - f^-) + (g^+ - g^-) \\ \Rightarrow (f+g)^+ + f^- + g^- &= f^+ + g^+ + (f+g)^-. \end{aligned}$$

From a previous proposition, this gives

$$\int (f+g)^+ \, d\lambda + \int f^- \, d\lambda + \int g^- \, d\lambda = \int f^+ \, d\lambda + \int g^+ \, d\lambda + \int (f+g)^- \, d\lambda,$$

and so

$$\int (f+g)^+ \, d\lambda - \int (f+g)^- \, d\lambda = \int f^+ \, d\lambda - \int f^- \, d\lambda + \int g^+ \, d\lambda - \int g^- \, d\lambda.$$

Therefore, $\int (f+g) \, d\lambda = \int f \, d\lambda + \int g \, d\lambda$.

(iii) $|f|$ is clearly measurable, as f is measurable. Also, $\int |f| \, d\lambda = \int f^+ \, d\lambda + \int f^- \, d\lambda < \infty$, as f is integrable. So $|f|$ is integrable. We also see

$$\left| \int f \, d\lambda \right| = \left| \int f^+ \, d\lambda - \int f^- \, d\lambda \right| \leq \int f^+ \, d\lambda + \int f^- \, d\lambda = \int |f| \, d\lambda.$$

□

Corollary 4.13. *$|f|$ is Lebesgue integrable and f is measurable $\iff$ f is integrable.*

Recall: Fatou's Lemma (see 4.2)

Note: While we use $\mathbb{R}$ in the following theorem, it should hold on any subset $E \subseteq \mathbb{R}$ as well.

Theorem 4.14 (Lebesgue Dominated Convergence Theorem)**.** *Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable; let $g : \mathbb{R} \to [0, \infty)$ be integrable. Suppose g dominates each f_n in the sense that $|f_n| \leq g$ pointwise. Then if $f_n \to f$ as $n \to \infty$ pointwise a.e., we have*

(i) f *is measurable*
(ii) $\lim_{n \to \infty} \int f_n \, d\lambda = \int f \, d\lambda$

Remark:

- $f_n \to f$ a.e. on E $\iff$ $f_n \chi_E \to f \chi_E$ a.e.
- f_n is measurable on E $\iff$ $f_n \chi_E$ is measurable
- f_n is integrable on E $\iff$ $f_n \chi_E$ is integrable

Proof. Assume $E = \mathbb{R}$ (by the comments above, for any $E \subset \mathbb{R}$ just multiply by the characteristic function over E to get the proof). Let

$$N = \bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R} : |f_n(x)| > g(x)\} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \neq f(x), \text{ or DNE}\}.$$

We know $\lambda(N) = 0$. On N^C, we have $|f_n| \le g$ pointwise, and so $|f| \le g$. We also see $\int\limits_N f_n \, d\lambda = 0$ and $\int\limits_N f \, d\lambda = 0$, as $\int\limits_N f_n \, d\lambda = \int f_n \chi_N \, d\lambda = 0$. Furthermore, $\int\limits_N g \, d\lambda = 0$. Note that

$$\int\limits_{\mathbb{R}} fn \, d\lambda = \int f_n\chi_N + f_n(\chi_{N^C}) \, d\lambda = \int\limits_N f_n \, d\lambda + \int\limits_{N^C} f_n \, d\lambda.$$

Similar identities hold for f and g. First, we know $f_n \to f$ pointwise a.e., and each f_n is measurable; hence, f is measurable. Also note that $|f| = \lim\limits_{n\to\infty} |f_n| \le g$ on N^C- therefore

$$\int\limits_{N^C} |f| \, d\lambda \le \int\limits_{N^C} g \, d\lambda \le \int g \, d\lambda < \infty$$

since g is integrable- this shows f is integrable. Consider $f_n + g$- for each $n \in \mathbb{N}$, this is a measurable function, with $f_n + g \ge 0$ on N^C. By Fatou's Lemma, we have

$$\int\limits_{N^C} \liminf_{n\to\infty}(f_n + g) \, d\lambda \le \liminf_{n\to\infty} \int\limits_{N^C} f_n + g \, d\lambda \Rightarrow \int\limits_{N^C} f + g \, d\lambda \le \left(\liminf_{n\to\infty} \int\limits_{N^C} f_n \, d\lambda\right) + \int\limits_{N^C} g \, d\lambda.$$

Since g is integrable, we can subtract $\int\limits_{N^C} g \, d\lambda$ from both sides, getting $\int\limits_{N^C} f \, d\lambda \le \liminf\limits_{n\to\infty} \int\limits_{N^C} f_n \, d\lambda$. If we repeat the same argument with $g - f_n$ instead (which are again positive), we once again note that by Fatou's Lemma

$$\int\limits_{N^C} g - f_n \, d\lambda \le \liminf_{n\to\infty} \int\limits_{N^C} g - f_n \, d\lambda \Rightarrow \int\limits_{N^C} g \, d\lambda - \int\limits_{N^C} f \, d\lambda \le \int\limits_{N^C} g \, d\lambda - \limsup_{n\to\infty} \int\limits_{N^C} f_n \, d\lambda$$

$$\Rightarrow \int\limits_{N^C} f \, d\lambda \ge \limsup_{n\to\infty} \int\limits_{N^C} f_n \, d\lambda.$$

This proves $\int\limits_{N^C} f \, d\lambda = \lim\limits_{n\to\infty} \int\limits_{N^C} f_n \, d\lambda$, which completes the proof. □

5. General L^p spaces

5.1. Construction and properties.

Note: For an L^p space, we have $p \in \mathbb{R}$ with $1 \le p \le \infty$.

Let $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$ (usually $A = [a, b], [0, \infty)$, or $\mathbb{R}$). Let $f : A \to \mathbb{C}$ be measurable. Fix $1 \le p < \infty$. We define

$$\|f\|_{L^p(A)} = \left(\int\limits_A |f|^p \, d\lambda\right)^{1/p}.$$

If A is fixed, we just write $\|f\|_p$.

Example: Let $A = [0, 1]$. Then $\|f\|_2 = \left(\int\limits_0^1 f(x)\overline{f(x)} \, d\lambda\right)^{1/2}$.

Question: Is $\| \cdot \|_p$ a norm on measurable functions?

Answer: No- if $f \neq g$ but $f = g$ a.e., then $\|f - g\|_p = 0$ even though $f - g \neq 0$ on A.

Example: $\|\chi_{\mathbb{Q}\cap[0,1]} - 0\|_{L^2([0,1])} = 0$, but $\chi_{\mathbb{Q}\cap[0,1]} \neq 0$.

Definition 5.1. *Define an equivalence relation $\sim$ by $f \sim g$ if $f = g$ a.e.*

Lemma 5.2. *Let $f, g : A \to \mathbb{R}$ be measurable. Then $f \sim g \iff \|f - g\|_p = 0$.*

Proof.
($\Rightarrow$) If $f = g$ a.e. on A, then $f - g = 0$ a.e. on A, and so $\|f-g\|_p = \left(\int_A |f-g|^p \, d\lambda\right)^{1/p} = (0)^{1/p} = 0$.

($\Leftarrow$) If $\|f - g\|_p = 0$, then

$$\left(\int_A |f-g|^p \, d\lambda\right)^{1/p} = 0 \Rightarrow \int_A |f-g|^p \, d\lambda = 0 \Rightarrow |f-g|^p = 0 \Rightarrow f = g \text{ a.e.}$$

□

Exercise: If $f : A \to [0, \infty)$ is measurable and $\int_A f \, d\lambda = 0$, then $f = 0$ a.e. on A.

Definition 5.3. $\mathcal{L}^p(A) = \{$*equivalence classes of* $f : A \to \mathbb{C}$ *s.t.* $\int_A |f|^p \, d\lambda < \infty\}$.

Remark: If $f : A \to \mathbb{C}$ is measurable and $\int_A |f|^p \, d\lambda < \infty$ then $[f] \in \mathcal{L}^p(A)$.

Proposition 5.4. *Let $1 \leq p < \infty$; let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$.*
(i) $\mathcal{L}^p(A)$ is a vector space;
(ii) $\|\cdot\|_p$ satisfies the properties of a norm.

Proof.
(i) First, note that $\|\cdot\|_p$ is well-defined- that is, if $f \sim g$ then $\|f\|_p = \|g\|_p$ (**exercise!**). Suppose $f, g \in L^p(A)$ and $c \in \mathbb{C}$. We want to show that $cf \in L^p(A)$ and $f + g \in L^p(A)$. First, note that $cf, f+g$ are both measurable, as f, g are measurable functions. Moreover,

$$\int_A |(cf)(x)|^p \, d\lambda(x) = \int_A |cf(x)|^p \, d\lambda(x) = \int_A \Big(|c||f(x)|\Big)^p \, d\lambda(x) = |c|^p \int_A |f(x)|^p \, d\lambda(x);$$

as $\int_A |f|^p \, d\lambda < \infty$ and $|c|^p$ is a finite value, this shows $\int_A |cf|^p \, d\lambda < \infty$ as well. Therefore, $cf \in L^p(A)$. Note that

$$\|cf\|_p = \left(\int_A |cf|^p \, d\lambda\right)^{1/p} = \left(|c|^p \int_A |f|^p \, d\lambda\right)^{1/p} = |c|\left(\int_A |f|^p \, d\lambda\right)^{1/p} = |c|\|f\|_p.$$

Next, note that the function $\psi(x) = x^p$ is convex, as $p \geq 1$. For fixed points $a, b \in \mathbb{C}$, we have $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p+b^p}{2}$. So

$$|f(x) + g(x)|^p \leq \Big(|f(x)| + |g(x)|\Big)^p \leq \frac{2^p}{2}\Big(|f(x)|^p + |g(x)|^p\Big) = 2^{p-1}\Big(|f(x)|^p + |g(x)|^p\Big).$$

Then we see

$$\int_A |f(x)+g(x)|^p \, d\lambda \le 2^{p-1}\left(\int_A |f(x)|^p \, d\lambda + \int_A |g(x)|^p \, d\lambda\right) < \infty.$$

Therefore, $f+g \in L^p(A)$ as well. This shows $L^p(A)$ is a vector space.

(ii) We want to show that $\|\cdot\|_p$ satisfies the properties of a norm. To start, suppose $\|f\|_p = 0$; we see

$$\|f\|_p = 0 \iff \int_A |f|^p \, d\lambda = 0 \iff |f|^p = 0 \text{ a.e. on } A$$
$$\iff |f| = 0 \text{ a.e. on } A \iff f = 0 \text{ a.e. on } A.$$

So $f = 0$ in $L^p(A)$. For the second condition of a norm, this has been proved already- see our work in the proof for part (i). For the triangle inequality, this was proved in MATH600- it follows from Hölder's Inequality. These three together show that $\|\cdot\|_p$ is a norm on $L^p(A)$. □

Definition 5.5 (L^∞ space)**.** *Let $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$. Let*

$$\|f\|_{L^\infty(A)} = \inf_{\substack{E\in\mathcal{L}(\mathbb{R})\\ E\subseteq A\\ \lambda(A\setminus E)=0}} \left\{\sup|f(x)| : x \in E\right\}.$$

This is called the essential sup of f.

Examples:

(i) If $f : A \to \mathbb{C}$, $f = 0$ a.e. then $\|f\|_\infty = 0$.
(ii) If $f : A \to \mathbb{C}$ is continuous, then $\|f\|_\infty = \|f\|_{\sup}$.

Definition 5.6. *Define $L^\infty(A) = \Big\{$ equivalence classes of $f : A \to \mathbb{C}$ measurable and $\|f\|_\infty < \infty\Big\}$.*

Lemma 5.7. *Let $g \in L^\infty(A)$. Define*

$$B = \{x \in A : |g(x)| > \|g\|_\infty\}.$$

Then $\lambda(B) = 0$.

Proof. (Essentially trivial) For $n \in \mathbb{N}$, define

$$B_n = \{x \in A : |g(x)| \ge \|g\|_\infty + 1/n\}.$$

It is clear that $B = \bigcup_{n\in\mathbb{N}} B_n$. Fix $n \in \mathbb{N}$, and let $E \subseteq A$ be such that $\lambda(A \setminus E) = 0$, and $\sup\{|g(x)| : x \in E\} \le \|g\|_\infty + \frac{1}{2n}$. We note that for every $x \in B_n$, then $x \notin E$, as $|g(x)| \ge \|g\|_\infty + \frac{1}{n}$. So $B_n \subseteq A \setminus E$, meaning $\lambda(B_n) = 0$. Unfixing n, this holds for every $n \in \mathbb{N}$. This shows

$$\lambda(B) = \lambda\left(\bigcup_{n\in\mathbb{N}} B_n\right) \le \sum_{n=1}^{\infty} \lambda(B_n) = 0.$$

□

Proposition 5.8. *$L^\infty(A)$ is a vector space, and $\|\cdot\|_\infty$ is a norm on $L^\infty(A)$.*

Proof. Suppose $f \in L^\infty(A)$, and $c \in \mathbb{C}$. We want to show that $cf \in L^\infty(A)$. First, note that as f is measurable, then so is cf. Let $E = \{x \in A : |f(x)| > \|f\|_\infty\}$. By the previous lemma, $\lambda(E) = 0$. This means

$$\|cf\|_\infty \leq \sup\left\{|cf(x)| : x \in A \setminus E\right\} = |c|\|f\|_\infty < \infty.$$

Therefore, $cf \in L^\infty(A)$. For the rest of the proof, **exercise!** □

Remark:

- If A is a compact set with $\lambda(A) > 0$, then $\lambda(A) < \infty$. It is easy to show that

$$L^\infty(A) \subseteq L^{p_1}(A) \subseteq L^{p_2}(A) \subseteq L^1(A),$$

where $p_1 > p_2$.
- If $A = \mathbb{R}$, and $p_1 < p_2$, then neither $L^{p_1}(\mathbb{R}) \subseteq L^{p_2}(\mathbb{R})$ nor $L^{p_2}(\mathbb{R}) \subseteq L^{p_1}(\mathbb{R})$.

Hölder's Inequality

Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (we use the convention that $\frac{1}{\infty} = 0$). Let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$. If $f \in L^p(A)$, and $g \in L^q(A)$ then

$$\int_A |fg| \, d\lambda \leq \|f\|_p \|g\|_q.$$

Note: This generalizes the Cauchy-Schwarz Inequality.

Proof. (Sketch) Suppose $\|f\|_p \neq 0, \|g\|_q \neq 0$ (this should be checked separately). Showing the Hölder Inequality is equivalent to showing

$$\int_A \left|\frac{f}{\|f\|_p}\right| \left|\frac{g}{\|g\|_q}\right| d\lambda \leq 1.$$

If we use the inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

for real numbers $a, b \geq 0$ and conjugate p, q, we see

$$\int_A \left|\frac{f}{\|f\|_p}\right| \left|\frac{g}{\|g\|_q}\right| d\lambda \leq \frac{1}{p}\int_A \left|\frac{f}{\|f\|_p}\right|^p d\lambda + \frac{1}{q}\int_A \left|\frac{g}{\|g\|_q}\right|^q d\lambda$$
$$= \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

□

Corollary 5.9 (Triangle inequality for L^p spaces)**.** *If $f, g \in L^p(A)$, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. We will prove the above for $1 < p < \infty$ (as we've proved this for $p = \infty$).

Exercise: prove for $p = 1$!

We first see that

$$\|f+g\|_p^p = \int_A |f+g|^p \, d\lambda = \int_A |f+g|^{p-1}|f+g| \, d\lambda \leq \int_A |f+g|^{p-1}\Big(|f|+|g|\Big) \, d\lambda$$
$$= \int_A |f+g|^{p-1}|f| \, d\lambda + \int_A |f+g|^{p-1}|g| \, d\lambda.$$

We know $f \in L^p(A)$ (as it is given); we claim that $|f+g|^{p-1} \in L^q(A)$, where p, q are conjugate. We note

$$\int_A \Big(|f+g|^{p-1}\Big)^q \, d\lambda = \int_A |f+g|^{pq-q} \, d\lambda.$$

As $\frac{1}{p} + \frac{1}{q} = 1$, then $pq = p + q$. Therefore, we have

$$\int_A |f+g|^{pq-q} \, d\lambda = \int_A |f+g|^{p+q-q} \, d\lambda = \int_A |f+g|^p \, d\lambda < \infty,$$

as $f+g \in L^p(A)$ (as $L^p(A)$ is a vector space). This shows $|f+g|^{p-1} \in L^q(A)$. By Hölder's Inequality, we have

$$\int_A |f+g|^{p-1}|f| \, d\lambda \leq \|f\|_p \| |f+g|^{p-1} \|_q = \|f\|_p \|f+g\|_p^{p/q}.$$

Similarly, we have

$$\int_A |f+g|^{p-1}|g| \, d\lambda \leq \|g\|_p \| |f+g|^{p-1} \|_q = \|g\|_p \|f+g\|_p^{p/q}.$$

This means

$$\|f+g\|_p^p \leq \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q}$$
$$= \Big(\|f\|_p + \|g\|_p\Big) \|f+g\|_p^{p/q}$$
$$\Rightarrow \|f+g\|_p^{p-p/q} \leq \|f\|_p + \|g\|_p.$$

As $pq = p + q$, then $p - \frac{p}{q} = \frac{pq-p}{q} = 1$. Therefore,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

□

Recall: A Banach space is a complete normed vector space.

Theorem 5.10. *Let $1 \leq p \leq \infty$. Let $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$. Then $L^p(A)$ is a Banach space.*

Proof. (Sketch of proof) We want to show that $L^p(A)$ is complete- so we'll show every Cauchy sequence in $L^p(A)$ converges to a function in $L^p(A)$.

Case 1: Let $1 \leq p < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(A)$. We want to show there exists an $f \in L^p(A)$ such that $f_n \to f$ in $L^p(A)$ (i.e. $\|f_n - f\|_p \to 0$ as $n \to \infty$). Since $\{f_n\}_{n\in\mathbb{N}}$ is $\|\cdot\|_p$-Cauchy, there exists a subsequence $\{f_{n_i}\}_i$ such that $\|f_{n_i} - f_{n_{i-1}}\|_p < \frac{1}{2^i}$ (**MATH600 exercise!**). For each $k \in \mathbb{N}$, define

$$h_k = f_{n_1} + \sum_{i=2}^{k}(f_{n_i} - f_{n_{i-1}}).$$

We note that $h_k = f_{n_k}$ (as it is a telescoping sum). We can see that $\lim\limits_{k\to\infty} h_k(x)$ exists a.e. for all $x \in A$ (i.e. pointwise- we'll finish this part of the proof in our next assignment). Define

$$f : A \to \mathbb{C}, \quad f(x) = \begin{cases} \lim\limits_{k\to\infty} h_k(x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

We want to prove that $f_{n_k} \to f$ as $k \to \infty$- if this holds, then $f_n \to f$ as $n \to \infty$ as $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Note: we "go down" to pointwise limits in order to define our limit- we now go back to showing convergence in $L^p(A)$. We claim $\|f_{n_k} - f\|_p \to 0$ as $k \to \infty$. Define

$$g_k = \sum_{i=2}^{k} |f_{n_i} - f_{n_{i-1}}|,$$
$$g = \sum_{i=2}^{\infty} |f_{n_i} - f_{n_{i-1}}|.$$

We note that $g : A \to [0, \infty]$. We have

$$\|g_k\|_p \leq \sum_{i=2}^{k} \|f_{n_i} - f_{n_{i-1}}\|_p \leq \sum_{i=2}^{k} \frac{1}{2^i} \leq 1,$$

(by the Triangle Inequality) for $k \in \mathbb{N}$. Therefore, $\|g_k\|_p \leq 1$ for all $k \geq 2$. We know $g_k \nearrow g$ pointwise on A, so $g_k^p \nearrow g^p$ (when $p \geq 1$) pointwise a.e. on A. Then by the Monotone Convergence Theorem,

$$\|g\|_p^p = \int_A g^p \, d\lambda = \lim_{k\to\infty} \int_A g_k^p \, d\lambda \leq 1.$$

This shows $g \in L^p(A)$, and so the set $\{x \in A : g(x) = \infty\}$ has measure 0. We now note that

(i) $|f_{n_k} - f|^p \leq g^p$ a.e. on A;
(ii) $|f_{n_k} - f|^p \to 0$ pointwise a.e. on A (by definition).

Exercise: check the two conditions listed above. Then by the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_A |f_{n_k} - f|^p \, d\lambda = 0.$$

Therefore, $\|f_{n_k} - f\|_p^p \to 0$, and so $\|f_{n_k} - f\|_p \to 0$ as $n \to \infty$. This completes the proof for Case 1.

Case 2: Let $p = \infty$. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^\infty(A)$; i.e., for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|f_n - f_m\|_\infty < \epsilon$. Let

$$A_k = \{x : |f_k(x)| > \|f_k\|_\infty\}.$$

We know that $\lambda(A_k) = 0$ (for each $k \in \mathbb{N}$). Let

$$B_{n,m} = \{x \in A : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$

As with A_k, we have $\lambda(B_{n,m}) = 0$ for $n, m \in \mathbb{N}$. If we let

$$F = \bigcup_{k=1}^{\infty} A_k \cup \bigcup_{n,m} B_{n,m},$$

then $\lambda(F) = 0$ as well. For all $x \in F^C$, and for all $n, m \in \mathbb{N}$ we have

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty.$$

We also have

$$\sup_{x \in F^C} |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty;$$

however, as $\|f_n - f_m\|_\infty \to 0$ as $n, m \to \infty$ on F^C, then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy on F^C. This means $f_n \to f$ uniformly on F^C, where

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x), & x \in F^C, \\ 0, & x \in F. \end{cases}$$

Therefore, $f_n \to f$ in $L^\infty(A)$ (Check- **exercise!**). Note that $f \in L^\infty(A)$, as f is measurable (as a pointwise limit of measurable functions) and

$$\|f\|_\infty \le \|f_n - f\|_\infty + \|f_n\|_\infty < \infty$$

for some fixed $n \in \mathbb{N}$. □

Corollary 5.11. *We proved:*

(i) If $f_n \to f$ in $\|\cdot\|_p$, then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \to f$ pointwise a.e.
(ii) $f_n \to f$ in $\|\cdot\|_\infty$ if and only if there exists a set F with $\lambda(F^C) = 0$ and $f_n \to f$ uniformly on F.

Remark: Let $1 \le p < \infty$. Then there exists sequences $\{f_n\}$ such that $f_n \to f$ in $\|\cdot\|_p$ but not pointwise almost everywhere.

Example: Take

$$\chi_{[0,1]}, \chi_{[0,1/4]}, \chi_{[1/2,1]}, \chi_{[0,1/2]}, \chi_{[1/2,2/3]}, \cdots$$

Then the sequence converges to 0 in L^p, but not pointwise a.e. (as the supremum is at least 1 sometimes).

Facts:

(i) $C_{[a,b]}$ is a dense subspace of $L^p[a,b]$ when $1 \le p < \infty$;
(ii) $C_{[a,b]}$ is a closed subspace of $L^\infty[a,b]$ (so not dense).

6. Complex analysis

6.1. Basics and definitions.

As expected, we'll be working over $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$.

Basic rules:

(i) $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$;
(ii) $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Under these operations, $\mathbb{C}$ becomes a field. When $z = x + iy$ and $z \neq 0$, we have inverses:

$$z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Notation: When $z = x + iy$, we say

(i) $|z| = \sqrt{x^2 + y^2}$;
(ii) $\Re z = x$;
(iii) $\Im z = y$;
(iv) $\overline{z} = x - iy$.

<u>Facts:</u> For $z, w \in \mathbb{C}$, we have
(i) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$;
(ii) $|zw| = |z||w|$;
(iii) $|z + w| \leq |z| + |w|$;
(iv) $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\overline{w}$.

Polar form: If $z \neq 0$, then $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg(z) \in (-\pi, \pi]$.

Example: Polar form is better for multiplication:

$$z_1 z_2 = \Big(r_1(\cos\theta_1 + i\sin\theta_1)\Big)\Big(r_2(\cos\theta_2 + i\sin\theta_2)\Big) = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Notation: We write $e^{i\theta} = \cos\theta + i\sin\theta$.

Note: Every $z \in \mathbb{C}$ where $z \neq 0$ admits n distinct n^{th} roots; if $z = re^{i\theta}$, then the n^{th} roots are $r^{1/n}e^{\frac{i(\theta+2k\pi)}{n}}$ for $0 \leq k \leq n-1$.

Definition 6.1. *We let*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : -x < \theta \leq x\}.$$

We call $\mathbb{T}$ *the torus (in one-dimension).*

<u>Fact:</u> The torus $\mathbb{T}$ endowed with multiplication as the operation is an abelian group. In fact, this is the first example of a *compact* group.

Examples: (Functions from $\mathbb{C}$ to $\mathbb{C}$)
(i) exp: $\mathbb{C} \to \mathbb{C}$, where $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The radius of convergence of exp is ∞.
(ii) $\sin z : \mathbb{C} \to \mathbb{C}$, where $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$.
(iii) $\cos z : \mathbb{C} \to \mathbb{C}$, where $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

Definition 6.2. *A region* Ω *in* $\mathbb{C}$ *is an open connected subset of* $\mathbb{C}$, *i.e.* Ω *is open and cannot be written as* $\Omega = A \cup B$ *where* $A \neq B \neq \emptyset$ *with* $A \cap \overline{B} = \emptyset$ *and* $\overline{A} \cap B = \emptyset$.

Exercise: Show that Ω is a region in $\mathbb{C}$ if and only if Ω is open and any two points in Ω can be connected by a piecewise linear curve in Ω.

Definition 6.3. *A piecewise linear curve is a finite union of line segments.*

<u>Hint:</u> Proving ($\Leftarrow$) is trivial, as if Ω is pathwise connected it is connected (see MATH600 homework). For ($\Rightarrow$), fix $x_0 \in \Omega$. Define

$$A = \{x \in \Omega : \text{ there exists piecewise linear path from } x \text{ to } x_0\}.$$

Take $B = \Omega \setminus A$, and prove that A, B are open.

Definition 6.4. *Suppose* Ω *is a region in* $\mathbb{C}$, *and* $a \in \Omega$. *Consider* $f : \Omega \to \mathbb{C}$; *we say* f *is differentiable at* a *if*

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. If the limit exists, we denote it by $f'(a)$. *We say* f *is analytic on* Ω *if it is differentiable at every point of* Ω.

Remark: The existence of the derivative for a complex function f is the subject of study in complex analysis.

Examples: The following are analytic functions:

(i) Any polynomial;
(ii) $e^z, \cos z, \sin z$ analytic on $\mathbb{C}$ (**check!**).

Example: The function $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = \overline{z}$ is *not* analytic, as

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{\overline{h}}{h} = \begin{cases} \frac{h}{h} = \ \ 1, & h \in \mathbb{R}, \\ -\frac{h}{h} = -1, & h \in i\mathbb{R}. \end{cases}$$

Therefore, the limit does not exist.

Proposition 6.5. *(i) If f is differentiable at a, then f is continuous at a;*
(ii) If f, g are differentiable at a then $f + g$, fg, f/g (provided $g(a) \neq 0$) are differentiable at a;
(iii) If f is differentiable at a, g differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$\frac{d}{dz}(g(f(z)))\big|_{z=a} = g'(f(a))f'(a).$$

Proof. Similar to the real case for metric spaces. □

Any $z \in \mathbb{C}$ is represented as $z = x + iy$ for $x, y \in \mathbb{R}$. Let Ω be a region in $\mathbb{C}$, and let $f : \Omega \to \mathbb{C}$ where $f(z) = u(z) + iv(z)$ with $u, v : \Omega \to \mathbb{R}$ are real-valued. So $f : \Omega \to \mathbb{C}$ is equivalent to

$$(x, y) \mapsto (u(x, y) + iv(x, y))$$

where we think of Ω as a subset of $\mathbb{R}^2$ instead.

Question: Suppose u, v have partial derivatives. Can we say that f is necessarily analytic?

Answer: No! The function $f(z) = \overline{z}$ is a counterexample; here $u(x, y) = x, v(x, y) = -y$ which clearly have partial derivatives. However, f is not analytic!

Suppose f is as above, and suppose f is differentiable at a point $a = p+iq \in \Omega$. Therefore,

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

exists (here h is a complex number). If we first just take h on x, we have

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to 0} \frac{u(p + x, q) + iv(p + x, q) - u(p, q) - iv(p, q)}{x}$$
$$= u_x(p, q) + iv_x(p, q).$$

If instead we take h on y, we have

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{y \to 0} \frac{f(a + iy) - f(a)}{iy} = \lim_{y \to 0} \frac{u(p, q + y) + iv(p, q + y) - u(p, q) - iv(p, q)}{iy}$$
$$= -i[u_y(p, q) + iv_y(p, q)].$$

As f is differentiable at a, this forces the previous two limits to be equal, which shows

$$u_x = v_y, \quad v_x = -u_y.$$

Proposition 6.6. *Suppose $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and u, v are real-valued. Suppose f is differentiable at a. Then u, v have partial derivatives at a, and*

$$u_x = v_y, \quad v_x = -u_y$$

at a. These are called the Cauchy-Riemann equations. Moreover, $f'(z) = u_x + iv_x = v_y - iu_y$.

Remark:

(i) If f is analytic on Ω, then so is f'. Hence, f is differentiable an infinite number of times on Ω.

(ii) An analytic function on Ω is uniquely determined by its values on a sequence convergent in Ω; i.e., if $p_n \to p$ in Ω and f, g are analytic on Ω with $f(p_n) = g(p_n)$ for all $n \in \mathbb{N}$, then $f = g$ on Ω.

(iii) If f is analytic on Ω and $a \in \Omega$, then f has a power series expansion around $z = a$.

Theorem 6.7 (Section 4.22)**.** *The function $f = u + iv$ is differentiable at $a = p + iq$ if and only if the functions u, v are differentiable at (p, q) and the Cauchy Riemann equations are satisfied at (p, q).*

Note: This does <u>not</u> imply the other direction of the previous proposition- there is a difference between u, v being differentiable at a point and u_x, v_x just existing.

Recall: Let Ω be a region in $\mathbb{C}$. Function $f : \Omega \to \mathbb{C}$ is called analytic if f is differentiable at every point of Ω. We say f is analytic at a point $a \in \Omega$ if f is analytic on a neighborhood containing a.

Examples:

(i) Let $f : \mathbb{C} \to \mathbb{C}$ where $f(z) = |z|^2$. We have $f(x+iy) = |x+iy|^2 = x^2+y^2 = u(x,y)+iv(x,y)$. This implies $u(x,y) = x^2 + y^2$, and $v(x,y) = 0$. If f is differentiable at $a = p + iq$, then by the proposition we have

$$\begin{aligned} u_x(p,q) = v_y(p,q), &\quad u_y(p,q) = -v_x(p,q) \\ \Rightarrow 2p = 0, &\quad 2q = 0 \\ \Rightarrow p = q &= 0. \end{aligned}$$

So f is not differentiable at $z \neq 0$. As $z = 0$, we see

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{|z|^2}{z} = \lim_{z\to 0} \overline{z} = 0.$$

So $f'(0) = 0$, and therefore f is differentiable at 0.

Note: f is not *analytic* at $z = 0$.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$, where

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

We see

$$f(x+iy) = \frac{(x+iy)^5}{|x+iy|^4} = \frac{(x+iy)^5}{(x^2+y^2)^2}$$

(Note that it is not easy to find u, v straight away). We know there exist $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x+iy) = u(x,y) + iv(x,y)$. If $y = 0$, then

$$f(x) = u(x,0) + iv(x,0) = \frac{x^5}{|x|^4} = x,$$

and therefore $f(x) \in \mathbb{R}$. This implies $u(x,0) = x, v(x,0) = 0$. Therefore, $u_x(0,0) = 1$, and $v_x(0,0) = 0$. Similarly, if we let $x = 0$, we get $v_y(0,0) = 1$ and $u_y(0,0) = 0$. Therefore, u_x, u_y, v_x, v_y exist at $(0,0)$ and the Cauchy-Riemann equations are satisfied. However, we will show that f is <u>not</u> differentiable at $z = 0$. We see

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\frac{z^5}{|z|^4}}{z} = \lim_{z\to 0} \frac{z^4}{z^2\overline{z}^2} = \lim_{z\to 0} \frac{z^2}{\overline{z}^2}.$$

Take $z = re^{\frac{2\pi i}{8}}$ for $r > 0$ and $r \in \mathbb{R}$. Then

$$\lim_{z\to 0} \frac{z^2}{\overline{z}^2} = \lim_{r\to 0} \frac{r^2 e^{\frac{\pi i}{2}}}{r^2 e^{\frac{-\pi i}{2}}} = -1.$$

However, checking on the x-axis we see

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1 \neq -1.$$

Therefore, the limit does not exist.

Theorem 6.8. *Let $\Omega \subseteq \mathbb{C}$ be a region, and $z_0 \in \Omega$ with $z_0 = x_0 + iy_0$. Let $f : \Omega \to \mathbb{C}$ where $f(x + iy) = u(x,y) + iv(x,y)$. Suppose*

(i) u_x, u_y, v_x, v_y exist on Ω and are continuous at (x_0, y_0).
(ii) u, v satisfy the Cauchy-Riemann equations at (x_0, y_0).

Then f is differentiable at z_0, and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Fix $z_0 = x_0 + iy_0$. Write

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = U_1(\Delta z) + iV_1(\Delta z),$$

where

$$U_1(\Delta z) = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}$$

and V_1 similar. We look at $U_1(\Delta z)$- we see

$$U_1(\Delta z) = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}$$
$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)}{\Delta z} + \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}.$$

Using the Real Mean Value Theorem for the first term in the latter sum, we see

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x u_x(x_0^*, y_0 + \Delta y)$$

for some $x_0 \leq x_0^* \leq x_0 + \Delta x$. Similarly,

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y u_y(x_0, y_0^*)$$

for some $y_0 \leq y_0^* \leq y_0 + \Delta y$. Then

$$U_1(\Delta z) = \frac{\Delta x u_x(x_0^*, y_0 + \Delta y)}{\Delta z} + \frac{\Delta y u_y(x_0, y_0^*)}{\Delta z}.$$

Let

$$E_1(\Delta z) := [u_x(x_0^*, y_0 + \Delta y) - u_x(x_0, y_0)]\frac{\Delta x}{\Delta z},$$
$$E_2(\Delta z) := [u_y(x_0, y_0^*) - u_y(x_0, y_0)]\frac{\Delta y}{\Delta z}.$$

Then

$$U_1(\Delta z) = u_x(x_0, y_0)\frac{\Delta x}{\Delta z} + u_y\frac{\Delta y}{\Delta z} + E_1(\Delta z) + E_2(\Delta z).$$

Let $\Delta z \to 0$; then $x_0^* \to x_0, y_0^* \to y_0$. Furthermore,

$$|\frac{\Delta x}{\Delta z}| \leq 1, \quad |\frac{\Delta y}{\Delta z}| \leq 1.$$

Then

$$\begin{aligned}
&\lim_{\Delta z \to 0} U_1(\Delta z) \\
&= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0)\frac{\Delta x}{\Delta z} + u_y(x_0, y_0)\frac{\Delta y}{\Delta z} \right) \\
&= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0)\frac{\Delta x}{\Delta z} - v_x(x_0, y_0)\frac{\Delta y}{\Delta z} \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{\Delta z \to 0} V_1(\Delta z) \\
&= \lim_{\Delta z \to 0} \left(v_x(x_0, y_0)\frac{\Delta x}{\Delta z} + u_x(x_0, y_0)\frac{\Delta y}{\Delta z} \right).
\end{aligned}$$

So

$$\begin{aligned}
&\lim_{\Delta z \to 0} U_1(\Delta z) + iV_1(\Delta z) \\
&= \lim_{\Delta z \to 0} \left(u_x(x_0, y_0)\left[\frac{\Delta x + i\Delta y}{\Delta z}\right] + iv_x(x_0, y_0)\left[\frac{\Delta x + i\Delta y}{\Delta z}\right] \right) \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0),
\end{aligned}$$

as $\Delta z = \Delta x + i\Delta y$. □

Example: Let $f : \mathbb{C} \to \mathbb{C}$ where

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We can prove $e^{z_1+z_2} = e^{z_1}e^{z_2}$ (**check!**). Then

$$\begin{aligned}
f(x + iy) &= e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \\
&= e^x \cos y + i(e^x \sin y) \\
&\Rightarrow u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y.
\end{aligned}$$

Then u_x, u_y, v_x, v_y exist on $\mathbb{R}^2$ and are continuous. We have

$$\begin{aligned}
u_x(x, y) &= e^x \cos y, \quad v_x(x, y) = e^x \sin y \\
u_y(x, y) &= -e^y \sin y, \quad v_y(x, y) = e^x \cos y.
\end{aligned}$$

So $u_x = v_y, u_y = -v_x$; so by our previous theorem, $f(z) = e^z$ is analytic on $\mathbb{C}$. Also,

$$f'(z) = u_x(x, y) + iv_x(x, y) = e^x \cos y + i(e^x \sin y) = e^z,$$

as we would expect.

Corollary 6.9. *Let $f(z) = u + iv$ be analytic on a region Ω.*

(i) If $f'(z) = 0$ on Ω, then f is constant.
(ii) If u or v are constant on Ω, then f is constant. In particular, a non-constant function cannot take only real or purely imaginary values.
(iii) If $|f|$ or arg(f) are constant, then f is contstant (here we assume $f \neq 0$ when looking at arg(f)).

Proof.
(i) Suppose $f'(z) = 0$, where $f'(x+iy) = u_x(x,y) + iv_x(x,y)$. This implies $u_x = 0$ and $v_x = 0$ on Ω. By the Cauchy-Riemann equations, this implies $u_y = 0$ and $v_y = 0$ as well. This means u is constant on Ω, and v is constant on Ω as well (sketch: fix (x_0, y_0) and take point (x_1, y_1). We know there is a piecewise path in Ω connecting these two points. **Exercise:** finish the rest of the proof). This means $f = u + iv$ is constant on Ω. □

6.2. **Complex integration.**

Definition 6.10. *A curve γ in the complex plane is described by a continuous map*

$$z : [a,b] \to \mathbb{C},$$
$$z \mapsto z(t)$$

such that there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ where $z(t)$ has a continuous derivative on each $[a_i, a_{i+1}]$ for $i = 0, \ldots, n-1$.

Recall:

$$z(t) = x(t) + iy(t),$$
$$z'(t) = x'(t) + iy'(t).$$

Note: The derivatives are one-sided at endpoints of the intervals.

Definition 6.11. *A reparameterization of γ is a continuous, increasing function $\tau : [\alpha, \beta] \to [a,b]$ where $\tau \mapsto \tau(t)$ such that $\tau(\alpha) = a, \tau(\beta) = b$, with τ being piecewise differentiable with a continuous derivative. A reparameterization of γ is given by*

$$[\alpha, \beta] \to \mathbb{C},$$
$$t \mapsto z(\tau(t)).$$

Definition 6.12 (Line Integral)**.** *If $f : [a,b] \to \mathbb{C}$ is continuous, then we define*

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

where $f(t) = u(t) + iv(t)$ (here the real and imaginary parts of f are given by u, v respectively).

Definition 6.13 (Integration over curves)**.** *Suppose $\Omega \subseteq \mathbb{C}$ is a region and $f : \Omega \to \mathbb{C}$, where f is continuous on Ω. Let γ be a curve in Ω described as $z : [a,b] \to \mathbb{C}$, where $z(t) = x(t) + iy(t)$. Then we define*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Remark: We will show later that

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a)) \text{ (generalizing proper integration).}$$

Note: The definition of the integral is invariant under change of parameter (or reparameterization) of the curve. This follows as if $\tau : [\alpha, \beta] \to [a, b]$ is a reparameterization of γ, and we let $w : [\alpha, \beta] \to \mathbb{C}$ where $w(t) = z(\tau(t))$, then

$$\int_{\alpha}^{\beta} f(w(t))w'(t)dt = \int_{\alpha}^{\beta} f(z(\tau(t)))z'(\tau(t))\tau'(t)dt$$
$$= \int_{a}^{b} f(z(s))z'(s)ds \text{ (where } s = \tau(t)).$$

Proposition 6.14. *Let $\Omega \subseteq \mathbb{C}$, γ as before. Suppose $f : \Omega \to \mathbb{C}$ is continuous.*

(i) If $-\gamma$ is the curve γ traced backwards, i.e. $-\gamma$ is described by the function $t \mapsto z(a + b - t)$, then

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

(ii) Suppose $a < c < b$ and we have curves γ_1 and γ_2 described respectively as $z_1 : [a, c] \to \mathbb{C}$ and $z_2 : [c, b] \to \mathbb{C}$ so that $z_1(c) = z_2(c)$. Let γ be the curve described by z as

$$z : [a, b] \to \mathbb{C},$$
$$z(t) = \begin{cases} z_1(t), & a \le t \le c, \\ z_2(t), & c \le t \le b \end{cases}$$

(We denote $\gamma = \gamma_1 + \gamma_2$). Then

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

(iii) $\left| \int_{\gamma} f(z)dz \right| \le \int_{a}^{b} |f(z(t))||z'(t)|dt.$

(iv) γ is called a closed curve if $\gamma(a) = \gamma(b)$.

Shift of parameter for closed curves: Suppose γ is described by $z : [a, b] \to \mathbb{C}$. Let $t_0 \in (a, b)$, and define

$$w : [t_0, t_0 + (b - a)] \to \mathbb{C},$$
$$w(t) = \begin{cases} z(t), & t_0 \le t \le b, \\ z(t - b + a), & b \le t \le t_0 + b - a. \end{cases}$$

This is called a shifting of the parameter for closed curves.

Integration along a closed curve is also invariant under shift of parameter; this follows as if $\gamma = \gamma_1 + \gamma_2$ and we obtain $\gamma_2 + \gamma_1$ after a shift of parameter, it is clear

$$\int_{\gamma_1+\gamma_2} = \int_{\gamma_2+\gamma_1}.$$

Proof.
(iii) We note that for any non-zero complex number $z \in \mathbb{C}$, we may write $z = |z|e^{i\theta}$, and

so $|z| = ze^{-i\theta}$. Then as $\int_\gamma f(z)dz$ is always a complex number, we may write

$$\left|\int_\gamma f(z)dz\right| = e^{-i\theta}\int_\gamma f(z)dz = \int_\gamma e^{-i\theta}f(z)dz = \int_a^b e^{-i\theta}f(z(t))z'(t)dt$$
$$= \Re\Big[\int_a^b e^{-i\theta}f(z(t))z'(t)dt\Big] = \int_a^b \Re\big[e^{-i\theta}f(z(t))z'(t)\big]dt$$
$$\leq \int_a^b \big|e^{-i\theta}f(z(t))z'(t)\big|dt = \int_a^b |f(z(t))||z'(t)|dt.$$

For $(i),(ii),(iv)$- **exercise!** □

We now take a brief detour to discuss:

Logarithm

Our goal is- when given a non-zero $z \in \mathbb{C}$, we want to define $\log z$ (i.e. the complex version of our regular log function).

We would like $\log z = w \iff z = e^w$ (as expected). Note that

$$|z| = |e^w| = |e^{\Re(w)+i\Im(w)}| = |e^{\Re(w)}||e^{i\Im(w)}| > 0,$$

as $z \neq 0$ and so $e^{\Re(w)} > 0$. This shows $\log z$ exists only if $z \neq 0$. Let $w = x + iy, z = re^{i\theta}$ where $\theta \in [0, 2\pi)$, and $z = e^w$. Then

$$re^{i\theta} = e^x e^{iy} = \begin{cases} r = e^x, \\ \theta = y + 2\pi n, \quad n \in \mathbb{Z}. \end{cases}$$

Then

$$\begin{cases} x = \ln r = \ln(|z|), \\ y = \arg(z) + 2\pi n \quad (n \in \mathbb{Z}). \end{cases}$$

Therefore, $w = \ln|z| + i(\arg(z) + 2\pi n)$, and so if w looks like the previous equation we have $e^w = z$.

Definition 6.15. $\ln z = \ln|z| + i(arg(z) + 2\pi n)$, $n \in \mathbb{Z}$. *(Note: $arg(z) \in [0, 2\pi)$).*

Observation: There is no continuous function $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $e^{L(z)} = z$.

Proof. **Exercise!** □

We need to delete a half-line. Take $\alpha \in [0, 2\pi)$, and let $H_\alpha = \{te^{i\alpha} : t \geq 0\}$. We define

$$\log : \mathbb{C} \setminus H_\alpha \to \mathbb{C},$$
$$z = re^{i\theta} \mapsto \ln r + i\theta, \ \ \alpha < \theta < 2\theta + \alpha.$$

Definition 6.16 (Principal branch of logarithm)**.** *Take $\alpha = \pi$, and define*

$$Log : \mathbb{C} \setminus H_\pi \to \mathbb{C},$$
$$Log(z) = \ln|z| + i\theta,$$

where $z = |z|e^{i\theta}$, and $-\pi < \theta < \pi$.

Exercise: Show that Log is continuous on $\mathbb{C} \setminus H_\pi = \Omega$.

Definition 6.17. *Suppose γ is a closed curve, and $p \in \mathbb{C}$ which is not on γ. We define the winding number of a curve as*

$$n(\gamma; p) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p}.$$

Proposition 6.18. *Suppose γ is a closed curve in $\mathbb{C}$, and $p \in \mathbb{C}$ is not on γ. Then*

(i) $n(\gamma; p) \in \mathbb{Z}$;
(ii) $n(\gamma; p)$ is constant as a function of p in each connected component of $\mathbb{C} \setminus \gamma$.

Proof.
(i) Suppose $w : [a, b] \to \mathbb{C}$ describes γ. Then

$$n(\gamma; p) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p} = \frac{1}{2\pi i} \int_a^b \frac{w'(t)}{w(t)-p} dt.$$

For $s \in [a, b]$, define

$$g(s) = \int_a^s \frac{w'(t)}{w(t)-p} dt.$$

The inside function is continuous for all but a finite number of points, and so it is Riemann integrable. Then $g : [a, b] \to \mathbb{C}$ is differentiable except at a finite number of points. Moreover, $g'(s) = \frac{w'(s)}{w(s)-p}$ for all but a finite number of $s \in [a, b]$. Note that everything we have done so far (i.e. derivation and integration) has been done coordinate-wise. Then

$$w'(s) - g'(s)(w(s) - p) = 0$$

for all but a finite number of $s \in [a, b]$. Therefore, for almost all of $[a, b]$ we have

$$\frac{d}{dt}\left(e^{-g(t)}(w(t) - p)\right) = 0$$

with $t \in [a, b]$. So $e^{-g(t)}(w(t) - p)$ is constant for all but a finite number of $t \in [a, b]$. Therefore, $e^{-g(t)}(w(t) - p)$ is constant on $[a, b]$ by the continuity of the function. We know $w(a) = w(b)$, as γ is closed. This means $e^{-g(a)} = e^{-g(b)}$; furthermore, as $g(a) = 0$, then $e^{-g(a)} = e^{-g(b)} = 1$. Therefore, $g(b) = 2\pi i n$ for $n \in \mathbb{Z}$. This means

$$\int_a^b \frac{w'(t)}{w(t)-p} dt = 2\pi i n,$$

and therefore

$$n(\gamma; p) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p} = n.$$

This shows $n(\gamma; p) \in \mathbb{Z}$.

(ii) Let γ be a closed subset of $\mathbb{C}$- then $\mathbb{C} \setminus \gamma$ is open. This means $\mathbb{C} \setminus \gamma$ decomposes into disjoint connected components, one of which is necessarily unbounded. Considering $n(\gamma; p)$ as a function of p, one can see that it is a continuous function of p by the Lebesgue Dominated Convergence Theorem (**exercise!**). Then $n(\gamma; -)$ must be constant on each bounded connected component of $\mathbb{C} \setminus \gamma$, as $n(\gamma; -)$ is a continuous integer valued function. Finally, we claim $n(\gamma; p) \to 0$ as $|p| \to \infty$. We note that as $w(t), w'(t)$ are piecewise continuous functions, then on each closed and compact subinterval they must be uniformly continuous, hence bounded at every $t \in [a, b]$. This means $w(t) - p \to \pm\infty$ as $|p| \to \infty$.

Let $|p|$ be large enough so that $|p|-|w(t)| > |p|/2$ for all $t \in [a,b]$. Furthermore, let $M > 0$ such that $\sup_{t\in[a,b]} |w'(t)| \leq M$. Then

$$\left|\frac{w'(t)}{w(t)-p}\right| \leq \frac{M}{|p|/2}$$

for all $t \in [a,b]$, and so

$$|n(\gamma;p)| \leq \int_a^b \left|\frac{w'(t)}{w(t)-p}\right| dt \leq \frac{2M(b-a)}{|p|} \to 0$$

as $|p| \to \infty$. This shows $n(\gamma;p) = 0$ if p belongs to the unbounded component of $\mathbb{C} \setminus \gamma$. □

Example: Let γ be a closed curve around point p such that $n(\gamma;p) = 1$. Then $n(-\gamma;p) = -1$. Similarly, if γ' is the curve of γ but traversed twice around, then $n(\gamma';p) = 2$.

Example: Let γ be the circle centered at p of radius R oriented counter-clockwise. We have

$$n(\gamma;q) = \begin{cases} 1 \text{ if } q \text{ is inside of } \gamma, \\ 0 \text{ if } q \text{ is not inside of } \gamma. \end{cases}$$

To show this, let $Z : [0, 2\pi] \to \mathbb{C}$ where $Z(t) = p + Re^{it}$. Take q on the inside of γ. Then

$$n(\gamma;q) = n(\gamma;p) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{Re^{it}}(Rie^{it})dt = \frac{1}{2\pi i}\int_0^{2\pi} idt = \frac{1}{2\pi i}(2\pi i) = 1,$$

by the previous proposition.

Observation 1: Suppose f is analytic in a region Ω, and γ is a curve in Ω described by

$$Z : [\alpha, \beta] \to \mathbb{C},$$
$$t \mapsto Z(t) = X(t) + iY(t)$$

(where X, Y denote the real and imaginary parts respectively). We note that $f(Z(t))$ first maps $t \mapsto X(t) + iY(t)$, and then maps $X(t) + iY(t) \mapsto U(X(t), Y(t)) + iV(X(t), Y(t))$ (where $f(x + iy) = u(x, y) + iv(x, y)$ by analyticity of f). This means

$$\frac{d}{dz}(f \circ Z) = \frac{d}{dt}U(X(t), Y(t)) + i\frac{d}{dt}(V(X(t), Y(t)))$$
$$= U_X(X(t), Y(t))X'(t) + U_Y(X(t), Y(t))Y'(t) + i\big(V_X(X(t), Y(t))X'(t) + V_Y(X(t), Y(t))Y'(t)\big)$$
$$= \big(U_X + iV_X\big)(X(t), Y(t))X'(t) + \big(U_Y + iV_Y\big)(X(t), Y(t))Y'(t)$$
$$= f'(X(t) + iY(t))X'(t) + if'(X(t) + iY(t))Y'(t)$$
$$= f'(X(t) + iY(t))(X'(t) + iY'(t))$$

by the Cauchy Riemann equations. So

$$\frac{d}{dt}(f \circ Z)(t) = f'(Z(t))Z'(t).$$

Observation 2: Suppose f is analytic on Ω and γ is a closed curve inside of Ω, described by $Z : [a, b] \to \mathbb{C}$. Then

$$\int_\gamma f'(z)dz = 0.$$

Remark:

(i) Ω is any region;
(ii) Not every analytic function is of the form f' for some analytic function f.

It is these conditions that show such a statement does not contradict Cauchy's Theorem.

Exercise: (Very important!!!!) Prove that $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$ but there is no analytic function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $g' = f$ on $\mathbb{C} \setminus \{0\}$. To prove the observation above:

Proof. We see

$$\int_\gamma f'(z)dz = \int_a^b f'(Z(t))Z'(t)dt = \int_a^b \frac{d}{dt}(f \circ Z)(t)dt$$

by Observation 1. Then

$$\int_a^b \frac{d}{dt}(f \circ Z)(t)dt = (f \circ Z)(b) - (f \circ Z)(a) = 0,$$

as γ is a closed curve, and so $Z(a) = Z(b)$. $\square$

6.3. **Cauchy's theorems and their implications.**

Theorem 6.19 (Cauchy-Goursat Theorem)**.** *If f is analytic at all points inside and on a closed curve γ (i.e. f is analytic on a simply connected region containing γ), then $\int_\gamma f dz = 0$.*

Theorem 6.20 (C1: Cauchy's Theorem for Rectangles)**.** *Suppose f is analytic on a region Ω and suppose R is a rectangle contained in Ω. Let γ be the boundary of R oriented counter-clockwise. Then $\int_\gamma f(z)dz = 0$.*

Proof. Split R into 4 equal parts by bisecting the length and width; this gives us 4 rectangles R^a, R^b, R^c, and R^d. Let $\gamma^a, \gamma^b, \gamma^c, \gamma^d$ be their boundaries, oriented counter clockwise. Note that

$$\int_\gamma f = \int_{\gamma^a} f + \int_{\gamma^b} f + \int_{\gamma^c} f + \int_{\gamma^d} f.$$

Assume towards contradiction that $|\int_\gamma f| = c > 0$; therefore, at least one of the 4 integrals above has absolute value larger than $\frac{c}{4}$. Call that curve γ_1, and its corresponding rectangle R_1. Then $|\int_{\gamma_1} f| \geq \frac{c}{4}$. Repeating this process on each subsequent rectangle, we obtain

$$R \supseteq R_1 \supseteq R_2 \supseteq \cdots$$

which is a sequence of nested rectangles and their associated boundary curves $\gamma, \gamma_1, \gamma_2, \ldots$ oriented clockwise such that

(i) $\left| \int_{\gamma_{n+1}} f \right| \geq \frac{1}{4} \left| \int_{\gamma_n} f \right|$, and so by induction $\left| \int_{\gamma_n} f \right| \geq \frac{c}{4^n}$ for each $n \in \mathbb{N}$;
(ii) $\ell(\gamma_n) = \frac{1}{2^n}\ell(\gamma)$, and $\text{diameter}(R_n) = \frac{1}{2^n}\text{diameter}(R)$.

By compactness, $\bigcap_{n=1}^{\infty} R_n \neq \emptyset$. Moreover, there exists some $a \in R$ such that $\bigcap_{n=1}^{\infty} R_n = \{a\}$, as diameter$(R_n) \to 0$. We know that f is differentiable at a, so for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| < \epsilon$$

when $|z - a| < \delta$. Let N be large enough so that for all $n \geq N$, we have $|z - a| < \delta$ for all $z \in R_n$. Then for $n \geq N$, we have

$$\left| \int_{\gamma_n} f(z)dz \right| = \left| \int_{\gamma_n} [f(z) - f(a) - (z - a)f'(a)]dz \right|,$$

as $\int_{\gamma_n} f(a)dz = 0$ (as it is equal to $\int_{\gamma_n} 1dz$, which is 0 on the closed curve as the derivative of analytic z by the 2nd observation). We have a similar case for $f(z) - f(a) - (z - a)f'(a)$. Then

$$\left| \int_{\gamma_n} [f(z) - f(a) - (z - a)f'(a)]dz \right| \leq \int_{\gamma_n} \left| f(z) - f(a) - (z - a)f'(a) \right| dz$$
$$\leq \int_{\gamma_n} \epsilon|z - a|dz \leq \epsilon \text{diag}(R_n)\ell(\gamma_n) \leq \frac{\epsilon}{4^n} \text{diag}(R)\ell(\gamma).$$

So

$$\left| \int_{\gamma} f(z)dz \right| \leq 4^n \left| \int_{\gamma_n} f(z)dz \right| \leq 4^n \left(\frac{\epsilon}{4^n} \text{diag}(R)\ell(\gamma) \right) = \epsilon \text{diag}(R)\ell(\gamma).$$

As diag$(R), \ell(\gamma)$ are fixed, if we let $\epsilon \to 0$ (as it was arbitrary) we see

$$\left| \int_{\gamma} f(z)dz \right| = 0.$$

Therefore, $\int_{\gamma} f(z)dz = 0$. □

Theorem 6.21 (C2: Cauchy's Theorem for a Disk)**.** *If f is analytic in an open disk D, and γ is a closed curve in D then $\int_{\gamma} f(z)dz = 0$.*

Proof. Suppose γ is described by $Z : [a, b] \to \mathbb{C}$ where $t \mapsto Z(t)$, with $Z(a) = Z(b)$. From observation 2, we note that to prove the theorem it is enough to find an analytic function F on D such that $F' = f$ everywhere on D. Let z_0 be the center of D. For every $z \in D$, define

$$F(z) = \int_{\gamma_z} f(\xi)d\xi$$

where γ_z is the unique curve from z_0 to z which is the union of first a vertical line and then a horizontal line. As γ_z is unique, $F(z)$ is well-defined. Note that as D is a disk, we can safely claim that γ_z always lies inside D. Also, note that

$$F(z) = \int_{\gamma_{z,1}} f(\xi)d\xi + \int_{\gamma_{z,2}} f(\xi)d\xi$$

where $\gamma_{z,1}$ is our vertical line and $\gamma_{z,2}$ is our horizontal line. Consider the parameterization

$$\begin{aligned} [y_0, y] &\to \mathbb{C} \qquad & [x_0, x] &\to \mathbb{C} \\ t &\mapsto x_0 + it \qquad & t &\mapsto t + iy \end{aligned}$$

of $\gamma_{z,1}, \gamma_{z,2}$ respectively. Then

$$F(z) = \int_{x_0}^{x} f(t+iy)dt + i\int_{y_0}^{y} f(x_0+it)dt.$$

By definition, we have

$$F_x(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{F(x+iy+h)-F(x+iy)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{h}\left[\int_{x_0}^{x+h} f(t+iy)dt - \int_{x_0}^{x} f(t+iy)dt\right].$$

Let $f = u(\cdot,\cdot) + iv(\cdot,\cdot)$. Then

$$F_x(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{h}\left[\int_{x_0}^{x+h} f(t+iy)dt - \int_{x_0}^{x} f(t+iy)dt\right] = u(x,y) + iv(x,y),$$

by application of the Fundamental Theorem of Calculus to u and v (for a fixed y). This is possible, as f is analytic and therefore u and v are differentiable. This shows $F_x(z) = f(z)$.

Repeat all of the above, but this time starting with curve μ_z instead of γ_z; here, μ_z is the unique curve joining z_0 and z by a horizontal and then a vertical line. Note that

$$F(z) = \int_{\gamma_z} f(\xi)d\xi = \int_{\mu_z} f(\xi)d\xi$$

by Theorem $C1$, as $\int_{\gamma_z - \mu_z} f(\xi)d\xi = 0$ (here $\gamma_z - \mu_z$ form a rectangle). Similar to before, we have

$$F(z) = \int_{\mu_z} f(\xi)d\xi = \int_{x_0}^{x} f(t+iy_0)dt + i\int_{y_0}^{y} f(x+it)dt.$$

Then

$$\begin{aligned} F_y(z) &= \lim_{h\to 0} \frac{i}{h} \int_{y}^{y+h} f(x+it)dt \\ &= if(x+iy) = if(z). \end{aligned}$$

To show F is analytic, let f be as before and $F = U(\cdot,\cdot) + iV(\cdot,\cdot)$. Then $U_x = u, V_x = v, U_y = -v$, and $V_x = u$ as $F_x = f$, $F_y = if$. This means $U_x = V_y$ and $U_y = -V_x$, where U and V both have continuous partial derivatives. This means F must be analytic. It is now clear that $F' = f$ on D, and therefore $\int_{\gamma} f(z)dz = 0$. □

Remark:

(i) If f is analytic on an open disk, then it has an analytic anti-derivative; i.e., there exists an F which is analytic on the disk with $F' = f$.

(ii) An almost identical proof of the previous theorem works if we replace "disk" with "rectangle" instead.

Definition 6.22. *Suppose Ω is a region, $a \in \Omega$, and f is analytic on $\Omega \setminus \{a\}$. Suppose $\lim_{z\to a}(z-a)f(z) = 0$. Then a is called a removable singularity of f.*

Lemma 6.23. *Suppose Ω, a, f are as above, and a is a removable singularity for f. Let R be a rectangle contained in Ω, and suppose $a \notin \partial R$ (where the boundary is oriented counter clockwise). Then $\int_{\partial R} f(z)dz = 0$.*

Proof. If a does not lie inside R, then take $R \subseteq \Omega' \subseteq \Omega$ where $a \notin \Omega'$. Then f is analytic on Ω' and R is a rectangle inside Ω'. By Theorem $C1$, this means $\int_{\partial R} f(z)dz = 0$.

Therefore, suppose a lies inside R; we know that $a \notin \partial R$. Split R into 9 total rectangles $R_1, \dots, R_9$, where a is contained in R_1. Let $\gamma_1, \dots, \gamma_9$ be their boundaries, oriented counter-clockwise. We know

$$\sum_{i=1}^{9} \int_{\gamma_i} f(z)dz = \int_{\gamma_1} f(z)dz$$

by the previous case. We also know that

$$\int_{\partial R} f(z)dz = \sum_{i=1}^{9} \int_{\gamma_i} f(z)dz,$$

so $\int_{\partial R} f(z)dz = \int_{\gamma_1} f(z)dz$. As a is a removable singularity for f, given $\epsilon > 0$ there exists a $\delta > 0$ such that $|z-a||f(z)| < \epsilon$ whenever $|z-a| < \delta$. Suppose R_1 is chosen so that it satisfies:

(i) R_1 is centered at a;
(ii) $\ell(\gamma_1) = \delta$.

Note that when computing $\int_{\gamma_1} f(z)dz$, we look at $z \in \gamma_1$. For such a point, we have

$$\frac{\delta}{8} \leq |z-a| < \delta,$$

as R_1 is centered at a. Then for $z \in \gamma_1$, we have

$$|f(z)| = \frac{|z-a||f(z)|}{|z-a|} \leq \frac{\epsilon}{\frac{\delta}{8}} = \frac{8\epsilon}{\delta}.$$

This means

$$\left| \int_{\gamma_1} f(z)dz \right| \leq \left(\max_{z \in \gamma_1} |f(z)| \right) \ell(\gamma_1) \leq \left(\frac{8\epsilon}{\delta} \right) \delta = 8\epsilon.$$

As $\epsilon > 0$ was arbitrary, letting it go to zero we have $\int_{\gamma_1} f(z)dz = 0$. This completes the proof. □

Theorem 6.24 (Analytic extension)**.** *Let Ω, a, f be as before, where a is a removable singularity for f. Then there exists an analytic function f^* on Ω such that $f(z) = f^*(z)$ for all $z \in \Omega \setminus \{a\}$. The function f^* is called an analytic extension of f.*

Question: Is analytic extension f^* unique?

Answer: Yes! This holds, as

(i) $f^*(z) = f(z)$ on all $z \in \Omega \setminus \{a\}$;
(ii) $f^*(a) = \lim_{z \to a} f^*(z) = \lim_{z \to a} f(z)$.

Example: Let $\Omega = \mathbb{C}$, and $f : \mathbb{C} \to \mathbb{C}$ where

$$f(z) = \begin{cases} \frac{1}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

The function f is analytic on $\Omega \setminus \{0\}$, but does not have an analytic extension to all of $\mathbb{C}$.

Proof. (Proof of the theorem above) Let R be a rectangular curve in Ω around a. We note that f is continuous on γ. Define

$$g(\xi) = \int_\gamma \frac{f(\xi)}{\xi - z} d\xi,$$

for all $z \in \Omega \setminus \gamma$. Suppose $z \neq a$, and fix z for now. Define

$$h_z(\xi) = \frac{f(\xi) - f(z)}{\xi - z}.$$

We note that h_z is analytic on $\Omega \setminus \{a, z\}$, but h_z has removable singularities at a and z. This holds, as

$$\lim_{\xi \to a} (\xi - a) h_z(\xi) = \lim_{\xi \to a} (\xi - a) \frac{f(\xi) - f(a)}{\xi - a} = \lim_{\xi \to a} \frac{-f(z)}{\xi - z} (\xi - a)$$

as a is a removable singularity of f. However,

$$\lim_{\xi \to a} \frac{-f(z)}{\xi - a} (\xi - a) = 0;$$

this establishes a is a removable singularity for h_z. We also see

$$\lim_{\xi \to z} (\xi - z) h_z(\xi) = \lim_{\xi \to z} f(\xi) - f(z) = 0,$$

as f is analytic at z when $z \neq a$. Then by the previous lemma (applied twice), we have

$$\int_\gamma h_z(\xi) d\xi = 0.$$

Then

$$\int_\gamma \frac{f(\xi)}{\xi - z} d\xi - \int_\gamma \frac{f(z)}{\xi - z} d\xi = 0,$$

and so

$$\int_\gamma \frac{f(\xi)}{\xi - z} d\xi = f(z) n(\gamma; z).$$

So if z is in γ, then

$$\int_\gamma \frac{f(\xi)}{\xi - z} d\xi = f(z).$$

This shows $g(z) = f(z)$ for z inside γ. Next, define

$$f^*(z) = \begin{cases} f(z), & z \in \Omega \setminus \{a\} \\ g(z), & z = a. \end{cases}$$

Since f^* and g are equal on a small neighborhood of a, f^* is analytic at a as g is analytic at a. For all other points, $f^* = f$ where f is analytic. This shows f^* is our analytic extension. □

Remark: The lemma works if f has a finite number of removable singularities inside the rectangle R.

Theorem 6.25 (Cauchy's Integral Formula for a Disk)**.** *Suppose f is analytic on an open disk D. Let γ be a closed curve inside D. Then for all $a \in D$ where $a \notin \gamma$, we have*

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)} dz.$$

Proof. Define $g(z) = \frac{f(z)-f(a)}{z-a}$. Function g is analytic on $D \setminus \{a\}$, and a is a removable singularity for g. By the previous theorem, there exists an analytic extension of g, say g^*. Then

$$\int_\gamma g^*(z)dz = 0,$$

by Theorem $C2$. However, we also note

$$\int_\gamma g^*(z)dz = \int_\gamma g(z)dz,$$

as $g^* = g$ on γ. So

$$\int_\gamma \frac{f(z)}{z-a}dz = f(a)\int_\gamma \frac{1}{z-a}dz = 2\pi i f(a)n(\gamma; a).$$

□

Remark: Theorem C2 is enough to study local behavior of analytic functions.

Theorem 6.26. *If f is analytic on a region Ω, then so is f'; hence, f has analytic derivatives of all orders.*

Proof. Fix arbitrary $a \in \Omega$. We know there exists an $R > 0$ such that $B_R(a) \subseteq \Omega$ (where $B_R(a)$ is the open ball of radius R centered at a). Let γ be the counter-clockwise oriented circle centered at a with radius r, where $r < R$. From Cauchy's Integral Formula for a disk, we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi \qquad \text{for all } z \in B_r(a).$$

From Assignment 5, the right hand side of the equation above is analytic on $B_r(a)$, and

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}d\xi, \qquad \text{for all } z \in B_r(a).$$

□

Recall- suppose γ is a closed curve in $\mathbb{C}$, and φ is a complex valued function which is continuous on γ. For all $n \in \mathbb{N}$, define

$$F_n(z) = \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n}d\xi, \qquad \text{for all } z \in \mathbb{C} \setminus \gamma.$$

Then F_n is analytic and $F_n'(z) = nF_{n+1}(z)$.

Corollary 6.27 (Generalized Cauchy Integral Formula on a Disk)**.** *Let f be analytic on an open disk D, and γ a closed curve in D. Then for all $n \in \mathbb{N}$, for $z \in D \setminus \gamma$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Theorem 6.28. *Suppose $\{f_n\}$ is a sequence of analytic functions on Ω, and suppose $f_n \to f$ uniformly on every compact subset of Ω. Then f is analytic on Ω. Moreover, $f_n' \to f'$ uniformly on every compact subset of Ω.*

Proof. Let $a \in \Omega$ be fixed. Let $\overline{B_R(a)} \subseteq \Omega$. We know that $f_n \to f$ uniformly on $\overline{B_R(a)}$- so f is continuous on $B_R(a)$. Let γ be a circle centered at a of radius r oriented counterclockwise, with $r < R$. By Cauchy's Integral Formula, for all $n \in \mathbb{N}$ we have

$$f_n(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f_n(\xi)}{\xi - z} d\xi, \qquad \text{for all } z \in \gamma.$$

Note that $z \in \gamma$ means z is *within* γ but not on γ itself. Since $f_n \to f$ uniformly on γ, by properties of Riemann integration we have

$$\begin{aligned} \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i} \lim_{n\to\infty} \int\limits_\gamma \frac{f_n(\xi)}{(\xi - z)} d\xi &= \frac{1}{2\pi i} \int\limits_\gamma \lim_{n\to\infty} \frac{f_n(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int\limits_\gamma \frac{f(\xi)}{\xi - z} d\xi. \end{aligned}$$

Here we note we are considering our complex integral of f as the sum of the integrals of its real and imaginary parts. So

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f(\xi)}{\xi - z} d\xi,$$

for all $z \in B_r(a)$. By Assignment 5, the right hand side of the equation above is analytic, so f is analytic on $B_r(a)$. As $a \in \Omega$ was an arbitrary point, this shows f is analytic everywhere on Ω. Furthermore, again by Assignment 5 for all $n \in \mathbb{N}$ we have

$$f_n'(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f_n(\xi)}{(\xi - z)^2} d\xi$$

and

$$f'(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f(\xi)}{(\xi - z)^2} d\xi$$

for all $z \in B_r(a)$. So

$$|f_n'(z) - f(z)| \leq \frac{1}{2\pi} \int\limits_\gamma \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^2} d\xi.$$

As $f_n \to f$ uniformly on γ (by assumption, as γ is a compact subset of Ω and $|\xi - z|^2 \geq \delta$ if $z \in \overline{B_{r-\delta}(a)}$). Therefore, $f_n' \to f'$ uniformly on $\overline{B_{r-\delta}(a)}$ (for $0 < \delta < r$).

Exercise: check the details of the last part!

By definition of compactness, this carries to uniform convergence on every compact subset of Ω.

Exercise: check that! □

Corollary 6.29. *Let $f(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$. Let $\frac{1}{R} = \limsup_{n\in\mathbb{N}}(|c_n|^{1/n})$. We know partial sum $f_k(z) = \sum_{n=0}^{k} c_n(z-a)^n$ is absolutely and uniformly convergent on $\overline{B_r(a)}$ with $r < R$.*

So f_k is uniformly convergent on compact subsets of $B_R(a)$. If $|z-a| > R$, then our partial sums diverge. So f is analytic on $B_R(a)$; also, $f'(z) = \sum\limits_{n=1}^{\infty} nc_n(z-a)^{n-1}$ on $B_R(a)$.

Theorem 6.30 (Taylor's Theorem)**.** *Suppose f is analytic on region Ω, and let $a \in \Omega$. If $B_R(a) \subseteq \Omega$, then $f(z)$ has a power series representation centered at a, with radius of convergence at least R. The representation is given by*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

for all $z \in B_R(a)$.

Proof. Let $z \in B_R(a)$. Let γ be the circle centered at a of radius r, with $|z-a| < r < R$, oriented counterclockwise. By Cauchy's Integral Formula, we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\xi)}{\xi - z} d\xi, \quad \text{for all } z \in B_r(a).$$

Look at $\frac{1}{\xi - z}$, for $\xi \in \gamma$. We see

$$(*) \frac{1}{\xi - z} = \frac{1}{(\xi - a) - (z-a)} = \frac{1}{\xi - a}\left(\frac{1}{1 - \frac{z-a}{\xi-a}}\right) = \frac{1}{\xi - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n.$$

The last equality holds, as $\left|\frac{z-a}{\xi-a}\right| \le \frac{|z-a|}{r} < \frac{r}{r} = 1$ and $\sum\limits_{n=0}^{\infty} \frac{|z-a|^n}{r^n}$ converges. Then by the Weierstrass M-test, we see

$$\frac{1}{\xi - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\xi-a}\right)^n$$

is uniformly convergent (as a function of γ). So

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(\xi)}{\xi - a}\left(\sum_{n=0}^{\infty}(\frac{z-a}{\xi-a})^n\right)d\xi = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\xi)}{(\xi-a)^{n+1}}d\xi\right)(z-a)^n$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n,$$

by the Generalized Cauchy Integral Formula. Note that the above series converges for every $z \in B_R(a)$- so the radius of convergence is at least R. This completes the proof. □

Exercise: Suppose $f_n, f : \Omega \to \mathbb{C}$ for each $n \in \mathbb{N}$, where Ω is a region. Let $f_n \to f$ uniformly on Ω, and let γ be a curve in Ω. Show $\int\limits_{\gamma} f_n \to \int\limits_{\gamma} f$.

<u>Hint:</u> Parameterize γ as $Z : [a,b] \to \mathbb{C}$. Note that $f_n \circ Z, f \circ Z : [a,b] \to \mathbb{C}$ and $f_n \circ Z \to f \circ Z$ uniformly. Prove that $(f_n \circ Z)Z' \to (f \circ Z)Z'$ uniformly (use the fact that our functions are bounded). Note that

$$\int\limits_{\gamma} f_n = \int\limits_a^b f_n(Z(t))Z'(t)dt = \int\limits_a^b \Re\Big((f_n \circ Z)(t)Z'(t)\Big)dt + i\int\limits_a^b \Im\Big((f_n \circ Z)(t)Z'(t)\Big)dt.$$

Use the inequality $|\Re(w)| \le |w|$ and $|\Im(w)| \le |w|$ for any $w \in \mathbb{C}$ to prove $(f_n \circ Z)Z' \to (f \circ Z)Z'$ uniformly for the real and imaginary parts of the function.

Remark: For a, R as in Taylor's Theorem, for every $r < R$, the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ converges uniformly to f on $\overline{B}_r(a)$. By Corollary 2.20, we have

$$f'(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(z-a)^{n-1}, \qquad \text{for all } z \in B_R(a).$$

Theorem 6.31 (Liouville's Theorem)**.** *If f is analytic on $\mathbb{C}$ (i.e. f is entire) and if f is bounded, then f must be a constant function.*

Proof. Homework exercise! □

Application: Suppose f is entire and bounded below- i.e., there exists a $\delta > 0$ such that $|f(z)| \geq \delta$ for all $z \in \mathbb{C}$. Then f is constant.

Proof. Take $0 < \delta_0 < \delta$. Define $g : \mathbb{C} \to \mathbb{C}$, where $g(z) = \frac{1}{f(z)-\delta_0}$. We know g is entire, and

$$|g(z)| = \frac{1}{|f(z) - \delta_0|} \leq \frac{1}{|f(z)| - \delta_0} \leq \frac{1}{\delta - \delta_0}.$$

By Liouville's Theorem, we know that g must be constant, as a bounded entire function. So there exists a $c \in \mathbb{C}$ such that $\frac{1}{f(z)-\delta_0} = c$ for all $z \in \mathbb{C}$. This implies f must be constant as well. □

Theorem 6.32 (Zeros of analytic functions)**.** *Suppose f is analytic on a region Ω, and $f(a) = 0$. Then either there exists a $\delta > 0$ such that $B_\delta(a) \subseteq \Omega$ and $|f(z)| > 0$ for all $z \in B_\delta(a)\backslash\{a\}$, or for all $R > 0$ such that $B_R(a) \subseteq \Omega$, we have $f|_{B_R(a)} = 0$. Consequently, zeros of non-constant analytic functions on Ω have no accumulation point in Ω.*

Proof. Let $R > 0$ such that $B_R(a) \subseteq \Omega$. By Taylor's Theorem, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ for all $z \in B_R(a)$. If $f^{(n)}(a) = 0$ for every $n \in \mathbb{N}$, then $f = 0$ on $B_R(a)$ (as $\frac{f^{(n)}(a)}{n!} = 0$ when $f^{(n)}(a) = 0$). If not, let $m \in \mathbb{N}$ be such that $f^{(m)}(a) \neq 0$, but $f^{(k)}(a) = 0$ for all $0 \leq k < m$. So

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = (z-a)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^{n-m} = (z-a)^m g(z).$$

Note that $g(z)$ as defined above is analytic, and has a radius of convergence at least R. So $f(z) = (z-a)^m g(z)$ for some $m \geq 1$, where g is analytic. Note that

$$g(a) = \frac{f^{(m)}(a)}{m!} \neq 0.$$

As g is continuous at a, there exists a $\delta > 0$ such that $|g(z)| > 0$ for every $z \in B_\delta(a)$. So $|f(z)| = |z-a|^m |g(z)| > 0$ on $B_\delta(a) \setminus \{a\}$. To finish the proof, we need to show that zeros of analytic functions which are nonconstant are isolated. Let $\{z_n\}$ be a sequence of zeros of f in Ω- so $f(z_n) = 0$ for all $n \in \mathbb{N}$. Suppose $\{z_n\}$ has ann accumulation point $w \in \Omega$. Without loss of generality (as we can just pass to a subsequence), assume $z_n \to w$.
Exercise: finish the proof! □

Notation: If f is analytic on $B_R(a), f(a) = 0$, then $f \neq 0$ on $B_R(a)$. As we have shown above, there exists an $m > 0$ such that $f^{(m)}(a) \neq 0$ and $f^{(k)}(a) = 0$ for all $k < m$. We say a is a zero of order m in this case. If so, there exists an analytic function g on Ω such that $f(z) = (z-a)^m g(z)$, where $g(a) \neq 0$.

The following theorem is a direct application of the previous theorem.

Theorem 6.33. *Suppose f is analytic on a region Ω, and suppose $\{a_n\}$ is a sequence of distinct points in Ω (converging to $a \in \Omega$). Assume $f(a_n) = 0$ for all $n \in \mathbb{N}$. Then $f = 0$ on Ω.*

Proof. If we suppose $a_n \to a$ and $f(a_n) = 0$, as f is analytic, it is continuous- hence $f(a) = 0$. By the previous theorem, this means a cannot be an isolated zero, so there exists an $R > 0$ such that $B_R(a) \subseteq \Omega$ and $f = 0$ on $B_R(a)$. Let $b \in \Omega$ be arbitrary. As Ω is a region, there exists a piecewise linear path γ from a to b inside Ω. Suppose γ is parameterized as $Z : [\alpha, \beta] \to \mathbb{C}$. Let

$$T = \sup\{\tau \in [\alpha, \beta] : f(Z(t)) = 0, \text{ for all } t \in [a, \tau]\}.$$

We note that the supremum is defined, as $\tau = \alpha$ belongs to T (as $Z(\alpha) = a$). We wish to show that $T = \beta$. If we suppose not, let $T < \beta$. By continuity of f, we have $f(Z(T)) = 0$; then by the previous theorem, as $Z(T)$ is not an isolated zero, there exists an $r > 0$ such that $B_r(Z(T)) \subseteq \Omega$ and $f\big|_{B_r(T)} = 0$. However, this would then violate our choice of T, as T is the supremum- but we are able to now find a $T' > T$ such that $f(Z(T')) = 0$. As we have reached a contradiction, this implies that $T = \beta$, so $f(Z(\beta)) = f(b) = 0$. □

Remarks:

(i) If f is analytic on Ω and $f\big|_{B_R(a)} = 0$ for some open disk $B_R(a) \subseteq \Omega$, then $f = 0$ on Ω.

(ii) If f and g are analytic on a region Ω and $f = g$ on $B_R(a)$ for some disk $B_R(a) \subseteq \Omega$, then $f = g$ on Ω.

(iii) If f and g are analytic on region Ω and $f(a_n) = g(a_n)$ where $\{a_n\} \subseteq \Omega$ is a sequence of distinct points converging to $a \in \Omega$ then $f = g$ on Ω.

Theorem 6.34 (Maximum Modulus Principle)**.** *Suppose f is analytic on a region Ω. Suppose f is not constant. Then $\sup\limits_{z\in\Omega} |f(z)|$ is not attained on Ω.*

Proof. Suppose $M = \sup\limits_{z\in\Omega} |f(z)|$. Suppose M is attained at $a \in \Omega$ (i.e. $|f(a)| = M$). Let γ be the circle of radius $r < R$ centered at a and oriented counterclockwise. By Cauchy's Integral Formula, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - a} d\xi.$$

Then

$$M = |f(a)| = \frac{1}{2\pi} \left| \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \right|.$$

We claim for all $\xi \in \gamma$, $|f(\xi)| = M$. Suppose not- i.e., suppose there exists a $\xi_0 \in \gamma$ such that $|f(\xi_0)| < M$. Let $Z : [0, 1] \to \mathbb{C}$ be the parameterization of γ; so $Z(t_0) = \xi_0$ for some $t_0 \in [0, 1]$. As Z is continuous, there exists a $\delta > 0$ such that for all $t \in [t_0 - \delta, t_0 + \delta] \cap [0, 1]$ we have $|f(Z(t))| < M$. Let $\xi_1 = Z(0), \xi_2 = Z(1)$. Decompose $\gamma = \gamma_1 + \gamma_2$, where γ_1 is the arc where $|f(Z(t))| < M$ and γ_2 is everything else. From this, we have

$$M = |f(a)| = \frac{1}{2\pi} \left| \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \right| \le \frac{1}{2\pi} \left[\left| \int_{\gamma_1} \frac{f(\xi)}{\xi - a} d\xi \right| + \left| \int_{\gamma_2} \frac{f(\xi)}{\xi - a} d\xi \right| \right]$$

$$\le \frac{1}{2\pi} (\max_{\xi\in\gamma_1} |f(\xi)|) \left(\frac{1}{r}\right) \ell(\gamma_1) + \frac{1}{2\pi} (\max_{\xi\in\gamma_2} |f(\xi)|) \left(\frac{1}{r}\right) \ell(\gamma_2) < \frac{1}{2\pi} (M) \left(\frac{1}{r}\right) \ell(\gamma_1) + \frac{1}{2\pi} (M) \left(\frac{1}{r}\right) \ell(\gamma_2)$$

$$= \frac{M}{2\pi r} (\ell(\gamma_1) + \ell(\gamma_2)) = \frac{M}{2\pi r} (2\pi r) = M.$$

This shows $M < M$- a clear contradiction. Therefore, we conclude that $|f(\xi)| = M$ for all $\xi \in \gamma$. As $r < R$ was arbitrary, this means $|f(\xi)| = M$ for all $\xi \in B_R(a)$. Note that f is analytic on $B_R(a)$, and $|f|$ is constant on $B_R(a)$- so by Assignment 4, this means f is constant on $B_R(a)$. Then by the previous corollary (where we consider the constant value c which is equal to f as an analytic function on a disk), this means $f = c$ on Ω. However, this contradicts f being non-constant on Ω, and so we conclude that $|f|$ does not attain its maximum in Ω. ☐

Definition 6.35. *Let $\gamma_1, \ldots, \gamma_n$ be curves in a region Ω. A formal sum $\gamma_1 + \cdots + \gamma_n$ is called a chain. We define*

$$\int_{\gamma_1+\cdots+\gamma_n} f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

for any continuous function f which is defined on $\gamma_1, \ldots, \gamma_n$. We say two chains $\gamma_1 + \cdots + \gamma_n$ and $\sigma_1 + \cdots + \sigma_m$ are identical if the integrals of f over each are equal for all continuous functions f defined on all individual curves.

Definition 6.36. *A chain is a cycle if it can be represented as a sum of closed curves in Ω.*

Definition 6.37. *If $\gamma = \gamma_1 + \cdots + \gamma_n$ is a cycle represented using closed curves $\gamma_1, \ldots, \gamma_n$ then for all $a \in \mathbb{C} \setminus \gamma$ we define*

$$n(\gamma; a) = n(\gamma_1; a) + \cdots + n(\gamma_n; a).$$

Note: The winding number $n(\gamma; a)$ is independent of our representation of the cycle.

Theorem 6.38 (Cauchy's Theorem- Final Form)**.** *Suppose f is analytic on a region Ω and γ is a cycle in Ω. If $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus \gamma$, then*

$$\int_{\gamma} f(z)dz = 0.$$

Remarks:

(i) If Ω is simply connected (i.e. $\sim \Omega$ has no holes), then for any closed curve γ_i in Ω and $a \in \mathbb{C} \setminus \gamma_i$ we have $n(\gamma_i; a) = 0$.

Proof. We have

$$n(\gamma_i; a) = \frac{1}{2\pi i} \int_{\gamma_i} \frac{d\xi}{\xi - a} = 0,$$

as $\frac{1}{\xi - a}$ is analytic on Ω if $a \in \mathbb{C} \setminus \gamma_i$ (and by Cauchy-Goursat). So $n(\gamma; a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$. From this, we see the conditions for the previous theorem are satisfied if Ω is simply connected. ☐

(ii) Suppose Ω is not simply connected and contains two "holes". Let γ_2 contain the first hole, γ_3 contain the second hole, and γ_1 enclose both γ_2 and γ_3 but oriented opposite of the two. Then $n(\gamma_1 + \gamma_2 + \gamma_3; a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$.

Theorem 6.39 (Cauchy's Integral Formula- Final Form)**.** *Suppose f is analytic on Ω and γ is a cycle in Ω such that $n(\gamma;a)=0$ for all $a\in\mathbb{C}\setminus\Omega$. Then*

$$n(\gamma;z)f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-z}d\xi, \qquad \text{for all } z\in\Omega\setminus\gamma.$$

Proof. Use the final form of Cauchy's Theorem; it is identical to the proof of Cauchy's Integral Formula in the simplified form. □

6.4. **Laurent series and the Residue Theorem.**

Proposition 6.40. *Let $R:=\limsup_{n\to\infty}|c_n|^{1/n}$. Then $\sum_{n=0}^{\infty}\frac{c_n}{(z-a)^n}$ is convergent if $|z-a|>R$ and divergent if $|z-a|<R$. The above series is uniformly convergent on the set*

$$\{z\in\mathbb{C}:|z-a|\geq R+\delta\},$$

for every $\delta>0$. Furthermore, if we define f as $f(z)=\sum_{n=0}^{\infty}\frac{c_n}{(z-a)^n}$ for $|z-a|>R$ then f is analytic on $\mathbb{C}\setminus\overline{B_R(a)}$ and $f'(z)=\sum_{n=0}^{\infty}\frac{-nc_n}{(z-a)^{n+1}}$.

Proof. We note that

$$\limsup_{n\to\infty}\Big|\frac{c_n}{(z-a)^n}\Big|^{1/n}=(\limsup_{n\to\infty}|c_n|^{1/n})\frac{1}{|z-a|}=\frac{R}{|z-a|}.$$

So by the Root Test, the above series converges for every $z\in\mathbb{C}$ satisfying $\frac{R}{|z-a|}<1$, and diverges for every $z\in\mathbb{C}$ satisfying $\frac{R}{|z-a|}>1$. Also, note that if $|z-a|\geq R+\delta$ for some fixed $\delta>0$, then

$$\Big|\frac{c_n}{(z-a)^n}\Big|\leq\frac{|c_n|}{(R+\delta)^n}.$$

Then as $\limsup_{n\to\infty}\frac{|c_n|^{1/n}}{R+\delta}<1$, by the Weierstrass M-test we have uniform convergence of the series on $\mathbb{C}\setminus B_{R+\delta}(a)$. Note that when we say "uniform convergence", we mean the sequence of partial sums converges uniformly to our function f. Finally, by a previous proposition we know that as f is uniformly convergent on every compact subset of $\mathbb{C}\setminus\overline{B_R(a)}$, then f is analytic with f' as described above. □

Remark: The proposition mentioned previously deals with $f_n\to f$ pointwise on Ω such that

(i) f_n is analytic;
(ii) for all $K\subseteq\Omega$, K compact we have $f_n\to f$ uniformly on K.

Definition 6.41. *Consider the Laurent series $\sum_{n=-\infty}^{\infty}A_n(z-a)^n$ where $A_n,a\in\mathbb{C}$ and z is a complex variable. For fixed z, if $\sum_{n=0}^{\infty}A_n(z-a)^n$ and $\sum_{n=-\infty}^{-1}A_n(z-a)^n$ both converge, we say the Laurent series converges. The sum of the Laurent series is the sum of the two subseries.*

Proposition 6.42. *Let $R_1=\limsup_{n\to\infty}|A_{-n}|^{1/n}$ and $R_2=\frac{1}{\limsup_{n\to\infty}|A_n|^{1/n}}$. If $R_1<R_2$, the Laurent series is absolutely convergent on*

$$\{z\in\mathbb{C}:R_1<|z-a|<R_2\}.$$

Furthermore, the series is uniformly convergent on the domain

$$\{z \in \mathbb{C} : R_1 + \delta \leq |z-a| \leq R_2 - \delta\},$$

for any $\delta > 0$*. If* $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ *on* $R_1 < |z-a| < R_2$ *then* f *is analytic, and* $f'(z)$ *is given by termwise differentiation.*

Proposition 6.43. *Suppose* f *is analytic on* Ω *and there exists* R_1, R_2 *where* $0 < R_1 < R_2$ *such that*

$$\{z \in \mathbb{C} : R_1 \leq |z-a| \leq R_2\} \subseteq \Omega.$$

Then $f(z)$ *has a unique Laurent expansion valid in* $R_1 < |z-a| < R_2$*.*

Proof. Let γ_1 be the circle of radius R_1 centered at a oriented counter-clockwise, and let γ_2 be the circle of radius R_2 centered at a and oriented clockwise. By assumption, we know γ_1, γ_2 lie in Ω. Consider $\gamma = \gamma_1 + \gamma_2$ as a cycle in Ω. Then for all $p \notin \gamma$, we have

$$n(\gamma; p) = n(\gamma_1; p) + n(\gamma_2; p) = \begin{cases} 1 \text{ if } R_1 < |a-p| < R_2, \\ 0 \text{ otherwise.} \end{cases}$$

So $n(\gamma; p) = 0$ for all $p \in \mathbb{C} \setminus \Omega$. Then by Cauchy's Integral Formula (final form), we have for all $z \in \Omega \setminus \gamma$

$$n(\gamma; z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\xi)}{\xi - z} d\xi + \int_{\gamma_2} \frac{f(\xi)}{\xi - z} d\xi.$$

In particular, for $R_1 < |z-a| < R_2$ then $f(z) = f_1(z) + f_2(z)$ (where $f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\xi)}{\xi - z} d\xi$). We note that f_1 is analytic on $\mathbb{C} \setminus \gamma_1$, and in particular on $|z-a| > R_1$. Similarly, f_2 is analytic on $|z-a| < R_2$. Since f_2 is analytic on a disk, it has a unique Taylor series expansion. To deal with f_1, we look at the change of variable $w = \frac{1}{z-a}$. Define $g_1(w) = f_1(a + \frac{1}{w})$; g_1 is analytic on $\left|a + \frac{1}{w} - a\right| = \left|\frac{1}{w}\right| > R_1$, i.e. on $0 < |w| < \frac{1}{R_1}$. We wish to show that $w = 0$ is a removable singularity for g_1. To do this, we check:

$$\lim_{z \to \infty} |f_1(z)| \leq \lim_{z \to \infty} \frac{1}{2\pi} \int_{\gamma_1} \frac{|f(\xi)|}{|\xi - z|} |d\xi| = 0,$$

as $|f(\xi)|$ is bounded (as f is a continuous function on a compact disk) but $|\xi - z| \to \infty$ as $z \to \infty$. So

$$\lim_{w \to 0} |g_1(w)| = \lim_{w \to 0} |f_1(a + \frac{1}{w})| = 0.$$

This shows $w = 0$ is a removable singularity. Then g_1 has an analytic extension on $|w| < \frac{1}{R_1}$; through a slight abuse of notation, let this analytic extension be denoted by g_1. By Taylor's Theorem, $g_1(w) = \sum_{n=0}^{\infty} a_n w^n$ on $|w| < \frac{1}{R_1}$- so

$$f_1(z) = g_1(\frac{1}{z-a}) = \sum_{n=0}^{\infty} a_n (z-a)^{-n}$$

on $|z-a| > R_1$. This means

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

where $c_0 = a_0 + b_0$ and the other coefficients are taken from their corresponding Taylor subseries (depending on whether $n > 0$ or $n < 0$). ☐

Definition 6.44. *Let $\Omega \subseteq \mathbb{C}$ be a region, and $a \in \Omega$. Suppose f is analytic on $\Omega \setminus \{a\}$- then a is an isolated singularity of f.*

Types of isolated singularities

(i) If $\lim_{z\to a} f(z)(z-a) = 0$, then a is a removable singularity.
(ii) If a is not a removable singularity but there exists an $m \in \mathbb{N}$ such that $\lim_{z\to a}(z-a)^{m+1}f(z) = 0$, then a is called a pole. If m is the smallest such number satisfying the limit above, we say a is a pole of order m.
(iii) If a is neither a removable singularity or a pole, then a is called an essential singularity.

Remark: If a is an isolated singularity, then there exists an $R > 0$ such that f is analytic on $0 < |z-a| < R$. So f has Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ on $0 < |z-a| < R$.

(i) If a is a removable singularity, $c_n = 0$ for all $n < 0$.
(ii) If a is a pole of order m (for $m \in \mathbb{N}$), then $c_n = 0$ for all $n < -m$, while $c_{-m} \neq 0$.
(iii) If $c_n \neq 0$ for an infinite number of integers less than 0, then a is an essential singularity.

Exercise! Check the remark above.

Recall:

(i) Let f be analytic on $B_R(z_0)$. Suppose the Taylor series of f above z_0 is given by $a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots$, where $a_n \neq 0$. We say z_0 is a zero of order n.
(ii) Suppose f has an isolated singularity at z_0; i.e., f is analytic on $0 < |z-z_0| < r$. Suppose the Laurent series of f is

$$\frac{b_{-n}}{(z-z_0)^n} + \frac{b_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{b_{-1}}{z-z_0} + b_0 + b_1(z-z_0) + \cdots ,$$

where $b_{-n} \neq 0$. Then z_0 is a pole of f of order n.

Definition 6.45. *If f has an isolated singularity at a, then the residue of f at $z = a$ is defined to be c_{-1}, where $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ is the Laurent expansion of f about a.*

Theorem 6.46 (Residue Theorem)**.** *Suppose f is analytic on Ω except for the isolated singularities $a_j, j = 1, 2, 3, \ldots,$. If γ is any cycle in $\Omega \setminus \{a_j : j \in \mathbb{N}\}$ such that $n(\gamma; p) = 0$ for all $p \in \mathbb{C} \setminus \gamma$, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{j\in\mathbb{N}} n(\gamma; a_j) Res_{a_j} f,$$

where $Res_{a_j} f$ is the residue of f at $z = a_j$.

Example: Let γ be the circle centered at 0 of radius 5 oriented counter clockwise. Suppose we wish to compute

$$\int_\gamma \frac{\sin z}{z^3(z-2)^2(z-7)} dz.$$

We note that as $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, then $\sin z$ is entire. Let $g(z) = \frac{\sin z}{z^3(z-2)^2(z-7)}$. We see g has singularities at $z_0 = 0, z_1 = 2$, and $z_3 = 7$, all of which are poles.

Exercise! Find the order of the poles.

So

$$\int_\gamma g(z)dz = 2\pi i\Big(n(\gamma;z_0)\text{Res}_{z_0}g + n(\gamma;z_1)\text{Res}_{z_1}g\Big).$$

To find $\text{Res}_{z_0}g$, define $f(z) = \frac{\sin z}{(z-2)^2(z-7)}$. This function is clearly analytic at $z=0$, so f has a Taylor expansion at z_0 on some ball $B_R(z_0)$ given by $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}z^n$. The Laurent series of g about z_0 (on some annulus $0 < |z - z_0| < R$) is given by

$$g(z) = \frac{1}{z^3}f(z) = \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n.$$

This means c_{-1} is equal to the coefficient of z^2 in the Taylor series of f: $\text{Res}(f;z_0) = \frac{f^{(2)}(0)}{2!}$.

For $z_1 = 2$, define $h(z) = \frac{\sin z}{z^3(z-7)}$. Then $g(z) = \frac{h(z)}{(z-2)^2}$, and h is analytic at z_1. We therefore have a Taylor series expansion about 2 on some ball $B_{R'}(z_1)$, given by

$$h(z) = \sum_{n=0}^{\infty}\frac{h^{(n)}(2)}{n!}(z-2)^n.$$

Then

$$g(z) = \frac{1}{(z-2)^2}h(z) = \sum_{n=0}^{\infty}\frac{h^{(n)}(2)}{n!}(z-2)^{n-2}.$$

This tells us $\text{Res}_{z_1}g = h'(2)$.

Example: Take γ as before, and try to compute $\int_\gamma e^{1/z}dz$. Let $f(z) = e^{1/z}$. Clearly f is analytic on $\mathbb{C}\setminus\{0\}$, and $z_0 = 0$ is an essential isolated singularity. The Taylor series of e^z is given by

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

This means the Laurent series for $e^{1/z}$ is

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!z^n}.$$

This means $\text{Res}_{z_0}f = 1$ (as the coefficient when $n = 1$). So

$$\int_\gamma f(z)dz = 2\pi i.$$

Example: We will look at the integral

$$\int_0^{\infty}\frac{\sin x}{x}dx.$$

Note: $\lim_{x\to 0}\frac{\sin x}{x} = 1$, and is continuous; however, $f(x) = \frac{\sin x}{x}$ is not Lebesgue integrable. This is because $\int_0^{\infty}\left|\frac{\sin x}{x}\right|dx = \infty$.

We first note that as $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$, then

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

Additionally, we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x} dx = \lim_{R\to\infty} \lim_{\epsilon\to 0} \Big[\int_{-R}^{-\epsilon} \frac{\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\sin x}{x} dx \Big].$$

Let γ_1 be the line segment on the real axis connecting $-R$ and $-\epsilon$, let γ_2 be the half arc connecting $-\epsilon$ and ϵ, let γ_3 be the line segment connecting ϵ and R, and let γ_4 be the half arc connecting R and $-R$. If we let $\gamma = \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4$, we have a closed half-annulus (with a specific orientation). Let $g(z) = \frac{e^{iz}}{z}$. We note $\int_{\gamma} g(z)dz = 0$, as g is analytic inside and on γ. This means

$$\int_{\gamma_1+\gamma_3} g(z)dz = \int_{\gamma_2+\gamma_4} g(z)dz.$$

Additionally, we note

$$\int_{\gamma_1+\gamma_3} g(z)dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x} dx.$$

Clearly, we then see

$$\Im\Big[\int_{\gamma_1+\gamma_3} g(z)dz \Big] = \int_{-R}^{-\epsilon} \frac{\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\sin x}{x} dx$$

(as we are working on γ_1, γ_3 which are entirely real). To calculate $\int_{\gamma_2} g(z)dz$ and $\int_{\gamma_4} g(z)dz$, we parameterize the curves as follows:

$$\gamma_2 : Z_2 : [0, \pi] \to \mathbb{C}, \quad \theta \mapsto \epsilon e^{i\theta},$$
$$\gamma_4 : Z_4 : [0, \pi] \to \mathbb{C}, \quad \theta \mapsto R e^{i\theta}.$$

From this parameterization, we see

$$\int_{\gamma_2} g(z)dz = \int_{0}^{\pi} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} (i\epsilon e^{i\theta}) d\theta = i \int_{0}^{\pi} e^{i\epsilon(\cos\theta + i\sin\theta)} d\theta.$$

Similarly, we have

$$\int_{\gamma_4} g(z)dz = i \int_{0}^{\pi} e^{iR(\cos\theta + i\sin\theta)} d\theta.$$

We first take

$$\lim_{\epsilon\to 0} \int_{\gamma_2} g(z)dz = i \lim_{\epsilon\to 0} \int_{0}^{\pi} e^{i\epsilon(\cos\theta + i\sin\theta)} d\theta = \pi i.$$

We also calculate

$$\lim_{R\to\infty}\left|\int\limits_{\gamma_4} g(z)dz\right| = \lim_{R\to\infty}\left|i\int\limits_0^\pi e^{iR(\cos\theta+i\sin\theta)}d\theta\right|$$

$$\leq \lim_{R\to\infty}\int\limits_0^\pi \left|e^{iR\cos\theta-R\sin\theta}d\theta\right| = \lim_{R\to\infty}\int\limits_0^\pi e^{-R\sin\theta}d\theta = 0.$$

Therefore, $\lim\limits_{\gamma_4} g(z)dz = 0$. Putting everything together, this tells us that

$$\Im\left[\lim_{\substack{R\to\infty\\ \epsilon\to 0}}\int\limits_{\gamma_1+\gamma_3} g(z)dz\right] = \lim_{\substack{R\to\infty\\ \epsilon\to 0}}\left[\int\limits_{-R}^{-\epsilon}\frac{\sin x}{x}dx + \int\limits_{\epsilon}^{R}\frac{\sin x}{x}dx\right]$$

$$= \int\limits_{-\infty}^{\infty}\frac{\sin x}{x}dx = \Im(\pi i) = \pi.$$

Therefore,

$$\int\limits_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$