QUANTUM MORPHISMS

G.H.

Contents

1. Introduction

These notes were taken in the University of Waterloo's "Quantum Morphisms" course, taught by Dr. David Roberson in Summer 2021. I typed them based on hand-written notes taken during class each week- the hope was that a typed version would provide a better record in the future and be much more useful. Material for the lectures was taken at least partially from:

- `arxiv.org/abs/1910.06958` Mancinska, Roberson. "Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs"
- `arxiv.org/abs/2004.10893` Mancinska, Roberson, Varvitsiotis. "Graph isomorphism: Physical resources, optimization models, and algebraic characterizations"
- `https://uwspace.uwaterloo.ca/handle/10012/7814` David Roberson's Ph. D. thesis

These notes are a work in progress; all mistakes are mine and mine alone (either through mistyping or a misunderstanding of the material). If you have any error corrections, tips, or general comments, please reach out to me at: ghoefer@udel.edu.

2. Quantum basics, and nonlocal games

2.1. Quantum states and measurements.

Basic notation:

(i) $|\phi\rangle$- a column vector (a "ket");
(ii) $\langle\phi|$- a row vector, the conjugate transpose of $|\phi\rangle$ (a "bra");

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(iii) $|i\rangle \in \mathbb{C}^n$- the i^{th} standard basis vector (where usually $i \in \{1, \dots, n\}$ or $\{0, \dots, n-1\}$);
(iv) $|x\rangle \in \mathbb{C}^Y$- the standard basis vector indexed by $x \in Y$ for some set Y;
(v) $\langle \phi|\varphi\rangle$- the inner product of $|\phi\rangle$ and $|\varphi\rangle$ (the result is a scalar);
(vi) $|\phi\rangle\langle\varphi|$- the outer product of $|\phi\rangle$ and $|\varphi\rangle$ (the result is a matrix);
(vii) $\mathbb{F}^{m\times n}$- the space of $m \times n$ matrices over $\mathbb{F}$.

Matrices in $\mathbb{C}^{n\times n}$.
The majority of the operators we'll be working with are one of the following types:

- Hermitian- for matrix M, we have $M^* = M$ (i.e. $M_{ij} = \overline{M_{ji}}$);
- Positive semi definite (psd): for matrix M, we have M is Hermitian and
$$\langle\phi|M|\phi\rangle \geq 0$$
for all $|\phi\rangle \in \mathbb{C}^n$;
- Unitary: for matrix M, we have $M^*M = MM^* = I$.

Note: These matrices are all *normal*, i.e. $M^*M = MM^*$.

Spectral Decomposition.
Using basics of linear algebra, for a complex normal matrix M in $\mathbb{C}^{n\times n}$ there exists an orthonormal basis $|\phi_1\rangle, \dots, |\phi_n\rangle \in \mathbb{C}^n$ such that
$$M = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i|,$$
where λ_i are the eigenvalues of M. We will often write $M = \sum_\lambda \lambda E_\lambda$, where
$$E_\lambda = \sum_i |\phi_i\rangle\langle\phi_i|.$$
Then
$$E_\lambda^2 = E_\lambda^* = E_\lambda,$$
$$E_\lambda E_\mu = \delta_{\lambda\mu} E_\lambda,$$
$$\sum_\lambda E_\lambda = I.$$
As may be familar to us, if a matrix M is Hermitian, then it has solely real eigenvalues. If a matrix M is positive semi definite, all eigenvalues are non-negative, and hence $M^{1/2}$ is well-defined. Finally, if M is unitary, the eigenvalues all have unit modulus, and so they can all be written in the form $\lambda = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$ (here λ is an eigenvalue of M).

Quantum States.

Definition 2.1. *A quantum state is a description of a quantum system.*

Definition 2.2. *A pure quantum state is a unit vector $|\phi\rangle \in \mathbb{C}^n$; note that here we use $\mathbb{C}^n$ as a representation of our system, and so it is the "state space" for any possible state our system may be in.*

Basic physics fact: An electron can have spin up $|\uparrow\rangle$, spin down $|\downarrow\rangle$, or any superposition of the two: $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$.

The fact above is really just a specific re-statement of the more general *Principle of Superposition.*

Principle of Superposition: If a quantum system can be in orthogonal states ϕ_1 and ϕ_2, then it can be in any state of the form $\alpha\phi_1 + \beta\phi_2$, where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$.

Question: What linear operations map states to states within our space?

Answer: Any linear operation which maps states to states must be of the form $M : \mathbb{C}^n \to \mathbb{C}^n$ which satisfies

$$\langle\phi|M^*M|\phi\rangle = 1, \qquad \text{for all unit vectors } |\phi\rangle \in \mathbb{C}^n.$$

Exercise: Show this implies that M^*M must be I, and so M is a unitary operator.

General Quantum States.
Suppose we know a given system is in a state $|0\rangle \in \mathbb{C}^2$ with probability $\frac{1}{2}$, and is in state $|1\rangle \in \mathbb{C}^2$ with probability $\frac{1}{2}$. How would we describe this?

Example 2.3. *Try and let ϕ be the state of the system above, and represent it as*

$$\phi = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Note that ϕ is not a unit vector, and so such a representation doesn't work.

To represent the state of a given system (when it is not necessarily a pure quantum state), we use density matrices instead.

Definition 2.4. *Let $\rho \in \mathbb{C}^{n\times n}$ be a matrix such that ρ is positive semi definite, and* $\mathrm{Tr}(\rho) = 1$.

For a pure quantum state $|\phi\rangle \in \mathbb{C}^n$, its typical density matrix can be obtained by letting $\rho = |\phi\rangle\langle\phi| \in \mathbb{C}^{n\times n}$. For mixed states $|\phi_1\rangle, \ldots, |\phi_n\rangle$ with corresponding probabilities $p_1, \ldots, p_n$, the density matrix is given by $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$. Similarly, for an ensemble with $\rho_1, \ldots, \rho_n$ and corresponding probabilities $p_1, \ldots, p_n$ the density matrix is given by $\rho = \sum_i p_i\rho_i$.

Example 2.5. *As an example of a mixed state, look at*

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise: Show that if $|\phi_0\rangle$ and $|\phi_1\rangle$ are an orthonormal basis for $\mathbb{C}^2$, then the density matrix for the ensemble $|\phi_i\rangle$ with probabilities $\frac{1}{2}$ for $i = 0, 1$ is still $\frac{1}{2}I$. Generalize this to $\mathbb{C}^n$.

Measurements.

Definition 2.6. *A quantum measurement is a tuple $\mathcal{M} = (M_1, \ldots, M_k)$ with $M_i \in \mathbb{C}^{m\times n}$ such that*

$$\sum_{i=1}^{k} M_i^* M_i = I.$$

When a state $\rho \in \mathbb{C}^{n\times n}$ is measured with $\mathcal{M}$ it results in:

(i) A classical outcome $i \in [k]$ with probability

$$\mathrm{Tr}(M_i \rho M_i^*) = \mathrm{Tr}(\rho M_i^* M_i).$$

Note that $\mathrm{Tr}(\rho M_i M_i^*) \geq 0$, as $M_i^* M_i$ is always positive semi definite; as the trace of a complex valued matrix is the sum of its eigenvalues, all eigenvalues are non-negative, which yields our comment. We need to check as well that this is a legitimate probability distribution if any of this will work- by the previous comment, we know it is always non-negative. Furthermore, we see

$$\sum_i \mathrm{Tr}(\rho M_i^* M_i) = \mathrm{Tr}(\rho \sum_i M_i^* M_i) = \mathrm{Tr}(\rho I) = \mathrm{Tr}(\rho) = 1,$$

by choice of ρ. So this is indeed a probability distribution like we need.

In the pure case (i.e. for pure state ϕ) we see

$$\mathrm{Tr}(M_i \left|\phi\right\rangle \left\langle\phi\right| M_i^*) = \langle\phi|M_i^* M_i|\phi\rangle = \|M_i \left|\phi\right\rangle\|^2.$$

(ii) We get a corresponding post-measurement state

$$\frac{1}{\mathrm{Tr}(M_i \rho M_i^*)} M_i \rho M_i^*.$$

This will be a density matrix, as $M_i \rho M_i^* = (M_i \rho^{1/2})(M_i \rho^{1/2})^*$ is positive semi definite. We include the constant factor of $\frac{1}{\mathrm{Tr}(M_i \rho M_i^*)}$ as a normalization factor, so we ensure we receive trace 1 always. Note that if $\mathrm{Tr}(M_i \rho M_i^*) = 0$, our probability for this outcome is always 0, so we have nothing to normalize in the first place (and therefore may disregard technical issues arising when dividing by 0). In the pure case, our post-measurement state is

$$\frac{M_i \left|\rho\right\rangle}{\|M_i \left|\rho\right\rangle\|}.$$

Note:

- We implicitly assume our measurements are compatible with state ρ (i.e. the sizes of matrices match up)
- The outcome probabilities only depend on $M_i^* M_i$, and not the operators M_i themselves.

Definition 2.7. *For M_i as above, the collection of $M_i^* M_i$ form a positive valued measure (POVM), i.e. a tuple $(P_1, \ldots, P_k)$ with $P_i \in \mathbb{C}^{n \times n}$ such that P_i is p.s.d. for all i and*

$$\sum_{i=1}^k P_i = I.$$

Example 2.8. *If we have POVM $(P_1, \ldots, P_k)$, we can get a measurement $\mathcal{M}$ where $\mathcal{M} = (P_1^{1/2}, \ldots, P_k^{1/2})$ (note that there are many other possible ones based on the given POVM).*

Example 2.9. *We look at a full basis measurement of a pure state. Let $\left|\phi\right\rangle \in \mathbb{C}^n$, $\left|\phi_1\right\rangle, \ldots, \left|\phi_n\right\rangle \in \mathbb{C}^n$ be an orthonormal basis and take the measurement given by $M_i = \left|\phi_i\right\rangle \left\langle\phi_i\right|$. We note*

$$M_i^* M_i = \left|\phi_i\right\rangle \langle\phi_i|\phi_i\rangle \left\langle\phi_i\right| = \left|\phi_i\right\rangle \left\langle\phi_i\right| = M_i = M_i^*.$$

The equations above hold, as $\langle\phi_i|\phi_i\rangle = 1$. We obtain the outcome i with probability

$$\langle\phi|M_i^* M_i|\phi\rangle = \langle\phi|\phi_i\rangle \langle\phi_i|\phi\rangle = |\langle\phi_i|\phi\rangle|^2.$$

It has a post-measurement state of

$$\frac{1}{\|M_i \left|\phi\right\rangle\|} M_i \left|\phi\right\rangle = \frac{1}{\|\left|\phi_i\right\rangle \langle\phi_i|\phi\rangle\|} \left|\phi_i\right\rangle \langle\phi_i|\phi\rangle.$$

Composite Systems.

Suppose we have System A, in state $\left|\phi_A\right\rangle \in \mathbb{C}^{d_A}$ (or $\rho_A \in \mathbb{C}^{d_A \times d_A}$ if not pure) and System

B in state $|\phi_B\rangle \in \mathbb{C}^{d_B}$ (or $\rho_B \in \mathbb{C}^{d_B \times d_B}$). Their combined system is represented by the combined state

$$|\phi_A\rangle \otimes |\phi_B\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathbb{C}^{d_A d_B}.$$

In the event the states aren't pure, we have

$$\rho_A \otimes \rho_B \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B} \cong \mathbb{C}^{d_A d_B \times d_A d_B}$$

instead.

Note: To find the tensor product of matrices, just take their Kronecker product.

Definition 2.10. *A pure state $|\phi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is called entangled if it cannot be written as*

$$|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle\,,$$

for states $|\phi_A\rangle \in \mathbb{C}^{d_A}$ and $|\phi_B\rangle \in \mathbb{C}^{d_B}$. A state $\rho \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B}$ is entangled if it cannot be written in the form

$$\rho = \sum_{i=1}^{k} p_i(\rho_A^i \otimes \rho_B^i)$$

for states $\rho_A^i \in \mathbb{C}^{d_A \times d_A}, \rho_B^i \in \mathbb{C}^{d_B \times d_B}$ and probability distribution $p_1, \ldots, p_k$.

Exercise: Show that

$$\phi = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

is entangled. Here $|00\rangle = |0\rangle \otimes |0\rangle$ and $|11\rangle = |1\rangle \otimes |1\rangle$.

Joint Measurements.
Let $\rho \in \mathbb{C}^{d_A \times d_A} \otimes \mathbb{C}^{d_B \times d_B}$. Let Alice have state space $\mathbb{C}^{d_A \times d_A}$, and Bob have state space $\mathbb{C}^{d_B \times d_B}$. Suppose Alice performs measurement $\mathcal{A} = (E_1, \ldots, E_k)$ on her system. This is equivalent to the "global" measurement $(E_1 \otimes I, \ldots, E_k \otimes I)$ being performed on the shared system. This has outcome i with probability

$$\mathrm{Tr}((E_i \otimes I)\rho(E_i^* \otimes I)),$$

and post-measurement state

$$(E_i \otimes I)\rho(E_i^* \otimes I)$$

(where we normalize the last bit). If Bob measures his space with $\mathcal{B} = (F_1, \ldots, F_r)$ "at the same time", this has outcome i, j with probability

$$\mathrm{Tr}((E_i \otimes F_j)\rho(E_i^* \otimes F_j^*)),$$

and post-measurement state

$$(E_i \otimes F_j)\rho(E_i^* \otimes F_j^*)$$

(again, we have to normalize the last bit for things to work out).

Exercise: Show that doing both measurements "at the same time" is equivalent to doing one and then the other. (Hint: write out what the post-measurement state is after Alice performs only her measurement. Then apply Bob's. Does this match with the joint measurement?)

Example 2.11. *Let $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$. We'll measure the first qubit in the standard basis. We have probability*

$$\frac{1}{2}\,\mathrm{Tr}\left((|0\rangle\langle 0| \otimes I)(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\right) = \frac{1}{2}\,\mathrm{Tr}(|0\rangle\langle 0|) = \frac{1}{2}\,\mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \frac{1}{2}.$$

It has post-measurement state

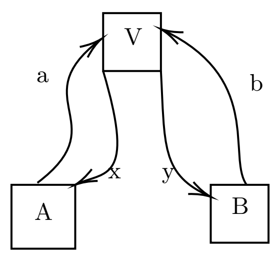
$$M_i |\phi\rangle = (|0\rangle \langle 0| \otimes I)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) = |00\rangle .$$

For outcome 1*, we similarly have a probability of* $\frac{1}{2}$ *and post-measurement state* $|11\rangle$*.*

Exercise: Flesh out the actual details of the last example.

Exercise: Show we can use a full-standard basis measurement with any orthonormal basis, and still receive the same result.

2.2. **Nonlocal games.** We will begin with an informal discussion on non-local games, before delving into a formal definition and more involving strategies. The framework for describing a specific example of a non-local game (the CHSH game) can be illustrated using the following diagram:



There are a few conditions and rules we require for the CHSH game. We should have:

(i) A verifier/referee V who sends Alice and Bob (A and B) the bits x and y, respectively.
(ii) Alice and Bob should respond with bits a and b, respectively.
(iii) We have all bits $x, y, a, b \in \{0, 1\}$.
(iv) The game is played for exactly one round, and Alice and Bob win if $x \wedge y = a \oplus b$.
(v) They *cannot* communicate during the game.

Example 2.12. *Suppose* V *sends* A *the bit* 0 *and* B *the bit* 1*, and both* A *and* B *respond with* 1*. Then*

$$x \wedge y = 0 \wedge 1 = 0,$$
$$a \oplus b = 1 \oplus 1 = 0.$$

This means A *and* B *win the game.*

If we now suppose V *sends* A *and* B *the bit* 1*, and they both respond with the bit* 1*, then*

$$x \wedge y = 1 \wedge 1 = 1,$$
$$a \oplus b = 1 \oplus 1 = 0.$$

This means A *and* B *lose the game. Based on the previous inputs, it is fairly easy to check that if both* A *and* B *respond with the same answer unless both of their inputs are the* 1 *bit, then they'll win the game. This is known as a winning strategy for the CHSH game.*

We can take the framework for a nonlocal game given in the example, and generalize it to deal with any nonlocal game- just remove the specific choice of sets for x, y, a, and b, along with the winning condition. This leads to the formal definition of a nonlocal game.

Definition 2.13. *A nonlocal game is a tuple* $(I_A, I_B, O_A, O_B, \pi, \lambda)$ *where:*

- I_A, I_B, O_A, O_B *are all finite sets;*
- $\pi : I_A \times I_B \to [0, 1]$ *is a probability distribution (often chosen as the uniform distribution);*

- $\lambda : O_A \times O_B \times I_A \times I_B \to \{0,1\}$ *is a "verification" function such that*

$$\lambda(a,b|x,y) = \begin{cases} 1, & \text{if they win,} \\ 0, & \text{if they lose.} \end{cases}$$

Note: For the specific example of the CHSH game, we have

- $I_A = I_B = O_A = O_B = \{0,1\}$;
- $\pi(x,y) = \frac{1}{4}$ (uniformly);
- The function

$$\lambda(a,b|x,y) = \begin{cases} 1, & x \wedge y = a \oplus b, \\ 0, & \text{otherwise.} \end{cases}$$

Classical Strategies:

Definition 2.14. *A deterministic strategy for the game is a pair of functions $f_A : I_A \to O_A$ and $f_B : I_B \to O_B$ such that if Alice receives x, they respond with $f_A(x)$ while if Bob receives y, they respond with $f_B(y)$.*

Example 2.15. *For the CHSH game- let $f_A(x) = f_B(y) = 0$ for all $x, y \in \{0,1\}$. Then*

$$a \oplus b = 0 \oplus 0 = x \wedge y$$

for all $x, y \in \{0,1\}$ unless $x = y = 1$. This is known as the "always 0" strategy. Note that with this strategy they win the game with a probability of $\frac{3}{4}$. Furthermore, this is actually the best possible case for the game when using a classical strategy- we cannot win with higher probability using any other strategy.

Correlations:
As the notation might suggest, the correlation $p(a,b|x,y)$ is the probability that Alice and Bob respond with answers a and b, given they have received x and y as inputs.

Definition 2.16. *A classical probability strategy consists of*

- *A random variable Z taking values in a (finite) set S;*
- *A pair of functions $f_A : I_A \times S \to O_A$, and $f_B : I_B \times S \to O_B$.*

We say Alice and Bob "share" Z. If Z takes the value $s \in S$, then A and B play with deterministic strategy $f_A(\cdot, s), f_B(\cdot, s)$ (so we can "pick" randomly from a collection of strategies overall). This gives the corresponding correlation

$$p = \sum_{s \in S} P(Z = s) p_s,$$

$$p_s(a,b|x,y) = \begin{cases} 1, & \text{if } f_A(x,s) = a \text{ and } f_B(y,s) = b, \\ 0, & \text{otherwise.} \end{cases}$$

Note that p is a convex combination of individual probability correlations.

Definition 2.17. *We say $C_{loc}(O_A, O_B, I_A, I_B)$ is the set of all classical correlations- i.e. the convex hull of classical deterministic correlations.*

Note: C_{loc} is closed.

Definition 2.18. *For a given nonlocal game G, we say the game G has a value $\omega(G)$ where*

$$\omega(G) = \sup_{p \in C_{loc}} \sum_{\substack{x \in I_A, y \in I_B \\ a \in O_A, b \in O_B}} \pi(x,y) p(a,b|x,y) \lambda(a,b|x,y).$$

The product inside the sum gives the probability that a chosen strategy wins the game.

Note:

- As C_{loc} is closed, we may actually replace the supremum in the previous definition with a maximum instead.
- While it is not entirely obvious, the above implies that the best we can do with any classical strategy is winning with probability of $\frac{3}{4}$.

Quantum Strategies:

Definition 2.19. *A quantum strategy consists of:*

- *A shared state* $|\phi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$*;*
- *A POVM* $\mathcal{E}_X = \{E_{xa} \in \mathbb{C}^{d_A \times d_A} : a \in O_A\}$ *for each* $x \in I_A$*;*
- *A POVM* $\mathcal{F}_Y = \{F_{yb} \in \mathbb{C}^{d_B \times d_B} : b \in O_B\}$ *for each* $y \in I_B$*.*

The strategy has the corresponding correlation

$$p(a, b|x, y) = \langle\phi|(E_{xa} \otimes F_{yb})|\phi\rangle .$$

Definition 2.20. *We say* $C_q(O_A, O_B, I_A, I_B)$ *is the set of all quantum correlations. This set is also convex, but this is not obvious.*

Definition 2.21. *For a given nonlocal game* G*, we say* G *has a quantum value* $\omega^*(G)$ *where*

$$\omega^*(G) = \sup_{p \in C_q} \sum_{\substack{x \in I_A, y \in I_B \\ a \in O_A, b \in O_B}} \pi(x, y) p(a, b|x, y) \lambda(a, b|x, y).$$

Note: The set C_q is not closed (this is highly non-trivial to actually show this).

Example 2.22. *For the CHSH game, start by letting* $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$*. Let*

$$E_0 = (|0\rangle, |1\rangle), E_1 = (|+\rangle, |-\rangle),$$
$$F_0 = (\cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle, -\sin\left(\frac{\pi}{8}\right)|0\rangle + \cos\left(\frac{\pi}{8}\right)|1\rangle),$$
$$F_1 = (\cos\left(\frac{-\pi}{8}\right)|0\rangle + \sin\left(\frac{-\pi}{8}\right)|1\rangle, -\sin\left(\frac{-\pi}{8}\right)|0\rangle + \cos\left(\frac{-\pi}{8}\right)|1\rangle).$$

Vectors $|+\rangle, |-\rangle$ *are the basis vectors for the "plus-minus" basis- it is commonly used in quantum information theory. They are defined such that*

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

Note that F_0 *is the standard basis rotated by* $\frac{\pi}{8}$ *radians, while* F_1 *is the standard basis rotated* $\frac{-\pi}{8}$ *radians.*

Exercise: Show that this strategy wins the CHSH game with probability $\cos^2(\frac{\pi}{8}) \approx 0.854$.

Note: The previous example/exercise shows that a quantum strategy can actually have a higher probability of winning the game than the best possible classical strategy.

The Non-Signalling Property:
For a quantum strategy

$$p(a, b|x, y) = \langle\phi|(E_{xa} \otimes F_{yb})|\phi\rangle ,$$

the probability that Alice responds with a on input x, y is given by the *marginal* $p_A(a|x)$- similarly for Bob. If Alice's marginal satisfies the condition

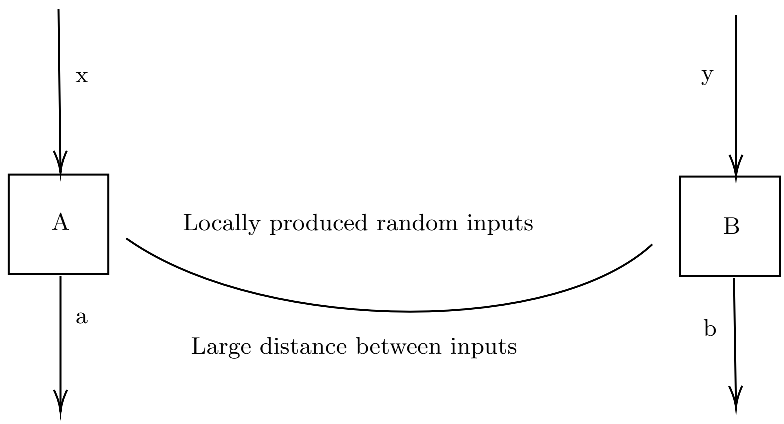
$$p_A(a|x) = \sum_b p(a, b|x, y) = \sum_b \langle\phi|E_{xa} \otimes F_{yb}|\phi\rangle = \langle\phi|E_{xa} \otimes I|\phi\rangle$$

for each a, x and Bob's marginal satisfies

$$p_B(b|y) = \langle\phi|I \otimes F_{yb}|\phi\rangle$$

for each b, y, then the strategy is *non-signalling*.

Physical Significance:
For a little bit of the intuition (in terms of physical principles) behind what is happening with quantum strategies, we turn to the following diagram:



Suppose we have our game, where we interpret it as an experiment set up according to the diagram above. If we require A and B to respond "quickly", the speed of light gives a limit which ensures that A and B cannot detect each other's inputs fast enough before they must answer. This means they cannot communicate during the course of the experiment. If A and B consistently win with high enough probability (i.e. higher than possible with a classical strategy), we can conclude that something non-local is happening within our experiment. In equivalent terms, the behavior of reality cannot be explained by a local hidden variable theory.

Note: There has been experimental verification of this- Aspect et all (2015).

Homomorphisms:

Definition 2.23. *A homomorphism from a graph G to a graph H is a function $f : V(G) \to V(H)$ satisfying*

$$u \sim v \Rightarrow f(u) \sim f(v).$$

In other words, the function is adjacency-preserving.

Notation:

- $f : G \to H$ means that f is a homomorphism from graph G to graph H.
- $G \to H$ means there exists a homomorphism from G to H.

Example 2.24. *The following are a few examples of graph homomorphisms:*

(i) Let $f : C_7 \to C_5$. Label the vertices of C_7 in ascending order, so we have $V(C_7) = \{v_1, \dots, v_7\}$. Similarly, label C_5 so that $V(C_5) = \{w_1, \dots, w_5\}$. Let f map

$$\begin{aligned} v_1 &\mapsto w_1, \\ v_2 &\mapsto w_2, \\ v_3 &\mapsto w_3, \\ v_4 &\mapsto w_4, \\ v_5 &\mapsto w_5, \\ v_6 &\mapsto w_4, \\ v_7 &\mapsto w_5. \end{aligned}$$

We can think of our homomorphism as "collapsing" two vertices onto another.

(ii) $f : G \to K_n$ *is an n-coloring of G. We define*

$$\chi(G) = \min\{n \in \mathbb{N} : G \to K_n\}.$$

(iii) $f : K_n \to G$ *is a clique of size n in G. We define*

$$\omega(G) = \max\{n \in \mathbb{N} : K_n \to G\}.$$

Note: We are aware of the abuse of notation with the definition of the clique number and also the value of a game- context should make it clear which we are using.

Properties of Homomorphisms:
Graph homomorphisms possess:

(i) Transitivity: if $f_1 : G \to H$ and $f_2 : H \to K$, then $f_2 \circ f_1 : G \to K$.
(ii) Reflexivity: $id : G \to G$.
(iii) Homomorphically equivalence: if $G \to H$ and $H \to G$, then we say $G \leftrightarrow H$.

Example 2.25. *The graphs P_4 and P_2 are homomorphically equivalent- just collaps P_4 onto the two vertices in P_2, and similarly just map P_2 onto any two adjacent vertices in P_4.*

Note that homomorphic equivalence does not imply equality. Graph homomorphisms give us a pre-order on the space of all graphs, and a partial order on homomorphic equivalence classes.

(iv) If H is loopless, then $f^{-1}(v)$ is an independent set.

Definition 2.26. *We say $f^{-1}(v)$ is the fiber of vertex v.*

(G, H)**- Homomorphism Game:**

Definition 2.27. *The graph homomorphism game is defined as follows: for two graphs G, H, let $I_A = I_B = V(G)$ and $O_A = O_B = V(H)$. Let*

$$\lambda(h, h'|g, g') = \begin{cases} 1, & \text{if } g = g' \Rightarrow h = h' \text{ and } g \sim g' \Rightarrow h \sim h', \\ 0, & \text{otherwise.} \end{cases}$$

Suppose we have $f_A, f_B : V(G) \to V(H)$, and suppose they give a perfect classical deterministic strategy. The consistency condition of λ in the homomorphism game implies $f_A(g) = f_B(g)$ for all $g \in V(G)$. The adjacency condition implies that f_A, f_B must *actually be homomorphisms.*

Theorem 2.28. *For graphs G, H, we have $G \to H$ if and only if there exists a perfect classical strategy for the (G, H)-homomorphism game.*

Quantum Strategies:
For a state $|\phi\rangle$ the POVM's $\mathcal{E}_g$ for $g \in V(G)$ and $\mathcal{F}_{g'}$ for $g' \in V(G)$ give a perfect quantum strategy for the (G, H)-homomorphism game if and only if

$$\langle \phi | E_{gh} \otimes F_{g'h'} | \phi \rangle = 0$$

whenever $g \sim g'$ and $h \not\sim h'$, or $g = g'$ and $h \not\sim h'$.

Theorem 2.29. *If $G \to_q H$, then there exists a perfect quantum strategy for the (G, H)-homomorphism game where:*

(i) $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$;
(ii) $E_{gh} \in \mathbb{C}^d \otimes \mathbb{C}^d$ *and* $F_{gh} \in \mathbb{C}^d \otimes \mathbb{C}^d$ *are projections for all* $g \in V(G), h \in V(H)$;
(iii) $F_{gh} = E_{gh}^T$.

Corollary 2.30. *We have $G \to_q H$ if and only if there exist projections $E_{gh} \in \mathbb{C}^{d\times d}$ for $g \in V(G)$ and $h \in V(H)$ such that $E_{gh}E_{g'h'} = 0$ whenever $g \sim g'$ and $h \not\sim h'$ or $g = g'$ and $h \neq h'$, and $\sum_{h\in V(H)} E_{gh} = I$.*

Proof. We'll look at the case when $d = 1$. By the second condition mentioned above, there exists a map $\varphi : V(G) \to V(H)$ such that

$$E_{gh} = \begin{cases} 1, & \text{if } \varphi(g) = h, \\ 0, & \text{otherwise.} \end{cases}$$

Then if $E_{g\varphi(g)}E_{g'\varphi(g)} = 1$, this implies φ is a homomorphism.

For the converse, if $\varphi : V(G) \to V(H)$ is a homomorphism, for each $g \in V(G)$ set

$$E_{gh} = \begin{cases} I, & \text{if } \varphi(g) = h, \\ 0, & \text{otherwise.} \end{cases}$$

These projections give a quantum homomorphism. □

Transitivity:

Proposition 2.31. *If $G \to_q H$ and $H \to_q K$, then $G \to_q K$.*

Proof. Exercise! □

Exercise: Prove that if the projections E_{gh} are pairwise commutative, then $G \to H$.

Returning to the topic of transitivity, we give a loose sketch of a proof for Proposition 2.20. Suppose $G \to_q H$ (with corresponding E_{gh}) and $H \to_q K$ (with corresponding F_{hk}). To show $G \to_q K$, the basic idea is to have the players A and B first play the (G, H) game, and then play the (H, K) game. In more detail, if A and B are sent g and $g' \in V(G)$ (respectively), by playing the (G, H) game we obtain $h, h' \in V(H)$ according to the quantum strategy. If we then act as if h, h' were *inputs* to the two player's (H, K) strategy, we obtain $k, k' \in V(K)$. This means

$$g = g' \Rightarrow h = h' \Rightarrow k = k',$$
$$g \sim g' \Rightarrow h \sim h' \Rightarrow k \sim k'.$$

Thinking about this in terms of projections, we set

$$P_{gk} = \sum_{h\in V(H)} E_{gh} \otimes F_{hk}.$$

Then

$$\sum_{k\in V(K)} P_{gk} = \sum_{k,h} E_{gh} \otimes F_{hk} = \sum_{h\in V(H)} E_{gh} \otimes \Bigg(\sum_{k\in V(K)} F_{hk}\Bigg)$$
$$= \sum_{h\in V(H)} E_{gh} \otimes I = I \otimes I = I.$$

If $g \sim g'$ but $k \not\sim k'$, we have

$$P_{gk}P_{g'k'} = \sum_{h,h'\in V(H)} E_{gh}E_{g'h'} \otimes F_{hk}F_{h'k'} = 0,$$

as either $E_{gh}E_{g'h'} = 0$ if $h \not\sim h'$ or $F_{hk}F_{h'k'} = 0$ if $h \sim h'$.

3. More on graphs

3.1. **Graph colorings. Quantum Colorings:**

By Corollary 2.19, we know we have $G \to_q K_n$ if and only if there exist projections $E_{gi} \in \mathbb{C}^{d\times d}$ which satisfy

(i) $\sum_{i=1}^{n} E_{gi} = I$ for all $g \in V(G)$;
(ii) $E_{gi}E_{g'i} = 0$ if $g \sim g'$.

Recall: $\chi(G) = \min\{n : G \to K_n\}$.

Definition 3.1 (Quantum Chromatic Number)**.** *For a graph G, we have*

$$\chi_q(G) := \min\{n : G \to_q K_n\}.$$

As we might wish for when defining a quantum analog of a previously established parameter for a graph, there are several similarities between χ and χ_q for a graph G. Among them are the following:

(i) $G \to H \Rightarrow \chi(G) \le \chi(H)$, while $G \to_q H \Rightarrow \chi_q(G) \le \chi_q(H)$ by transitivity. We say χ_q is *monotone* with respect to quantum homomorphism.
(ii) $G \to_q K_1$ if and only if $G \to K_1$; this means there is a quantum homomorphism between G and K_1 if and only if G is edgeless. In this case, we set $E_{g1} = I$.
(iii) $G \to_q K_2$ if and only if $G \to K_2$; this means there is a quantum homomorphism between G and K_2 if and only if G is bipartite.

 Proof. For a rough proof: Suppose we label the two vertices of K_2 with u and v. We have

$$E_{u1} = E_{u1}(E_{v1} + E_{v2}) = E_{u1}E_{v2} = (E_{u1} + E_{u2})E_{v2} = E_{v2}.$$

 Similarly, we have

$$E_{u2} = E_{v1} \Rightarrow (E_{v1}, E_{v2}) = (E_{u2}, E_{u1}).$$

 □

 Exercise: Show that $G \to_{NS} K_2$ for all graphs G (here $\to_{NS}$ means a non-signalling homomorphism).
(iv) $\chi_q(K_n) = n$.

 Proof. Exercise! □

Uniform Rank Colorings:

Suppose that $E_{gi} \in \mathbb{C}^{d\times d}$ give a quantum n-coloring of graph G. Define

$$F_{gi} = \bigoplus_{k=0}^{n-1} E_{g(i+k)} = \begin{pmatrix} E_{gi} & & & 0 \\ & E_{g(i+1)} & & \\ & & \ddots & \\ 0 & & & E_{g(i+n-1)} \end{pmatrix},$$

i.e. we look at the subscripts modulo n. Then $F_{gi} \in \mathbb{C}^{nd\times nd}$ give a quantum n-coloring of G with $\operatorname{rank}(F_{gi}) = d$ for all $g \in V(G)$, and $i \in [n]$.

Definition 3.2. *We define $\chi_q^r(G)$ to be the minimum natural number n such that G has a quantum n-coloring using only rank r projections (i.e. a rank-r quantum n-coloring of G).*

Remark: For all $r \in \mathbb{N}$, we have

$$\chi_q^r(G) \le \chi_q^1(G) \le \chi(G).$$

Suppose that $E_{gi} \in \mathbb{C}^{d\times d}$ gives a rank-r quantum n-coloring of G. Note that $d = rn$. Fix $i \in [n]$, and let $E_g = E_{gi}$ for all $g \in V(G)$. Then if $g \sim g'$, we have $E_gE_{g'} = 0$.

The map $g \mapsto E_g$ is a *d/r projective representation* of G. Its value is exactly the rational number d/r.

Note: It is important to note the difference between a d_1/r_1 representation and a d_2/r_2 representation, even if they are the same rational number. For example: a 3/1 representation is not the same as a 6/2 representation, even though both reduce to 3.

Definition 3.3 (Projective Rank)**.** *For a graph G,*

$$\xi_f(G) := \inf\{d/r : G \text{ has a } d/r \text{ representation}\}.$$

Note: We can think of the projective rank almost like a "fractional" quantum chromatic number. We also have the inequality

$$\xi_f(G) \leq \chi_q(G), \chi_f(G).$$

Rank-1 Quantum Colorings:
Suppose $E_{gi} \in \mathbb{C}^{n\times n}$ give a rank-1 quantum n-coloring of G. This means $E_{gi} = |\phi_{gi}\rangle\langle\phi_{gi}|$ for some $|\phi_{gi}\rangle \in \mathbb{C}^n$. The conditions on E_{gi} translate to:

(i) The set $\{|\phi_{gi}\rangle : i \in [n]\}$ is an orthonormal basis for all $g \in V(G)$;
(ii) $\langle\phi_{gi}|\phi_{g'i}\rangle = 0$ if $g \sim g'$.

Define the matrix $U_g := \sum_{i=1}^{n} |\phi_{gi}\rangle\langle i|$ for all $g \in V(G)$ (so the i^{th} column of U_g is just $|\phi_{gi}\rangle$). We note that:

(i) U_g is unitary;
(ii) $(U_g^* U_{g'})_{ii} = 0$ for $i \in [n]$ if $g \sim g'$. This means $U_g^* U_{g'}$ has 0's down the diagonal when $g \sim g'$.

Theorem 3.4. *A rank-1 quantum n-coloring is equivalent to a homomorphism to Cay$\big(U(n), \{U \in U(n) : U_{ii} = 0 \text{ for all } i\}\big)$ (this is sometimes known as the unitary derangement group).*

For a fixed $i \in [n]$, the map $g \mapsto |\phi_{gi}\rangle$ is an orthogonal representation of G in dimension n (i.e. an $n/1$- representation).

Definition 3.5 (Orthogonal Rank)**.** *For a graph G, we define the orthogonal rank of G as*

$$\xi(G) := \min\{n : G \text{ has an orthogonal representation in } \mathbb{C}^n\}.$$

Note: For any graph G, the following inequality always holds:

$$\xi_f(G) \leq \xi(G) \leq \chi_q^1(G).$$

However, neither $\xi(G) \leq \chi_q(G)$ nor $\chi_q(G) \leq \xi(G)$ holds for every graph G.

Constructions:

Definition 3.6. *We say that $|\phi\rangle \in \mathbb{C}^n$ is flat if all of its entries have the same modulus; i.e., $|\langle i|\phi\rangle|$ does not depend on i.*

Example 3.7. *The unit vector in $\mathbb{C}^n$ is flat, as each of its entries has modulus $\frac{1}{\sqrt{n}}$.*

Suppose that $g \mapsto |\phi_g\rangle \in \mathbb{C}^n$ is a flat orthogonal representation of G. Take F, a flat $n \times n$ unitary matrix (so $F_{ij} = \omega^{ij}$ where ω is a primitive n^{th} root of unity). Furthermore, let

$$Dg = \sqrt{n}\text{Diag}(|\phi_g\rangle),$$
$$Ug = DgF.$$

The notation "Diag" means we create a matrix where the entries on the diagonal are precisely the entries of $|\phi_g\rangle$, and 0 everywhere else. Note that both of the maps defined above are unitary.

Lemma 3.8. *If F is flat and unitary, and D is diagonal, then F^*DF has constant diagonal.*

We can actually use the lemma above to tell us more about $U_g^*U_{g'}$. By definition of each matrix, we have

$$\mathrm{Tr}\,(U_g^*U_{g'}) = \mathrm{Tr}(F^*D_g^*D_{g'}F) = \mathrm{Tr}(D_g^*D_{g'}) = \langle \phi_g|\phi_{g'}\rangle = 0.$$

Furthermore, by the lemma above we know that $F^*D_g^*D_{g'}F$ must have constant diagonal. As a matrix with constant diagonal has trace 0 if and only if all entries along the diagonal are 0, this shows

$$(U_g^*U_{g'})_{ii} = (F^*D_g^*D_{g'}F)_{ii} = 0.$$

Example 3.9. *Let Ω_n be the orthogonality graph of the ± 1 vectors in $\mathbb{R}^n$. Then Ω_n has a flat orthogonal representation in dimension n by construction. We also have*

$$\Omega_n = \begin{cases} \textit{edgeless, if } n \textit{ is odd;} \\ \textit{bipartite, if } n \equiv 2 \pmod 4). \end{cases}$$

We can calculate directly that

$$\chi_q(\Omega_{4n}) = 4n.$$

One remarkable fact is that $\chi_q(\Omega_{4n})$ is known exactly, while $\chi(\Omega_{4n})$ is not always known.

The Quaternion Trick:
Let $r = (r_0, r_1, r_2, r_3)^T \in \mathbb{R}^4$ be a unit vector. Then

$$\mathcal{B} = \left\{ \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}, \begin{pmatrix} -r_2 \\ r_3 \\ r_0 \\ -r_1 \end{pmatrix}, \begin{pmatrix} -r_1 \\ r_0 \\ -r_3 \\ r_2 \end{pmatrix}, \begin{pmatrix} -r_3 \\ -r_2 \\ r_1 \\ r_0 \end{pmatrix} \right\}$$

is an orthonormal basis of $\mathbb{R}^4$. We'll use the notation that $\mathcal{B} = \{r^0, r^1, r^2, r^3\}$. We note that for vectors $r, s \in \mathcal{B}$, then $r^i \perp s^i$ if $r \perp s$ (for all $i = 0, \ldots, 3$).

Theorem 3.10. *If G has an orthogonal representation in $\mathbb{R}^4$, then $\chi_q(G) \leq 4$.*

The question then is- what does all of this have to do with quaternions? We can actually create an association between unit vectors in $\mathbb{R}^4$ and quaternions in the following way: define $q(r) = r_0 + r_1 i + r_2 j + r_3 k$. Then

$$q(r^1) = iq(r),$$
$$q(r^2) = jq(r),$$
$$q(r^3) = kq(r).$$

Furthermore,

$$\langle r, s\rangle = \mathrm{Re}(q(r)\overline{q(s)}),$$
$$\langle r^1, s^1\rangle = \mathrm{Re}(iq(r)\overline{iq(s)}) = \mathrm{Re}(q(r)\overline{q(s)}),$$
$$\langle r^1, r^2\rangle = \mathrm{Re}(iq(r)\overline{jq(r)}) = \mathrm{Re}(-kq(r)\overline{q(r)}).$$

Note that the last value is the k coefficient of $q(r)\overline{q(r)}$. For more information about this general idea, see the paper "On the quantum chromatic number of a graph" by Cameron et all.

Based on all of these connections and strings of inequalities involving different quantum parameters of a graph G, we might ask ourselves: is $\xi(G) = \chi_q^1(G) = \chi_q(G)$ true? Fortunately (or unfortunately, depending on your point of view) this is not always the case.

In 2011, Fukawa, Imai and LeGall used a reduction from 3-SAT → 3-COLORING to obtain a graph G where $\chi_q(G) = 3 < \chi(G)$. As $\chi_q(G) \leq \chi_q^1(G)$, in this case this implies $\chi_q(G) < \chi_q^1(G)$ as $\chi_q^1(G) = 3 \iff \chi(G) = 3$. Additionally, Mancinska and Roberson were able to show that $\chi_q(G) < \xi(G)$ is also possible. Therefore, the string of equalities we posited does not always hold.

What about if $\chi_q(G) \leq \xi(G)$ for all graphs G? Again, this is (unfortunately) not always true either. Let G_{13} be the orthogonality graph of the vectors in $\mathbb{R}^3$ with entries from $\{0, -1, 1\}$. A picture of G_{13} can be found below:

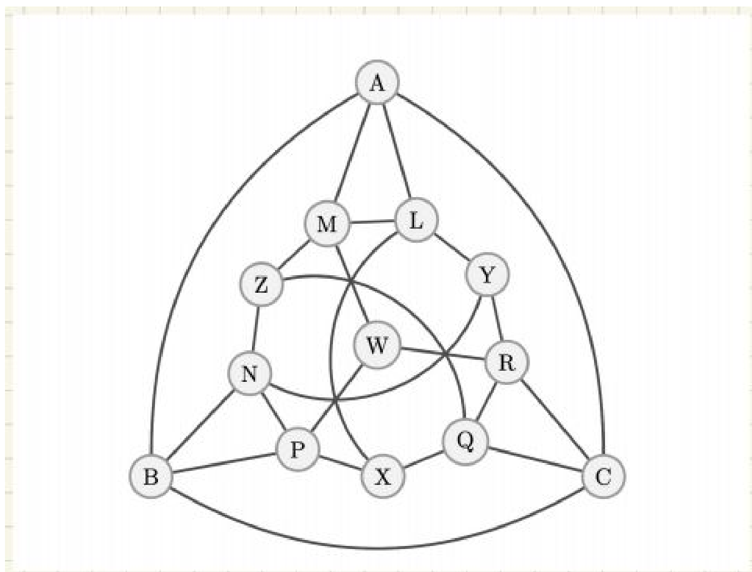


FIGURE 1. The orthogonality graph G_{13}

It can be shown that $\omega(G_{13}) = \xi(G_{13}) = 3 < 4 = \chi(G_{13}) = \chi_q(G_{13})$. Therefore, $\xi(G) < \chi_q(G)$ in this case.

What might we be able to say about the graph G_{14}, where

$$G_{14} = G_{13} + \text{ an apex vertex.}$$

This apex vertex is adjacent to all others. As $\chi(G_{13}) = 4$, this means $\chi(G_{14}) = 5$. We know that G_{13} has an orthogonal representation in $\mathbb{R}^3$, so G_{14} has an orthogonal representation in $\mathbb{R}^4$. By Theorem 3.10, this means $\chi_q(G_{14}) \leq 4$. Here, we have been able to add a vertex without increasing the quantum chromatic number.

Remarks:

(i) $\chi_q(G_{14}) < \chi(G_{14})$ is the smallest known example where we have the quantum chromatic number as strictly smaller than the chromatic number for a graph.

(ii) $\chi_q(G_{13}) = \chi_q(G_{14})$- so adding the apex vertex does nothing to increase χ_q.

Definition 3.11. *For $G \to_q H$ with corresponding E_{gh}'s, we say that the projections are locally commuting if*

$$E_{gh}E_{g'h'} = E_{g'h'}E_{gh}$$

whenever $g \sim g'$.

3.2. **Projective packings and independence numbers.**

Projective Packings:

Definition 3.12. *An assignment $g \mapsto E_g \in \mathbb{C}^{d\times d}$ of projections to the vertices of a graph G is called a projective packing if $g \sim g'$ implies $E_g E_{g'} = 0$. The value of a projective packing is given by $\frac{1}{d} \sum_{g\in V(G)} rank(E_g)$.*

Definition 3.13 (Projective Packing Number)**.** *For a graph G, we define*

$$\alpha_p(G) := \sup\{x \in \mathbb{Q} : G \text{ has a projective packing of value } x\}.$$

As with the regular clique number, we also define the projective clique number to be

$$\omega_p(G) := \alpha_p(\overline{G}).$$

Various Properties Involving Projective Packings:

(i) If $S \subseteq V(G)$ is an independent set, letting

$$E_g = \begin{cases} I, & g \in S, \\ 0, & \text{otherwise.} \end{cases}$$

gives a projective packing of value $|S|$. Therefore, $\alpha(G) \leq \alpha_p(G)$.

(ii) We have $\frac{|V(G)|}{\alpha_p(G)} \leq \xi_f(G)$, with equality if G is vertex transitive; this is in direct relation with the classic inequality involving α and χ for a graph G.

Proof. Suppose $g \mapsto E_g$ is a (d/r)-representation; then it also is a projective packing of value

$$\frac{1}{d} \sum_{g\in V(G)} \text{rank}(E_g) = \frac{r|V(G)|}{d}.$$

Therefore,

$$\alpha_p(G) \geq \frac{r|V(G)|}{d} = \frac{|V(G)|}{(d/r)}$$
$$\Rightarrow \frac{d}{r} \geq \frac{|V(G)|}{\alpha_p(G)}.$$

If we then take the infimum over all (d/r)-representations, we have the desired inequality. □

(iii) $\omega_p(G) \leq \xi_f(G)$.

(iv) If $G \to_q H$ then $\omega_p(G) \leq \omega_p(H)$ and $\xi_f(G) \leq \xi_f(H)$.

Proof. Let $g \mapsto E_g \in \mathbb{C}^{k\times k}$ be a projective clique of value $R/k = \frac{1}{k} \sum_{g\in V(G)} \text{rank}(E_g)$.

We have $F_{gh} \in \mathbb{C}^{d\times d}$ for $g \in V(G), h \in V(H)$ which corresponds to the quantum homomorphism between G and H. Define

$$P_h := \sum_{g\in V(G)} E_g \otimes F_{gh}.$$

We need to check that P_h is a projection, $P_h P_{h'} = 0$ if $h \sim h'$ in $\overline{H}$ (as we are working with a projective clique), and that the value of P_h is at least the value of E_g. To that end- exercise! □

Note: A (loose) analogy: α_p is to ξ_f as α is to χ_f.

Notation:

(i) $\alpha_p(G) \dot{\geq} x \in \mathbb{Q}$ means G has a projective packing of value at least x;

(ii) $\alpha_p(G) \doteq x \in \mathbb{Q}$ means $\alpha_p(G) = x$ and $\alpha_q(G) \geq x$.

Quantum Independence Number:
Classically, we may define

$$\omega(G) = \max\{n : K_n \to G\},$$
$$\alpha(G) = \omega(\overline{G}).$$

We recall that projections $E_{ig} \in \mathbb{C}^{d\times d}$ give a quantum homomorphism from K_n to G if

(i) $\sum_{g\in V(G)} E_{ig} = I$, for all $i \in [n]$;
(ii) $E_{ig}E_{jg'} = 0$ if either $i \neq j$ and $g = g'$ or $g \sim g'$, or $i = j$ and $g \neq g'$.

Then $F_g = \sum_{i=1}^{n} E_{ig}$ is a projective packing of value n. Therefore, $\alpha_p(G) \dot{\geq} \alpha_q(G)$.

Proof. Exercise! □

A Construction:
Suppose $g \mapsto |\phi_g\rangle \in \mathbb{C}^d$ is an orthogonal representation of G, and $f : V(G) \to [k]$ is a k-coloring of $\overline{G}$ where $k = \frac{|V(G)|}{d}$. Then $\omega(G) = d$, and $f^{-1}(i)$ is a clique of size d for each $i \in [k]$. Therefore, the set $\{|\phi_g\rangle : g \in f^{-1}(i)\}$ is an orthonormal basis for all $i \in [k]$. Define

$$E_{ig} = \begin{cases} |\phi_g\rangle\langle\phi_g|, & \text{if } g \in f^{-1}(i), \\ 0, & \text{otherwise.} \end{cases}$$

We claim that the E_{ig}'s give a quantum homomorphism from K_k to $\overline{G}$ (i.e. $K_k \to_q \overline{G}$).

Proof. We first note that as $E_{ig} = 0$ if $g \notin f^{-1}(i)$, then

$$\sum_{g\in V(G)} E_{ig} = \sum_{g\in f^{-1}(i)} |\phi_g\rangle\langle\phi_g| = I.$$

If $i \neq j$, then $E_{ig}E_{jg} = 0$ as $f^{-1}(i) \cap f^{-1}(j) = \emptyset$ (as the $f^{-1}(i)$'s partition our graph). Similarly, if $i \neq j$ and $g \sim g'$, then

$$E_{ig}E_{jg'} = \begin{cases} |\phi_g\rangle\langle\phi_g|\phi_{g'}\rangle\langle\phi_{g'}|, & \text{if } g \in f^{-1}(i), g' \in f^{-1}(j), \\ 0, & \text{otherwise.} \end{cases}$$

However, we must have $|\phi_g\rangle\langle\phi_g|\phi_{g'}\rangle\langle\phi_{g'}| = 0$ by orthogonality of our representation. This means $E_{ig}E_{jg'} = 0$. This completes the proof. □

Theorem 3.14 (Cubitt, Leung, Matthews, and Winter)**.** *If $\chi(\overline{G}) = \frac{|V(G)|}{\xi(G)}$, then $\alpha_q(G) = \chi(\overline{G})$.*

Note: We have $\alpha_q(G) \leq \chi(\overline{G})$ always.

Theorem 3.15 (Mancinska, Scarpa, and Severini)**.** *For a graph G, we have $\alpha_q(G) = \chi(\overline{G})$ if and only if $\alpha_p(G) \doteq \chi(\overline{G})$.*

How do we find such graphs that also satisfy parameter inequality $\alpha(G) < \chi(\overline{G})$?

Definition 3.16 (Kochen-Specker sets)**.** *A set $S \subseteq \mathbb{C}^d$ of non-zero vectors such that there is no subset $T \subseteq S$ of mutually non-orthogonal vectors containing one vector from every orthogonal basis $B \in \mathcal{B}$, where $\mathcal{B}$ is the set of all bases for S, is called a Kochen-Specker set.*

We will construct a graph using a Kochen-Specker set in the following way: we define

$$V(G(S)) = \{(|\phi\rangle, B) : |\phi\rangle \in B \subseteq S, \text{ where } B \text{ is an orthogonal basis.}\},$$

and let $(|\phi\rangle, B) \sim (|\phi'\rangle, B')$ if $\langle\phi|\phi'\rangle = 0$. We have $n := |V(G(S))|$, b as the number of orthogonal bases in S (so n/d). With this, we see map $(|\phi\rangle, B) \mapsto B$ gives a coloring of $\overline{G(S)}$ with $b = n/d$ colors. Furthermore, $(|\phi\rangle, B) \mapsto |\phi\rangle$ is an orthogonal representation of G in $\mathbb{C}^d$. Therefore, putting everything together we have

$$\alpha_q(G(S)) = \chi(\overline{G(S)}) = b = n/d.$$

However, if there were an independent set of size b we could use it to find a subset of S which would "hit" every orthogonal basis B exactly once (we can think of our orthogonal bases as partitioning our graph). As this cannot happen, this forces

$$\alpha(G(S)) < b = \alpha_q(G(S)) \Rightarrow \chi(G) \geq \frac{n}{\alpha(G)} > \frac{n}{b} = d = \xi(G).$$

To sum up our previous discussions, we have

$$\chi(\overline{G}) \geq \chi_q(\overline{G}), \chi_f(\overline{G}),$$
$$\xi(G) \geq \xi_f(\overline{G}) \geq \alpha_p(G) \geq \alpha_q(G) \geq \alpha(G).$$

Review

Last week, we:

- Introduced the projective packing and clique numbers α_p and ω_p
- Introduced the relation to projective rank:

$$\xi_f(G) \geq \frac{|V(G)|}{\alpha_p(G)}$$

  and $w_p(G) \leq \xi_f(G)$
- Introduced the quantum clique and independence numbers w_q and α_q, with

$$\alpha(G) \leq \alpha_q(G) \leq \alpha_p(G) \leq \chi(\overline{G}).$$

Using $G \to_q H$ and $G \nrightarrow H$ to get $\alpha(K) < \alpha_q(K)$

Definition 3.17. *For graphs G and H, we define their homomorphic product as the graph $G \ltimes H$, with vertex set $V(G \ltimes H) = V(G) \times V(H)$ where $(g,h) \sim (g',h')$ if $g = g'$ and $h \neq h'$, or $g \sim g'$ and $h \not\sim h'$.*

Note: The conditions for adjacency in the homomorphic product are exactly those needed for orthogonality between projections, where $E_{gh}E_{g'h'} = 0$.

Example 3.18. *Here are a couple of examples to illustrate the concept of the homomorphic product.*

(i) $G \ltimes K_n \cong G \square K_n$ *(Cartesian product $(=, \sim)$ or $(\sim, =)$);*
(ii) $K_n \ltimes H = \overline{K_n \times H}$ *(Categorical product $(\sim, \sim)$).*

Suppose we look at the homomorphic product of two graphs G and H- how may we be able to conceptualize $G \ltimes H$? Suppose we think of $V(G \ltimes H) = V(G) \times V(H)$ in terms of a matrix, where vertices in G label rows and the vertices of H are entries. For each fixed $g \in G$, the corresponding row has entries which vary over all vertices in H. This allows us to partition our set of vertices $V(G) \times V(H)$ into a collection of cliques- as a partition into cliques is the same as a coloring of the complement of a graph, this implies

$$\chi(\overline{G \ltimes H}) \leq |V(G)|.$$

Lemma 3.19. *For graphs G, H we have*

$$\alpha(G \ltimes H) \leq |V(G)|,$$

and equality holds if and only if $G \to H$.

Note: We have the following string of inequalities

$$\alpha_q(G \ltimes H) \leq \alpha_p(G \ltimes H) \leq \chi(\overline{G \ltimes H}) \leq |V(G)|.$$

Theorem 3.20 (Roberson, Mancinska)**.** *The following are equivalent:*

(i) $G \to_q H$;
(ii) $\alpha_p(G \ltimes H) \doteq |V(G)|$;
(iii) $\alpha_q(G \ltimes H) = |V(G)|$.

Proof. First, we'll show that *(i)* $\Rightarrow$ *(iii)* in terms of their corresponding games. Suppose A and B have a strategy for the (G, H)-homomorphism game. They can play the $(K_{V(G)}, \overline{G \ltimes H})$-homomorphism game as follows: upon receiving $g \in V(K_{V(G)}) = V(G)$, they act as if this is their input for the (G, H)-homomorphism game and obtain output $h \in V(H)$. They respond with $(g, h) \in V(G \ltimes H)$. With this in mind, if A and B get g, g' they'll have outputs (g, h) and (g', h') respectively. If $g = g'$, then $h = h'$ (by our strategy for the game), so $(g, h) = (g', h')$. If $g \neq g'$, then either:

(i) $g \sim g'$, which implies $h \sim h'$, and so $(g, h) \not\sim (g', h')$ in $G \ltimes H$; otherwise,
(ii) $g \not\sim g'$, which implies $(g, h) \not\sim (g', h')$ in $G \ltimes H$.

These together show that *(i)* $\Rightarrow$ *(iii)*.

Next, to show that *(iii)* $\Rightarrow$ *(ii)* is immediate, as

$$\alpha_q(G \ltimes H) = |V(G)| \Rightarrow \alpha_p(G \ltimes H) \doteq |V(G)|$$

by definition.

Finally, we'll show that *(ii)* $\Rightarrow$ *(i)*. Suppose $(g, h) \mapsto E_{gh} \in \mathbb{C}^{d \times d}$ is a projective packing of value $|V(G)|$. This means

$$\sum_{g,h} \mathrm{rk}(E_{gh}) = d|V(G)|.$$

We want to show that the E_{gh}'s satisfy the conditions for a quantum homomorphism between G and H. To that end, we need to check their orthogonality and show they sum to I. We first note that by the definition of $G \ltimes H$, adjacency in $G \ltimes H$ corresponds to orthogonal projections- so the orthogonality conditions hold. For a fixed $g \in V(G)$, by orthogonality we have that $\sum_h E_{gh}$ is a projection. As our $g \in V(G)$ was arbitrarily fixed, this holds for every vertex in G. Therefore,

$$\begin{aligned} d|V(G)| = \sum_{g,h} \mathrm{rk}(E_{gh}) = \sum_g \left(\sum_h \mathrm{rk}(E_{gh}) \right) = \sum_g \mathrm{rk}\left(\sum_h E_{gh} \right) \\ \leq \sum_g d = d|V(G)|. \end{aligned}$$

This means $\mathrm{rk}\left(\sum_h E_{gh} \right) = d$ for every $g \in V(G)$, which implies $\sum_h E_{gh} = I$ for all $g \in V(G)$. This completes the proof. $\square$

Lemma 3.21. $\alpha(G \ltimes K_n) = \alpha(G \square K_n) \leq n\alpha(G)$.

Example 3.22. *We'll return to the graph Ω_n, for $n \in \mathbb{N}$. Recall that Ω_n is the orthogonality graph of $\{\pm 1\}^n$. Furthermore, recall that $\chi_q(\Omega_{4n}) = 4n$, while $\chi(\Omega_{4n})$ is exponential with n. Frankl and Rodl were able to show that*

$$\alpha(\Omega_{4n}) \leq \gamma^{4n}$$

for some $\gamma < 2$. Using the previous lemma, Scarpa was able to show that

$$\alpha_q(\Omega_{4n} \Box K_{4n}) = |V(\Omega_{4n})| = 2^{4n},$$

but

$$\alpha(\Omega_{4n} \Box K_{4n}) \leq 4n\gamma^{4n}$$

for some $\gamma < 2$. This means we have

$$\frac{\alpha_q(\Omega_{4n} \Box K_n)}{\alpha(\Omega_{4n} \Box K_{4n})} \geq \frac{1}{4n}\left(\frac{2}{\gamma}\right)^{4n}.$$

Another KS Set

(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
(1, −1, 0, 0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

FIGURE 2. Kochen-Specker set

In the picture above, there are 9 bases, where each vector appears in 2 of them.

We can think of each basis as a vector, and each vector as an edge. This gives a graph G on 9 vertices, with degree 4. In fact,

$$G \cong \text{Payley}(9) \cong K_3 \Box K_3 \cong K_3 \times K_3 \cong L(K_{3,3}).$$

Note: Here $L(G)$ means the line graph of G.

We also have $|V(L(G))| = 8$, $\omega(L(G)) = 4$, and $\chi_q(L(G)) = 4$, with $4 < \frac{18}{4}$. Additionally, $\frac{18}{4} < \chi(L(G)) = 5$, with $\xi_f(L(G)) = 4$.

As $L(G)$ is vertex transitive, this implies

$$\chi_f(\overline{L(G)}) = \frac{|V(L(G))|}{\omega(L(G))} = \frac{18}{4} = 4.5.$$

By definition, this gives

$$\alpha_p(L(G)) = \frac{|V(L(G))|}{\xi_f(L(G))} = \frac{18}{4} \Rightarrow \alpha_q(L(G)) = 4 = \alpha(L(G)).$$

Theorem 3.23 (Roberson)**.** *For a graph G, $\alpha_p(G) \doteq \chi_f(\overline{G})$ if and only if there exists an $r \in \mathbb{N}$ such that $\alpha_p(G([\overline{K_r}])) = r\chi_f(\overline{G})$, if and only if there exists an $r \in \mathbb{N}$ such that $\alpha_q(rG) = r\chi_f(\overline{G})$.*

Note: When we have rG for a graph G, this means we take r copies of the graph G.

Using the previous theorem, this means

$$\alpha_q(2L(G)) = 9 > 2\alpha_q(L(G)).$$

This phenomenon is odd- it does not occur often with other graphs. We also note that

$$\chi_q(L(G)) < \frac{|V(L(G))|}{\alpha_q(L(G))}.$$

This is a "violation" of the classical inequality that we know holds for α and χ.

Lovász Theta
Let G be a graph with the adjacency matrix A.

Theorem 3.24 (Hoffman)**.** *We have*

$$\chi(G) \geq 1 - \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = \lambda_{max}\left(I + \frac{1}{-\lambda_{min}(A)}\right).$$

The only property of A that is needed for the proof of the theorem above is that $A_{uv} = 0$ if $u \not\sim v$, where $u, v \in V(G)$. Optimizing Hoffman's bound over "suitable" A gives us the Lovász theta function of $\overline{G}$.

Definition 3.25. *For a graph G, we define*

$$\vartheta(\overline{G}) = \overline{\vartheta}(G) = \max\{\lambda_{max}(A+I) : \; A_{uv} = 0 \textit{ if } u \not\sim v, \quad \textit{where } \lambda_{min}(A) \geq -1\}.$$

We add the stipulation on $\lambda_{min}(A)$ to ensure $A + I$ is positive semi-definite.

The previous definition of $\overline{\vartheta}(G)$ is equivalent to the minimum value for t such that $M_{uv} = t-1$ for all $u \in V(G)$, where $M_{uv} = -1$ if $u \sim v$ and $M \geq 0$. (This is, in fact, one of the many different formulations for the Lovász theta function). As we have both maximal and minimal formulations for $\overline{\vartheta}(G)$, this means we have "certificates" for both $\overline{\vartheta}(G) \geq x$ and $\overline{\vartheta}(G) \leq y$.

Theorem 3.26 (Lovász Sandwich Theorem)**.** *For a graph G, we have*

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}).$$

Properties of the Lovász theta function Let G be a graph.

(i) $\vartheta(G)\vartheta(\overline{G}) \geq |V(G)|$ with equality if G is vertex transitive.
(ii) If G is vertex and edge transitive, then

$$\vartheta(G) = 1 - \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

for A, the adjacency matrix of G.
(iii) $\overline{\vartheta}(K_n) = n$.
(iv) The function is "efficiently computable", in that we can sufficiently approximate it through computational help.
(v) It also has many different formulations.

Formulation in terms of projective packings
Recall that the value of a projective packing $g \mapsto E_g \in \mathbb{C}^{d \times d}$ is

$$\frac{1}{d}\sum_g \text{rk}(E_g) = \frac{1}{d}\sum_g \text{Tr}(E_g) = \frac{1}{d}\text{Tr}\left(\sum_g E_g\right)$$
$$= \text{the average of the eigenvalues of } \sum_g E_g.$$

Proposition 3.27. *For a graph G, $\vartheta(G) = \max\{\lambda_{max}\left(\sum_g E_g\right)\}$ such that $g \mapsto E_g$ is a projective packing.*

A more typical formulation would be the following:

$$\vartheta G = \max \sum_g |\langle \varphi | \psi_g \rangle|^2$$

such that $g \mapsto |\psi_g\rangle$ is

Is there anything we can say about a quantum version of the Sandwich Theorem? Suppose we wish to prove

$$\omega_q(G) \leq \overline{\vartheta}(G) \leq \chi_q(G).$$

To do so, it suffices to prove that if $G \to_q H$, then $\overline{\vartheta}(G) \leq \overline{\vartheta}(H)$. If this were the case- if we let $m = \omega_q(G)$ and $n = \chi_q(G)$, then

$$K_m \to_q G \to_q K_n \Rightarrow m = \overline{\vartheta}(K_m) \leq \overline{\vartheta}(G) \leq \overline{\vartheta}(K_n) = n.$$

3.3. **More results.**

Remarks:

(i) Variants ϑ^- and ϑ^+ are due to Schrijver and Szegedy, respectively. These variants give

$$\omega(G) \leq \omega_q(G) \leq \omega_p(G) \leq \overline{\vartheta}^-(G) \leq \overline{\vartheta}(G) \leq \overline{\vartheta}^+(G) \leq \xi_f(G) \leq \chi_q(G) \leq \chi(G).$$

(ii) Recently, Elphick and Wocjan (along with some others) have shown that classical spectral bounds can also apply to quantum analogues.

The Inertia Bound

Let $M \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Let

(i) $n^+(M)$ be the number of positive eigenvalues of M;
(ii) $n^0(M)$ be the number of zero eigenvalues of M;
(iii) $n^-(M)$ be the number of negative eigenvalues of M.

Definition 3.28. *With M as above, we define $(n^+(M), n^0(M), n^-(M))$ as the inertia of M.*

Theorem 3.29 (Cvetkovit)**.** *Let M be a Hermitian weighted adjacency matrix for G (i.e. $M_{uv} = 0$ if $u \nsim v$). Then*

$$\alpha(G) \leq n^0(M) + \min\{n^-(M), n^+(M)\}.$$

Notation: For a Hermitian matrix, $\lambda_i^{\uparrow}(M)$ is the i^{th} largest eigenvalue of M, while $\lambda_i^{\downarrow}(M)$ is the i^{th} smallest eigenvalue of M. We have

$$\lambda_1^{\downarrow}(M) \leq \lambda_2^{\downarrow}(M) \leq \cdots \leq \lambda_n^{\downarrow}(M),$$
$$\lambda_1^{\uparrow}(M) \geq \cdots \geq \lambda_n^{\uparrow}(M).$$

Theorem 3.30 (Interlacing Theorem)**.** *Let $M \in \mathbb{C}^{n\times n}$ be Hermitian, and let $R \in \mathbb{C}^{n\times k}$ be such that $R^*R = I \in \mathbb{C}^{k\times k}$. Then*

$$\lambda_i^{\downarrow}(M) \geq \lambda_i(R^*MR) \text{ for all } i = 1, \ldots, k$$
$$\lambda_i^{\uparrow}(M) \leq \lambda_i(R^*MR) \text{ for all } i = 1, \ldots, k.$$

Corollary 3.31. *Let $M \in \mathbb{C}^{n\times n}$ be Hermitian and let M' be a $k \times k$ principal submatrix of M. Then*

$$\lambda_i^{\downarrow}(M) \geq \lambda_i^{\downarrow}(M'),$$
$$\lambda_i^{\uparrow}(M) \leq \lambda_i^{\uparrow}(M').$$

Proof. (Cvetkovit) Suppose $S \subseteq V(G)$ is an independent set of size $\alpha(G)$. Let M be a Hermitian weighted adjacency matrix for G. Let M' be the principal submatrix of M, consisting of the rows and columns indexed by elements of S. Then as S is an independent set, and M is weighted, $M' = 0$ with $\lambda_i^{\downarrow}(M) \geq \lambda_i^{\downarrow}(M') = 0$ for $i = 1, \ldots, \alpha(G)$. Thus

$$n^0(M) + n^+(M) \geq \alpha(G).$$

Similarly,

$$n^0(-M) + n^+(-M) \geq \alpha(G) \Rightarrow n^0(M) + n^-(M) \geq \alpha(G),$$

as M is Hermitian. This implies

$$n^0(M) + \min\{n^-(M), n^+(M)\} \geq \alpha(G).$$

□

One question we may ask is- can we alway attain equality for the independence number usiing some real symmetric matrix M? This was a question posed by Chris Godsil, along with Elzinga and Gregory- and the answer (given by John Sinkovic in 2016) is no. If we use the adjacency matrix for Payley(17), we have a symmetric real matrix which does not attain equality. With this question settled, another we may ask is- what about for a complex Hermitian matrix?

Isotropic Subspaces

Definition 3.32. *Let $M \in \mathbb{C}^{n\times n}$. A subspace $U \subseteq \mathbb{C}^n$ is M-isotropic if $\langle x|My\rangle = 0$ for every $x, y \in U$.*

Lemma 3.33 (Elzinga and Gregory/Elphick and Wocjan)**.** *Let $M \in \mathbb{C}^{n\times n}$ be Hermitian. Then the maximum dimension of an M-isotropic subspace is $n^0(M)+\min\{n^+(M), n^-(M)\}$.*

Proof. Without loss of generality, let $n^- \leq n^+$. First, we show that this dimension can be attained. Let $|\psi_i^0\rangle$ for $i = 1, \ldots, n^0$, $|\psi_j^-\rangle$ for $j = 1, \ldots, n^-$, and $|\psi_k^+\rangle$ for $k = 1, \ldots, n^+$ be a full set of orthonormal eigenvectors for M, with $M|\psi_i^0\rangle = 0, M|\psi_j^-\rangle = \lambda_j^-|\psi_j^-\rangle$ where $\lambda_j^- < 0$ and $M|\psi_k^+\rangle = \lambda_k^+|\psi_k^+\rangle$ where $\lambda_k^+ > 0$. Define

$$0 \neq |\psi_i\rangle = |\psi_i^+\rangle + \left(\frac{\lambda_i^+}{-\lambda_i^-}\right)^{1/2}|\psi_i^-\rangle$$

for $i = 1, \ldots n^-$. Then

$$\langle\psi_i|\psi_j\rangle = \langle\psi_i|M\psi_j\rangle = 0$$

if $i \neq j$. Furthermore,

$$\langle\psi_i|M\psi_i\rangle = \lambda_i^+\langle\psi_i^+|\psi_i^+\rangle + \lambda_i^-\left(\frac{\lambda_i^+}{-\lambda_i^-}\right)\langle\psi_i^-|\psi_i^-\rangle = \lambda_i^+ - \lambda_i^+ = 0.$$

Thus, $\mathrm{span}(\{|\psi_i^0\rangle : 1 = 1, \ldots, n^0\}) \cup \{|\psi_i\rangle : i = 1, \ldots n^-\})$ is an M-isotropic subspace of dimension $n^0 + n^-$. Now, let U be any M-isotropic subspace, and let $V = \mathrm{span}\{|\psi_i^+\rangle : i = 1, \ldots, n^+\}$. Then $U \cap V = \{0\}$, and thus

$$\begin{aligned} n^0 + n^- + n^+ = n &\geq \dim(U + V) \\ &= \dim(U) + \dim(V) - \dim(U \cap V) \\ &= \dim(U) + n^+. \end{aligned}$$

Therefore, $\dim(U) \leq n^0 + n^-$. □

Alternative proof of Inertia Bound (Godsil):

Proof. If $S \subseteq V(G)$ is an independent set, then $\mathrm{span}(\{|v\rangle : v \in S\})$ is an M-isotropic subspace for any weighted adjacency M since $\langle u|Mv\rangle = M_{uv} = 0$ for $u, v \in S$. Thus, $\alpha(G) \leq n^0 + \min\{n^-, n^+\}$. □

Note: We don't require M to be Hermitian here- an interesting fact.

Definition 3.34 (Optimized Inertia Bound)**.**

$$\min\{\max\{dim(U) : U \text{ is } M - isotropic\} \text{ the matrix } M \text{ is a weighted adjacency matrix.}\}.$$

Elzinga and Gregory's question can be phrased as- is the optimized inertia bound equal to $\alpha(G)$? What about

$$\hat{\alpha}(G) = \max\{\dim(U) : \ U \text{ is } M - \text{isotropic for all weighted adjacency matrices } M\}?$$

Lemma 3.35 (Duan, Severini, Winter)**.** $\hat{\alpha}(G) = \alpha(G)$.

Proof. (Sketch) That $\alpha(G) \leq \hat{\alpha}(G)$ should be clear- just take S a maximal independent set, and $U = \text{span}\{|v\rangle : \ v \in S\}$. For the other direction, show that $\{v \in V(G) : \ v \notin U^{\perp}\}$ is an independent set. □

Note: Check the lecture notes- I am not sure I wrote down that last part right.

Results from Elphick and Wocjan

Recall: $\alpha_p(G) = \sup\{\frac{1}{d}\sum_v \text{rk}(E_v) : v \mapsto E_v \in \mathbb{C}^{d\times d}$ is a projective packing.$\}$ (so $E_u E_v = 0$ if $u \sim v$).

Theorem 3.36 (Elphick and Wocjan)**.** *Let M be a Hermitian weighted adjacency matrix for G. Then*

$$\alpha_p(G) \leq n^0(M) + \min\{n^-(M), n^+(M)\}.$$

Proof. Let $v \mapsto E_v \in \mathbb{C}^{d\times d}$ be a projective packing. For each $v \in V(G)$, spectrally decompose E_v as $E_v = \sum_{i=1}^{r_v} |\psi_i^v\rangle\langle\psi_i^v|$ where r_v is the rank of E_v. Define $|\varphi_i^v\rangle = |v\rangle \otimes |\psi_i^v\rangle \in \mathbb{C}^{V(G)} \otimes \mathbb{C}^d$. Then

$$\langle\varphi_i^v|\varphi_j^u\rangle = \langle v|u\rangle\langle\psi_i^v|\psi_j^u\rangle = 0,$$

unless $v = u$ and $i = j$. Furthermore, we see

$$\langle\varphi_i^v|(M \otimes I_d)\varphi_j^u\rangle = \langle v|Mu\rangle\langle\psi_i^v|\psi_j^u\rangle = 0,$$

as $\langle v|Mu\rangle = 0$ unless $v \sim u$, but $\langle\psi_i^v|\psi_j^u\rangle = 0$ if $v \sim u$. Therefore, $\text{span}\{|\varphi_i^v\rangle : \ v \in V(G), i \in [r_v]\}$ is an $(M \otimes I_d)$-isotropic subspace of dimension $\sum_v \text{rk}(E_v)$. Thus,

$$\sum_v \text{rk}(E_v) \leq n^0(M \otimes I_d) + \min\{n^-(M \otimes I_d), n^+(M \otimes I_d)\}$$
$$= d\Big(n^0(M) + \min\{n^-(M), n^+(M)\}\Big).$$

This implies

$$\frac{1}{d}\sum_v \text{rk}(E_v) \leq n^0(M) + \min\{n^-(M), n^+(M)\}.$$

□

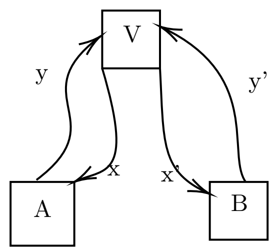
Corollary 3.37. *There is no Hermitian weighted adjacency matrix M for $L(Payley(9))$ which achieves equality in the previous bound.*

3.4. **Quantum isomorphisms.** We'll start off with a well known definition.

Definition 3.38. *An isomorphism from a graph G to a graph H is a function $f : V(G) \to V(H)$ such that*

(i) f is a bijection;
(ii) $g \sim g' \iff f(g) \sim f(g')$.

Quantum Isomorphism
How can we think of quantum isomorphisms between graphs? In order to do so, we'll have to define a non-local game that captures the information of an isomorphism, and apply perfect quantum strategies. The (G, H)-isomorphism game can be, roughly, modeled with the help of the following diagram.



As before, we have a few conditions we require must be satisfied:

(i) Assume $V(G)$ and $V(H)$ are disjoint.
(ii) V sends A and B the vertices $x, x' \in V(G) \cup V(H)$.
(iii) A and B respond with $y, y' \in V(G) \cup V(H)$.
(iv) As usual, they cannot communicate during the game.

The winning conditions for the game are as follows:

(i) We have $x \in V(G) \iff y \in V(H)$, and the same for x', y'. Thus, $\{x, y\} = \{g, h\}$ for some $g \in V(G), h \in V(H)$. We can define g', h' similarly.
(ii) $\text{rel}(g, g') = \text{rel}(h, h')$; that is, $g = g' \iff h = h', g \sim g' \iff h \sim h'$, and $g \not\sim g' \iff h \not\sim h'$.

Note: The $(G, H), (\overline{G}, \overline{H})$, and (H, G)-isomorphism game are all the same.

Proposition 3.39. *There is a perfect classical strategy for the (G, H)-isomorphism game if and only if $G \cong H$.*

To see part of the proof for the previous proposition, note that if $G \cong H$ and we let f be the isomorphism, a perfect strategy would be to respond to g with $f(g)$ and respond to h with $f^{-1}(h)$.

Definition 3.40. *We say that G and H are quantum isomorphic, denoted $G \cong_q H$, if there is a perfect quantum strategy for the (G, H)-isomorphism game.*

Note: For a perfect quantum strategy, we need $p(y, y'|x, x') = 0$ unless there exists $g, g' \in V(G)$ and $h, h' \in V(H)$ such that $\{x, y\} = \{g, h\}, \{x', y'\} = \{g', h'\}$, and $\text{rel}(g, g') = \text{rel}(h, h')$.

Theorem 3.41. *If $G \cong_q H$ then there is a winning quantum strategy such that*

(i) $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \in \mathbb{N}$;
(ii) E_{xy} and F_{xy} are projections for all $x, y \in V(G) \cup V(H)$;
(iii) $F_{xy} = E_{xy}^T$ for all $x, y \in V(G) \cup V(H)$;
(iv) $E_{xy} = 0$ if $x, y \in V(G)$ or $x, y \in V(H)$;
(v) $E_{gh} = E_{hg}$ for all $g \in V(G)$ and $h \in V(H)$.

Note: This last condition is analogous to saying that the strategy for inputs from $V(G)$ is "inverse" to the strategy for inputs from $V(H)$. If A got g and responded with h, and B got h, then B *must* respond with g.

Corollary 3.42. *$G \cong_q H$ if and only if there exists some $d \in \mathbb{N}$ and projections $E_{gh} \in \mathbb{C}^{d\times d}$ for all $g \in V(G), h \in V(H)$ satisfying:*

(i) $\sum_g E_{gh} = I$ *for all* $h \in V(H)$;

(ii) $\sum_h E_{gh} = I$ *for all* $g \in V(G)$;

(iii) $E_{gh}E_{g'h'} = 0$ *if* $rel(g, g') \neq rel(h, h')$.

Note: E_{gh} corresponds in some sense to mapping g to h- i.e., $E_{gh} = 0$ means g can never be mapped to h, while $E_{gh} = I$ would mean g always is mapped to h.

Matrix Formulations:
One known fact about graph isomorphisms is that $G \cong H$ if and only if there exists a permutation matrix P such that $P^T A_G P = A_H$, where A_G, A_H are the adjacency matrices for G and H respectively. One may ask- what is a permutation matrix?

Definition 3.43. *A matrix $P \in \mathbb{C}^{n\times n}$ is a permutation matrix if $P_{ij} \in \{0, 1\}$ for all $i, j \in [n]$ and each row and column of P contains exactly one 1.*

Suppose we relax this condition:

Definition 3.44. *Let $D \in \mathbb{C}^{n\times n}$. If D is real with $D_{ij} \in [0, 1]$ and $\sum_j D_{ij} = 1 = \sum_\ell D_{\ell k}$ for all $i, k \in [n]$, then D is a doubly stochastic matrix.*

Theorem 3.45 (Birkhoff- von Neumann Theorem)**.** *The set of doubly stochastic matrices is the convex hull of the permutation matrices.*

From this relaxation of matrix conditions, we obtain a relaxation of isomorphisms.

Definition 3.46. *Graphs G and H are fractionally isomorphic, denoted by $G \cong_f H$, if there exists a doubly stochastic D such that*

$$A_G D = D A_H \quad \textit{(i.e. we have a linear property).}$$

We note $D^T A_G D = A_H$ gives us the isomorphism.

Theorem 3.47 (Tinhofer/Ramana, Scheinerman and Ulman)**.** *Graphs G and H are fractionally isomorphic if and only if we have equitable partitions $(C_1, \ldots, C_r)$ and $(C'_1, \ldots, C'_r)$ such that $|C_i| = |C'_i|$ for all $i \in [r]$ and these partitions have the same quotient matrix.*

Note: The quotient matrix has entries of the degree/number of neighbors d_{ij} between C_i and C_j in position (i, j).

Theorem 3.48 (Atserias, Mankinska, Roberson, Samal, Severini, Varvitsiatis)**.** *We have $G \cong_f H$ if and only if there exists a non-signalling correlation $p : (V(G) \cup V(H))^4 \to [0, 1]$ which wins the (G, H)-isomorphism game.*

Corollary 3.49. *$G \cong_q H \Rightarrow G \cong_f H$, as any perfect quantum strategy induces a non-signalling strategy.*

What if we relax conditions again? We note that any permutation matrix P satisfies the following condition:

$$P^T P = P P^T = I.$$

What this means is, every permutation matrix is unitary. A well known fact in graph theory is that G and H are co-spectral if and only if there exists a unitary matrix U such that

$$U^* A_G U = A_H.$$

We'll introduce a "quantum" relaxation.

Definition 3.50. *A matrix $P = (P_{ij}) \in M_n(\mathbb{C}^{d\times d}) = \mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$ is a quantum permutation matrix if*

(i) $P_{ij} = P_{ij}^2 = P_{ij}^*$ *for all* $i,j \in [n]$;
(ii) $\sum_j P_{ij} = I = \sum_\ell P_{\ell k}$ *for all* $i,k \in [n]$.

Note that these imply $P_{ij}P_{ik} = 0$ if $j \neq k$, and $P_{ij}P_{\ell j} = 0$ if $i \neq \ell$.

Lemma 3.51. *Suppose $P = (P_{ij}) \in M_n(\mathbb{C}^{d\times d})$ such that $P_{ij} = P_{ij}^2 = P_{ij}^*$ for all $i,j \in [n]$. Then P is unitary if and only if it is a quantum permutation matrix.*

Proof. Exercise! □

Theorem 3.52 (Atserias et. all)**.** *We have $G \cong_q H$ if and only if there exists a quantum permutation matrix $P = (P_{ij}) \in M_n(\mathbb{C}^{d\times d})$ satisfying*

$$P^*(A_G \otimes I_d)P = A_H \otimes I_d.$$

Remark: Note that the latter condition holds if and only if

$$(A_G \otimes I_d)P = P(A_H \otimes I_d)$$
$$\Rightarrow \text{for } (g,h) - \text{blocks- } \sum_{g\sim g'} P_{g'h} = \sum_{h\sim h'} P_{gh'}.$$

Proof. Recall that $G \cong_q H$ if and only if there exist projections $P_{gh} \in \mathbb{C}^{d\times d}$ for all $g \in V(G), h \in V(H)$ satisfying:

(i) $\sum_h P_{gh} = I$ for all $g \in V(G)$;
(ii) $\sum_g P_{gh} = I$ for all $h \in V(H)$;
(iii) $P_{gh}P_{g'h'} = 0$ if $\mathrm{rel}(g,g') \neq \mathrm{rel}(h,h')$.

We claim $P = (P_{gh})$ is a quantum permutation matrix. As the first two conditions for a quantum permutation matrix are already covered by (i) and (ii) for our P_{gh}'s, it suffices to show that condition (iii) above is equivalent to

$$(A_G \otimes I_d)P = P(A_H \otimes I_d).$$

To that end, first suppose (iii) holds. Then for any $g \in V(G), h \in V(H)$:

$$\sum_{g\sim g'} P_{g'h} = \sum_{g\sim g'} P_{g'h} \sum_{h'} P_{gh'} = \sum_{g'\sim g} P_{g'h} \sum_{h'\sim h} P_{gh'} = \sum_{g'} P_{g'h} \sum_{h'\sim h} P_{gh'} = \sum_{h'\sim h} P_{gh'}.$$

So by our remark right before this proof, this shows $(A_G \otimes I_d)P = P(A_H \otimes I_d)$.

Now, suppose $\sum_{g'\sim g} P_{g'h} = \sum_{h'\sim h} P_{gh'}$ for all g,h. Note that as a sum of orthogonal projections, these two sums must also be projections. Thus

$$\sum_{g'\sim g} P_{g'h} \sum_{h'\sim h} P_{gh'} = \left(\sum_{g'\sim g} P_{g'h}\right)^2 = \sum_{g'\sim g} P_{g'h} = \sum_{g'\sim g} P_{g'h} \sum_{h'} P_{gh'}.$$

Therefore,

$$\sum_{g' \sim g} P_{g'h} \sum_{h' \not\sim h} P_{gh'} = 0$$
$$\Rightarrow \text{ if } g \sim \hat{g},\ \hat{h} \not\sim h, \text{ then}$$
$$P_{\hat{g}h} P_{g\hat{h}} = P_{\hat{g}h} \Big(\sum_{g' \sim g} P_{g'h} \sum_{h' \not\sim h} P_{gh'} \Big) P_{g\hat{h}} = 0.$$

The proof is similar for the other cases of $\mathrm{rel}(g, \hat{g}) \neq \mathrm{rel}(h, \hat{h})$. □

Corollary 3.53. *If $G \cong_q H$, then G and H must be co-spectral.*

Introduction to C^*-algebras

In order to work with C^*-algebras, we need to introduce the notion of a Hilbert space. This is a real or complex inner product space which is a complete metric space *with respect to* the metric induced by the inner product. We'll deal exclusively with the complex case.

What can we say about this inner product? Well- exactly what we know by definition; for all $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$:

(i) $\langle y, x\rangle = \overline{\langle x, y\rangle}$;
(ii) $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$;
(iii) $\langle x, x\rangle \geq 0$ with equality only if $x = 0$.
(iv) It has induced norm $\|x\| = \sqrt{\langle x, x\rangle}$;
(v) The induced metric is $d(x, y) = \|x - y\|$;
(vi) It must be complete- that is, every Cauchy sequence converges to a point in the space.

Example 3.54.

(i) $\mathbb{C}^n$
(ii) Sequence spaces: if S is a countable set, and we have

$$\ell^2(S) = \{(x_i)_{i\in S} : \sum_{i\in S} |x_i|^2 \text{ converges}\},$$
$$\langle x, y\rangle = \sum_{i\in S} x_i \overline{y_i}.$$

Of particular interest is when $S = \Gamma$, where Γ is a group.

Suppose we have some Hilbert space $\mathcal{H}$. We'll look at the set $\mathcal{B}(\mathcal{H})$, where we define

$$\mathcal{B}(\mathcal{H}) = \{L : \mathcal{H} \to \mathcal{H}\ :\ L \text{ is linear and bounded.}\}.$$

(i) We say L is bounded if there exists $\delta \geq 0$ such that for all $h \in \mathcal{H}$,

$$\|Lh\| \leq \delta\|h\|.$$

This happens if and only if L is continuous.
(ii) The infimum (minimum) such δ is the operator norm of L, denoted $\|L\|_{op}$.

With these, we can define a C^*-algebra. For $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space, then $\mathcal{A}$ is a C^*-algebra if

(i) If $X, Y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then $\lambda X + Y \in \mathcal{A}$, with $XY \in \mathcal{A}$.
(ii) If $X \in \mathcal{A}$, then $X^* \in \mathcal{A}$ where X^* is the adjoint of X- i.e it satisfies relation

$$\langle X\psi, \phi\rangle = \langle \psi, X^*\phi\rangle$$

for all $\psi, \phi \in \mathcal{H}$.

(iii) $\mathcal{A}$ is closed in the operator norm:

$$X_n \in \mathcal{A},\ X \in \mathcal{B}(\mathcal{H}) \text{ and } \|X_n - X\|_{op} \to 0 \Rightarrow X \in \mathcal{A}.$$

Example 3.55. *The space $\mathcal{B}(\mathcal{H})$ itself is a C^*-algebra of operators.*

We also have an abstract definition for a C^*-algebra.

Definition 3.56. *An algebra $\mathcal{A}$ over $\mathbb{C}$ is a C^*-algebra if it is equipped with a norm $\|\cdot\|$ and map $* : \mathcal{A} \to \mathcal{A}$ satisfying (for all x, y:*

(i) $(x^*)^* = x$;
(ii) $(x+y)^* = x^* + y^*$;
(iii) $(\lambda x)^* = \overline{\lambda} x^*$;
(iv) $(xy)^* = y^* x^*$;
(v) $\|xy\| \leq \|x\|\|y\|$;
(vi) *$\mathcal{A}$ is complete with respect to $\|\cdot\|$;*
(vii) $\|x^* x\| = \|x\|^2$.

Example 3.57. *Let S be a compact space, and set*

$$C(S) := \{f : S \to \mathbb{C} |\ f \text{ is continuous}\}.$$

This is a commutative C^*-algebra under pointwise multiplication.

Definition 3.58. *Let $\pi : \mathcal{A} \to \mathcal{A}'$ be a linear map between $*$-algebras $\mathcal{A}$ and $\mathcal{A}'$. We say*

(i) π is a homomorphism if $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in \mathcal{A}$;
(ii) π is a $*$-homomorphism if additionally $\phi(x^*) = \pi(x)^*$.

Theorem 3.59 (GNS Theorem (Gelfand, Naimark, Segal))**.** *Let $\mathcal{A}$ be an abstract C^*-algebra. Then there exists a Hilbert space $\mathcal{H}$ and a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\|\pi(x)\| = \|x\|$ for all $x \in \mathcal{A}$. Moreover, if $\mathcal{A}$ is unital (i.e has a unit 1) then π can be chosen so that $\pi(1) = I_{\mathcal{H}}$.*

The above conditions guarantee that π is injective. Thus, π is a $*$-isomorphism between $\mathcal{A}$ and its image $\pi(\mathcal{A})$. Moreover, the conditions on π ensure that $\pi(\mathcal{A})$ is a C^*-algebra of operators. This means every (abstract) C^*-algebra is $*$-isomorphic to a C^*-algebra of operators.

States on C^*-algebras

Definition 3.60. *A state on a unital C^*-algebra $\mathcal{A}$ is a linear functional $s : \mathcal{A} \to \mathbb{C}$ such that $s(1) = 1$ and $s(x^* x) \geq 0$ for all $x \in \mathcal{A}$. A state s is faithful if $s(x^* x) = 0$ implies $x = 0$. It is tracial if $s(xy) = s(yx)$ for all $x, y \in \mathcal{A}$.*

Example 3.61.

(i) If $|\psi\rangle \in \mathbb{C}^d$ is a unit vector, then $s(M) = \langle \psi | M \psi \rangle$ is a state on $\mathbb{C}^{d\times d}$.
(ii) $tr(M) = \frac{1}{d}\operatorname{Tr}(M) = \frac{1}{d}\sum_{i=1}^{d} M_{ii}$ is a faithful tracial state on $\mathbb{C}^{d\times d}$.

Theorem 3.62 (GNS State Theorem)**.** *Let $s : \mathcal{A} \to \mathbb{C}$ be a state on a unital C^*-algebra. Then there is a Hilbert space $\mathcal{H}$, a unit vector $|\psi\rangle \in \mathcal{H}$, and unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $s(x) = \langle \psi | \pi(x) \psi \rangle$ for all $x \in \mathcal{A}$, and the subspace*

$$\{\pi(x)\,|\psi\rangle :\ x \in \mathcal{A}\}$$

is dense in $\mathcal{H}$.

Quantum Commuting Strategies

Definition 3.63. *Let G, H be two graphs, and $V = V(G) \cup V(H)$. A quantum commuting strategy for the (G, H)-isomorphism game consists of*

(i) A unit vector $|\psi\rangle \in \mathcal{H}$ for some Hilbert space $\mathcal{H}$;
(ii) POVMs $\mathcal{E}_x = \{E_{xy} \in \mathcal{B}(\mathcal{H}) : y \in V\}$ for all $x \in V$ for player A;
(iii) POVMs $\mathcal{F}_x = \{F_{xy} \in \mathcal{B}(\mathcal{H}) : y \in V\}$ for all $x \in V$ for player B;
(iv) The POVMs must satisfy

$$E_{xy}F_{x'y'} = F_{x'y'}E_{xy}$$

for all $x, y, x', y' \in V$. This produces the correlation

$$p(y, y'|x, x') = \langle\psi|E_{xy}F_{x'y'}\psi\rangle .$$

We say $G \cong_{qc} H$ if there is a perfect quantum commuting strategy for the (G, H)-isomorphism game.

Note: From now on, we'll refer to quantum commuting isomorphisms as just quantum isomorphisms. This follows from the fact that $C_q \subseteq C_{qc}$. In the finite dimensional case, the two classes are equivalent. However, for infinite dimensional Hilbert spaces this no longer holds.

Theorem 3.64 (Paulsen, Severini, Stahlke, Todorov, Winter)**.** *$G \cong_{qc} H$ if and only if there exists a unital C^*-algebra $\mathcal{A}$ with (faithful) tracial state $\tau : \mathcal{A} \to \mathbb{C}$, and projections $E_{gh} \in \mathcal{A}$ for all $g \in V(G), h \in V(H)$ satisfying:*

(i) $\sum_h E_{gh} = 1$, for all $g \in V(G)$;
(ii) $\sum_g E_{gh} = 1$, for all $h \in V(H)$;
(iii) $E_{gh}E_{g'h'} = 0$ if rel$(g, g') \neq$ rel(h, h').

Theorem 3.65. *We have $G \cong_{qc} H$ if and only if there exists a unital C^*-algebra $\mathcal{A}$ with a (faithful) tracial state, and a quantum permutation matrix $\mathcal{P} = (P_{gh}) \in M_n(\mathcal{A})$ such that*

$$A_G\mathcal{P} = \mathcal{P}A_H,$$
$$i.e. \sum_{g'\sim g} P_{g'h} = \sum_{h'\sim h} P_{gh'}$$

for all $g \in V(G), h \in V(H)$.

4. More on C^*-algebras, quantum groups. and binary linear systems

4.1. **Binary Linear Systems.**

Definition 4.1. *A binary linear system (BLS) is simply a linear system of equations over $\mathbb{Z}_2$: we have*

$$Mx = b \text{ such that } M \in M_{m\times n}(\mathbb{Z}_2),\ b \in M_{m\times 1}(\mathbb{Z}_2),\ \text{and } x \text{ is a vector variable.}$$

We can write each individual equation as a constraint: for each C_ℓ, we have

$$\sum_i M_{\ell i}x_i = b_\ell, \text{ or } \sum_{i\in S_\ell} x_i = b_\ell$$

where $S_\ell = \{i \in [n] : M_{\ell i} = 1\}$.

Example 4.2. *Let*

$$\begin{aligned} x_1+x_2+x_3&=0 & x_1+x_4+x_7&=0\\ x_4+x_5+x_6&=0 & x_2+x_5+x_8&=0\\ x_7+x_8+x_9&=0 & x_3+x_6+x_9&=1 \end{aligned}$$

This system has <u>*no*</u> *solution (each variable appears twice; adding them all up would imply* $0 \equiv 1 \pmod 2$*).*

BLS Games

Let $M \in M_{m\times n}(\mathbb{Z}_2)$, and $b \in M_{m\times 1}(\mathbb{Z}_2)$. The rules for the (M,b)-game are as follows:

(i) A is sent $\ell \in [m]$, and responds with $f : S_\ell \to \mathbb{Z}_2$.
(ii) B is sent $k \in [m]$, and responds with $f' : S_k \to \mathbb{Z}_2$.

The winning conditions for the game are as follows:

(i) We have $\sum_{i\in S_\ell} f(i) = b_\ell$.
(ii) We have $\sum_{j\in S_k} f'(j) = b_k$.
(iii) If $i \in S_\ell \cap S_k$, then $f(i) = f'(i)$.

The first two conditions are known as the "constraint satisfaction" conditions. The third ensures that we have consistency between our two players.

Note: As usual, there is a perfect classical strategy for the (M,b)-game if and only if $Mx = b$ has a solution.

Quantum Solutions for $Mx = b$

We'll change the game slightly by reformulating it into multiplicative terms. We first change our variables from $\{0,1\}$ to $x_i \in \{\pm 1\}$. For $a \in \mathbb{Z}_2$, we send $a \mapsto (-1)^a$. Finally, our equation ℓ is now

$$\ell : \prod_{i=1}^{n} x_i^{M_{\ell i}} = (-1)^{b_\ell} \text{ or, } \prod_{i\in S_\ell} x_i = (-1)^{b_\ell}.$$

(Again, here S_ℓ denotes the indices where our assignment for equation ℓ is non-zero). We relax this to operator-valued variables as follows.

Definition 4.3. *A quantum solution to* $Mx = b$ *consists of a Hilbert space* $\mathcal{H}$ *and operators* $A_i \in \mathcal{B}(\mathcal{H})$ *for* $i \in [n]$ *satisfying:*

(i) $A_i^* = A_i$ *and* $A_i^2 = 1$ *for all* $i \in [n]$*;*
(ii) $A_iA_j = A_jA_i$ *if there exists an* $\ell \in [m]$ *such that* $i, j \in S_\ell$*;*
(iii) $\prod_{i\in S_\ell} A_i = (-1)^{b_\ell} I$ *for all* $\ell \in [m]$.

(Recall that M *is an* $m \times n$ *matrix).*

Theorem 4.4 (Cleve, Liu, Slofstra)**.** *The* (M,b)*-game has a perfect quantum commuting strategy if and only if* $Mx = b$ *has a quantum solution.*

Proof. (Sketch) "Standard techniques" show that the (M,b)-game has a perfect quantum strategy if and only if there exists a C^*-algebra $\mathcal{A}$ with a tracial state and projections $P_f^\ell \in \mathcal{A}$ for $\ell \in [m]$ and $f : S_\ell \to \{\pm 1\}$ satisfying:

(i) $\sum_f P_f^\ell = 1$, for all $\ell \in [m]$.
(ii) $P_f^\ell = 0$, if $\prod_{i\in S_\ell} f(i) \neq (-1)^{b_\ell}$.
(iii) $P_f^\ell P_{f'}^k = 0$ if there exists an $i \in S_\ell \cap S_k$ such that $f(i) \neq f'(i)$.

For all $\ell \in [m]$ and $i \in S_\ell$, define

$$A_i^\ell = \sum_{f(i)=1} P_f^\ell - \sum_{f(i)=-1} P_f^\ell = \sum_f f(i) P_f^\ell.$$

We claim $A_i^\ell = A_i^{\ell *}$, $(A_i^\ell)^2 = 1$, and A_i^ℓ does not depend on ℓ.

That our A_i^ℓ's are self-adjoint is immediate. We know that $P_f^\ell P_{f'}^\ell = 0$ if $f \neq f'$, and so

$$(A_i^\ell)^2 = \sum_{f(i)=1} P_f^\ell + \sum_{f(i)=-1} P_f^\ell = \sum_f P_f^\ell = 1.$$

Finally, suppose $i \in S_\ell \cap S_k$. We see

$$A_i^\ell A_i^k = \sum_{\substack{f:S_\ell \to \pm 1 \\ f':S_k \to \pm}} f(i) f'(i) P_f^\ell P_{f'}^k = \sum_{f,f'} P_f^\ell P_{f'}^k = 1,$$

as $f(i) \neq f'(i)$ implies $P_f^\ell P_{f'}^k = 0$- hence $f(i)f'(i) \neq 1$ implies orthogonality (this is used in the reduction above). So $A_i^\ell A_i^k = 1$, and so $A_i^\ell (A_i^k)^2 = A_i^k$. Therefore, $A_i^\ell = A_i^k$. Now, using the fact that $A_i = A_i^\ell$ for any ℓ such that $i \in S_\ell$, to show the remaining conditions hold is quite simple. □

Remarks:

(i) We have $A_i = P_i^+ - P_i^-$ (where each P_i^+ and P_i^- are the sums of the projectors for whether $f(i) = 1$ or -1, respectively). This means A_i is a sum of two projectors. We note that $\{P_i^+, P_i^-\}$ forms a projective measurement for the variable x_i. In this case A_i is called a (binary) observable.

(ii) We can *almost* "go back": given a quantum solution $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$, define

$$P_i^+ = \frac{1}{2}\Big(I + A_i\Big), \quad P_i^- = \frac{1}{2}\Big(I - A_i\Big),$$
$$P_f^\ell = \prod_{i \in S_\ell} P_i^{f(i)}.$$

However, our problem here is we don't have a tracial state, which is necessary.

Example 4.5. *The following is a quantum solution to the Mermin-Peres magic square game. Let*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, all three are self-adjoint and square to the identity. Furthermore,

$$XY = -YX,\ XZ = -ZX,\ YZ = -ZY.$$

If we set

$$x_1 = Z \otimes I,\ x_2 = I \otimes X,\ x_3 = Z \otimes X,$$
$$x_4 = I \otimes Z,\ x_5 = X \otimes I,\ x_6 = X \otimes Z,$$
$$x_7 = Z \otimes Z,\ x_8 = X \otimes X,\ x_9 = Y \otimes Y$$

we should have a perfect solution (I might have the indices wrong).

The Solution Group

Recall the conditions of solutions for our system of equations:

(i) $A_i^* = A_i$ and $A_i^2 = I$ for all $i \in [n]$;
(ii) $A_i A_j = A_j A_i$ if there exists an $\ell \in [m]$ such that $i, j \in S_\ell$;
(iii) $\prod_{i \in S_\ell} A_i = (-1)^{b_\ell} I$ for all $\ell \in [m]$.

We want to encode these conditions (or at least most of them) as group relations.

Definition 4.6. *Let $M \in M_{m\times n}(\mathbb{Z}_2)$, and $b \in M_{m\times 1}(\mathbb{Z}_2)$. The solution group of $Mx = b$, denoted $\Gamma(M,b)$, is the group with generators $g_1, \dots, g_n$ and J satisfying the relations:*

(i) $g_i^2 = e$ *for all* $i \in [n]$, *and* $J^2 = e$;
(ii) $g_i J g_i^{-1} J^{-1} = e$, *i.e. our* g_i*'s commute with* J *for all* $i \in [n]$;
(iii) $g_i g_j = g_j g_i$ *if there exists an* $\ell \in [m]$ *such that* $i, j \in S_\ell$;
(iv) $\prod_{i\in S_\ell} g_i = J^{b_\ell}$ *for all* $\ell \in [m]$.

Theorem 4.7 (Cleve, Liu, Slofstra)**.** *Let M, b be as before. The following are equivalent:*

(i) The (M,b)-game has a perfect quantum strategy.
(ii) $Mx = b$ has a quantum solution.
(iii) The solution group $\Gamma(M,b)$ has $J \neq e$.

Proof. (Sketch)
We've "seen" $(i) \Rightarrow (ii)$.

So assume (ii) holds. We note that a quantum solution for $Mx = b$ is a representation for the solution group, where $J \mapsto -I$. Then $J \neq e$, and so (iii) holds.

Now assume (iii) holds. We'll show $(iii) \Rightarrow (ii)$, and pick up a tracial state on the way to prove (i). Let $\Gamma = \Gamma(M,b)$, and let

$$\mathcal{H} = \ell^2(\Gamma) = \{\sum_{g\in\Gamma} \alpha_g |g\rangle : \sum_g |\alpha_g|^2 < \infty\}.$$

Here $\langle g|h\rangle = \delta_{gh}$. Define $L_g \in \mathcal{B}(\mathcal{H})$ by letting $L_g |h\rangle = |gh\rangle$ for all $g, h \in \Gamma$. Clearly, $L_g L_h = L_{gh}$. Define $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle - |J\rangle)$ (this is a unit vector, as $J \neq e$ by assumption).

We see

$$L_J |\psi\rangle = \frac{1}{\sqrt{2}}(|J\rangle - |e\rangle) = -|\psi\rangle .$$

Also, we have

$$L_J L_g |\psi\rangle = L_{Jg} |\psi\rangle = L_{gJ} |\psi\rangle = L_g L_J |\psi\rangle = -L_g |\psi\rangle .$$

Thus, L_J acts like $-I$ on $\mathcal{H}_0 = \operatorname{span}\{L_g|\psi\rangle : g \in \Gamma\}$. It follows that by letting $A_i = L_{g_i}\big|_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H}_0)$ for $i \in [n]$, we have a quantum solution for $Mx = b$ (the details here require a bit more working out). Moreover, $A \mapsto \langle \psi|A\psi\rangle$ is tracial on $\mathcal{B}(\mathcal{H}_0)$, since

$$\begin{aligned}\langle\psi|L_g L_h \psi\rangle &= \frac{1}{2}\Big((\langle e| - \langle J|)(|gh\rangle - |ghJ\rangle)\Big)\\ &= \frac{1}{2}\Big(\langle e|gh\rangle + \langle J|ghJ\rangle - \langle J|gh\rangle - \langle e|gh\rangle\Big)\\ &= \begin{cases} 1, & \text{if } gh = e \iff hg = e,\\ -1, & \text{if } gh = J \iff hg = J,\\ 0, & \text{otherwise.}\end{cases}\\ &= \langle\psi|L_h L_g \psi\rangle .\end{aligned}$$

□

Theorem 4.8 (Cleve, Mittal)**.** *Let M, b be as before. The following are equivalent:*

(i) The (M,b)-game has a perfect quantum tensor strategy.
(ii) $Mx = b$ has a finite dimensional quantum solution.
(iii) The solution group $\Gamma(M,b)$ has a finite dimensional representation ϕ such that $\phi(J) \neq \phi(e)$.

The following is a sort of "classical" analog for the previous theorem.

Theorem 4.9. *Let M, b be as before. The following are equivalent:*

(i) The (M, b)-game has a perfect classical strategy.
(ii) $Mx = b$ has a classical solution.
(iii) $Mx = b$ has a 1-dimensional quantum solution.
(iv) $Mx = b$ has a commutative quantum solution.
(v) The abelianization of the solution group $\Gamma(M, b)$ has $J \neq e$.

Solfstra's Embedding Theorem
The following is a quite important result which plays a role in applying previously stated results.

Theorem 4.10. *Let Γ' be a finitely presented group, let J' be a central element of Γ', and let $w_1, \ldots, w_n \in \Gamma'$ be such that $w_i^2 = e$ for all $i \in [n]$. Then there is a BLS $Mx = b$, distinct indices $i_1, \ldots, i_n$ and an embedding (injective homomorphism) $\phi : \Gamma' \to \Gamma(M, b)$ such that $\phi(J') = J$ and $\phi(w_i) = g_{i_k}$ for all $k \in [n]$.*

Corollary 4.11. *There is a BLS $Mx = b$ such that the (M, b)-game has a perfect qc-strategy but no perfect q-strategy.*

Corollary 4.12 (Strengthening (Slofstra))**.** *There is a BLS $Mx = b$ such that the (M, b)-game can be won with probability arbitrarily close to* 1 *using q-strategies, but has no perfect q-strategy.*

What we really notice, based on the previous corollary, is that *the set of q-correlations is not closed*. This is an extremely important result.

Corollary 4.13. *It is undecidable to determine if a BLS game has a perfect qc-strategy.*

What we get later (again by Slofstra) is that it is undecidable to determine if:

(i) A BLS game has a perfect q-strategy.
(ii) A BLS game can be won with probability arbitrarily close to 1 using q-strategies.

Recall that $\mathcal{C}_x$ is the set of x-correlations, where $x \in \{loc, q, qs, qa, qc, ns\}$. The definition for each is:

- loc- classical
- q- finite dimensional tensor-product framework
- qs- infinite dimensional tensor-product framework
- qa- closure of q, also closure of qs (as the two closures are equal)
- qc- commuting operator framework
- ns- non-signalling

Before Slofstra, all we knew was

$$\mathcal{C}_{loc} \subsetneq \mathcal{C}_q \subseteq \mathcal{C}_{qs} \subseteq \mathcal{C}_{qa} \subseteq \mathcal{C}_{qc} \subsetneq \mathcal{C}_{ns}.$$

After Slofstra, what we knew was

$$\mathcal{C}_{loc} \subsetneq \mathcal{C}_q \subseteq \mathcal{C}_{qs} \subsetneq \mathcal{C}_{qa} \subseteq \mathcal{C}_{qc} \subsetneq \mathcal{C}_{ns}.$$

With more work, we now know

$$\mathcal{C}_{loc} \subsetneq \mathcal{C}_q \subsetneq \mathcal{C}_{qs} \subsetneq \mathcal{C}_{qa} \subsetneq \mathcal{C}_{qc} \subsetneq \mathcal{C}_{ns}.$$

Slofstra's proof relies on the use of some tools from geometric group theory- an important one is stated in the following semi-colloquial way: we start with g_i for $i \in [n]$, which are J-generators for some solution group $\Gamma(M, b)$. We can construct/find a graph G, whose edges are assigned generators g_i for each edge and where the vertices are relations generated precisely by our g_i edges. If we take the product of generators incident to the boundary of the graph in counter-clockwise order, we end up with $J^{\#} = e$, where $\#$ denotes the number of generators which give $= J$ relations in the group. Furthermore, any relation in the group can be generated in this way.

Constructing $Mx = b$ with q-solutions but no classical solutions
Let G be a graph. The incidence matrix of G is the $V(G) \times E(G)$ matrix M such that

$$M_{v,e} = \begin{cases} 1, & \text{if } v \text{ is an endpoint of } e, \\ 0, & \text{otherwise.} \end{cases} .$$

Theorem 4.14 (Arkhipov)**.** *Let M be the incidence matrix of a connected graph G, and let $b \in \mathbb{Z}_2^{V(G)}$. Then*

(i) $Mx = b$ has a solution if and only if b has an even weight.
(ii) If b has odd weight, then $Mx = b$ has a quantum solution if and only if G is not planar.

Lemma 4.15 (Arkhipov)**.** *For G and M as above and $b, b' \in \mathbb{Z}_2^{V(G)}$ where the weight of b is equal to the weight of b' modulo 2, then $Mx = b$ has a classical/quantum solution if and only if $Mx = b'$ has a classical/quantum solution.*

Proof. (Sketch proof of the theorem) We note that $x \in \mathbb{Z}_2^{E(G)}$ satisfies $Mx = b$ if and only if the graph $(V(G), \{e \in E(G) : x_e = 1\})$ satisfies $\deg(v) \equiv b_v \pmod 2$.

Assume that G is not planar. Then by Kuratowski's theorem, we know G has either $K_{3,3}$ or K_5 as a subdivision. We take it for granted (for the moment) that for both $K_{3,3}$ and K_5 the system $Mx = b$ has a quantum solution (recall the Magic Square game previously mentioned; this is the solution for $K_{3,3}$. Graph K_5 has a "Magic Pentagram" solution). We may also assume $b = e$, by the lemma above. (Unsure of conclusion here?)

For the converse, we'll prove it by contrapositive. If G is planar, then draw G in the plane. If we assume there is exactly one J-generator, and all others are e-relation generators, using the geometric group theory trick mentioned previously we say $\# = 1$. Furthermore, as G is planar when drawing our graph in the plane none of our edges will be incident to the boundary. This means $J^{\#} = J^1 = e$. □

Corollary 4.16. *If M is the incidence matrix of a connected non-planar graph G and pick $b \in \mathbb{Z}_2^{V(G)}$ with odd weight, then $Mx = b$ has a quantum solution but no classical solution.*

Remark: When M is the incidence matrix of a graph, the system $Mx = b$ has a quantum solution if and only if it has a finite-dimensional quantum solution.

A Graph Associated to $Mx = b$
Let $M \in \mathbb{Z}_2^{m \times n}$ and $b \in \mathbb{Z}_2^m$. The graph $G(M, b)$ has vertex set

$$\bigcup_{\ell=1}^{m} \{f : S_\ell \to \mathbb{Z}_2 | \sum_{i \in S_\ell} f(i) = b_\ell\}$$

and

$$f : S_\ell \to \mathbb{Z}_2, \quad f' : S_k \to \mathbb{Z}_2$$

are adjacent if there exists $i \in S_\ell \cap S_k$ such that $f(i) \neq f'(i)$.

Remark: The sets $\{f : S_\ell \to \mathbb{Z}_2 | \sum_{i \in S_\ell} f(i) = b_\ell\}$ for $\ell \in [m]$ partition $V(G(M, b))$ and each such set induces a clique (complete subgraph). This implies

$$\alpha(G(M, b)) \leq \alpha_q(G(M, b)) \leq \alpha_{qc}(G(M, b)) \leq \chi(\overline{G(M, b)}) \leq m.$$

Atserias, Mancinska, Roberson, Samal, Severini, and Varvitsiotis

Theorem 4.17. *For $M \in \mathbb{Z}_2^{m\times n}$ and $b \in \mathbb{Z}_2^m$, the following are equivalent:*

(i) $Mx = b$ *has a solution;*
(ii) $G(M,b) \cong G(M,0)$*;*
(iii) $\alpha(G(M,b)) = m$*.*

Theorem 4.18. *For $M \in \mathbb{Z}_2^{m\times n}$ and $b \in \mathbb{Z}_2^m$, the following are equivalent:*

(i) $Mx = b$ *has a finite dimensional quantum solution;*
(ii) $G(M,b) \cong_q G(M,0)$*;*
(iii) $\alpha_q(G(M,b)) = m$*;*
(iv) $G(M,b)$ *has a projective packing of value* m*.*

Theorem 4.19. *For $M \in \mathbb{Z}_2^{m\times n}$ and $b \in \mathbb{Z}_2^m$, the following are equivalent:*

(i) $Mx = b$ *has a quantum solution;*
(ii) $G(M,b) \cong_{qc} G(M,0)$*;*
(iii) $\alpha_{qc}(G(M,b)) = m$*;*
(iv) $G(M,b)$ *has a tracial packing of value* m*.*

Proof. We will prove $(ii) \Rightarrow (iii)$ first. If $G(M,b) \cong_q G(M,0)$, this implies $\alpha_q(G(M,b)) = \alpha_q(G(M,0)) = m$, as

$$m = \alpha(G(M,0)) \leq \alpha_q(G(M,0)) \leq m.$$

The same holds for qc, and hence (iii) holds.

Next, we'll show $(iii) \Rightarrow (iv)$. If $\alpha_q(G) = k$, this implies G has a projective packing of value k. The same holds for qc. For $(i) \Rightarrow (ii)$ and $(iv) \Rightarrow (i)$, **exercise!** □

We pause to recall a few general facts we have discussed so far.

(i) If G is a connected graph with incidence matrix M, and $b \in \mathbb{Z}_2^{V(G)}$ has odd weight, then
  - $Mx = b$ has *no* solution.
  - $Mx = b$ has a (finite dimensional) quantum solution if and only if G is *not* planar.

  We view the edges of G as our variables and its vertices as our equations.

(ii) Given a BLS $Mx = b$, we construct the graph $G(M,b)$ where:
  - It has vertices: satisfying assignments $f : S_\ell \to \mathbb{Z}_2$ to the equations in $Mx = b$.
  - Vertices are adjacent if they disagree.

(iii) A previous theorem gives us
  - $Mx = b$ has a solution $\iff G(M,b) \cong G(M,0)$.
  - $Mx = b$ has a quantum solution $\iff G(M,b) \cong_{qc} G(M,0)$.
  - $Mx = b$ has a finite dimensional quantum solution $\iff G(M,b) \cong_q G(M,0)$.

  As a corollary, we get that: if G is a connected non-planar graph with incidence matrix M and $b \in \mathbb{Z}_2^{V(G)}$ with odd weight. Then $G(M,b) \cong_q G(M,0)$, but $G(M,b) \ncong G(M,0)$. (For an explicit example of this, see the paper by Atserias, Mancinska, Roberson, Samal, Severini, and Varvitsiotis (2018)).

(iv) We have the theorem that $G \cong_{qc} H$ if and only if there exists a C^*-algebra $\mathcal{A}$ that admits a tracial state, and there is a quantum permutation matrix (QPM) $\mathcal{P} \in M_n(\mathcal{A})$ such that

$$A_G \mathcal{P} = \mathcal{P} A_H,$$

where the (g,h)-entry of the equation above is given by

$$\sum_{g'\sim g} \mathcal{P}_{g'h} = \sum_{h'\sim h} \mathcal{P}_{gh'}.$$

For a QPM, we have the equality given above if and only if $\mathcal{P}_{gh}\mathcal{P}_{g'h'} = 0$ if $\mathrm{rel}(g,g') \neq \mathrm{rel}(h,h')$.

Quantum Groups

Definition 4.20. *A compact matrix quantum group (CMQG) $\mathbb{G}$ is a pair $(C(\mathbb{G}),\mathcal{U})$ where $C(\mathbb{G})$ is a unital C^*-algebra which is generated by the entries of the matrix $\mathcal{U} = (u_{ij}) \in M_n(C(\mathbb{G}))$. Moreover, the $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ given by*

$$u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$$

must exist, and $\mathcal{U}$ and its transpose $\mathcal{U}^T$ must be invertible.

Why introduce compact matrix quantum groups? We'll briefly discuss the motivation for their introduction with the following example.

Example 4.21. *Let $\mathbb{G}$ be a subgroup of $GL(n,\mathbb{C})$. We use $C(\mathbb{G})$ to denote the algebra of continuous functions from $\mathbb{G}$ to $\mathbb{C}$ under pointwise multiplication. Let $u_{ij} : \mathbb{G} \to \mathbb{C}$ denote the function that maps an element of $\mathbb{G}$ to its ij-entry. Then $(C(\mathbb{G}),\mathcal{U})$ where $\mathcal{U} = (u_{ij}) \in M_n(C(\mathbb{G}))$ is a CMQG. Conversely, any CMQG $\mathbb{G} = (C(\mathbb{G}),\mathcal{U})$ where $C(\mathbb{G})$ is commutative is isomorphic to a CMQG of this form.*

In the noncommutative case, $C(\mathbb{G})$ is still often referred to as "the algebra of functions on $\mathbb{G}$".

Automorphism Group of a Graph
Recall that for a group G, it has automorphism group

$$\mathrm{Aut}(G) = \{P \in \mathbb{C}^{V(G)\times V(G)} : P \text{ is a permutation matrix and } A_G P = P A_G\}.$$

Let $u_{ij} : \mathrm{Aut}(G) \to \mathbb{C}$ be defined as in the example above.
Claim: The u_{ij} generate $C(\mathrm{Aut}(G)) \cong \mathbb{C}^{\mathrm{Aut}(G)}$.

Proof. Let $P \in \mathrm{Aut}(G)$ and define $\pi : V(G) \to V(G)$ such that

$$P_{ij} = \begin{cases} 1, & \text{if } j = \pi(i), \\ 0, & \text{otherwise.} \end{cases}.$$

Then $u_\pi = \prod_{i\in V(G)} u_{i\pi(i)}$ is the characteristic function of P:

$$u_\pi(P') = \prod_{i\in V(G)} P'_{i\pi(i)} = \begin{cases} 1, & \text{if } P = P', \\ 0, & \text{otherwise.} \end{cases}$$

□

Our second claim is as follows:
Claim: $\mathcal{U} = (u_{ij})$ is a QPM. Moreover, $A_G\mathcal{U} = \mathcal{U}A_G$.

Proof. We see $u_{ij}(P) \in \{0,1\}$, which implies

$$u_{ij}(P) = u_{ij}(P)^2 = u_{ij}(P)^* \forall P \in \mathrm{Aut}(G)$$
$$\Rightarrow u_{ij} = u_{ij}^2 = u_{ij}^*.$$

We also have

$$\sum_k u_{ik}(P) = 1 \forall\ P \in \text{Aut}(G) \Rightarrow \sum_k u_{ik} = 1.$$

Similarly, $\sum_\ell u_{\ell j} = 1$. The proof of $A_G\mathcal{U} = \mathcal{U}A_G$ is left as an **exercise!** □

Defining $C(\textbf{Aut}(G))$ Abstractly

We want a quantum analog of Aut(G), i.e. a noncommutative version of $C(\text{Aut}(G))$.

Definition 4.22. *Define $\mathcal{A}(G)$ to be the universal C^*-algebra generated by the elements p_{ij} for $i, j \in V(G)$ satisfying the relations*

(i) $p_{ij} = p_{ij}^2 = p_{ij}^*$ *for all* $i, j \in V(G)$.
(ii) $\sum_k p_{ik} = 1 = \sum_\ell p_{\ell j}$ *for all* $i, j \in V(G)$.
(iii) When $\mathcal{P} = (p_{ij})$, *then* $A_G\mathcal{P} = \mathcal{P}A_G$.
(iv) The p_{ij}*'s all commute.*

The universal C^*-algebra construction above is analogous to defining groups using generators and relations. This means that if $\mathcal{A}'$ is a C^*-algebra generated by some elements $p'_{ij} \in \mathcal{A}'$ for $i, j \in V(G)$ and the p'_{ij} satisfy the relations in the definition above, then there is a *surjective $*$-homomorphism* $\phi : \mathcal{A}(G) \to \mathcal{A}'$ such that $\phi(p_{ij}) = p'_{ij}$.

Proposition 4.23. *There is a $*$-isomorphism $\phi : \mathcal{A}(G) \to C(Aut(G))$ such that $\phi(p_{ij}) = u_{ij}$.*

Proof. **Exercise!** □

The Quantum Automorphism Group

Definition 4.24. *Define $C(Qut(G))$ to be the universal C^*-algebra generated by elements u_{ij} for $i, j \in V(G)$ satisfying the relations*

(i) $u_{ij} = u_{ij}^2 = u_{ij}^*$ *for all* $i, j \in V(G)$.
(ii) $\sum_k u_{ik} = 1 = \sum_\ell u_{\ell j}$ *for all* $i, j \in V(G)$.
(iii) $A_G\mathcal{U} = \mathcal{U}A_G$ *(here* $\mathcal{U} = (u_{ij})$*).*

Then $Qut(G) = (C(Qut(G)), \mathcal{U})$ is a CMQG called the quantum automorphism group of G. The matrix $\mathcal{U}$ is called the fundamental representation.

We have the following special case if $G = \overline{K_n}$- i.e. the empty graph on n vertices. For such a G, we have Qut$(G) = S_n^+$, which is none other than the *quantum symmetric group*

$$K_1 = \mathcal{U} = (1), \quad K_2 = \mathcal{U} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

Remark: Sometimes $C(\text{Qut}(G))$ is commutative- e.g. $G = K_n$ for $n \leq 3$, or $G = C_n$ for $n \neq 4$, or if G is the Petersen graph. In this case, we write Qut$(G) =$ Aut(G) and say G *has no quantum symmetry.*

Remark: Qut$(G) =$ Qut$(\overline{G})$, since any QPM commutes with I and J and $A_{\overline{G}} = J - I - A_G$.

Properties of Qut(G)

We will list some properties of the quantum qutomorphism group as follows:

(i) Comultiplication: $\Delta : C(\text{Qut}(G)) \to C(\text{Qut}(G)) \otimes C(\text{Qut}(G))$, and

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

is a $*$-homomorphism.

(ii) Antipode: we have $S : C(\mathrm{Qut}(G)) \to C(\mathrm{Qut}(G))^{op}$, $S(ab) = S(b)S(a)$, and $S(u_{ij}) = u_{ji}$ is a $*$-homomorphism.
(iii) Counit: $\epsilon : C(\mathrm{Qut}(G)) \to \mathbb{C}$, where $\epsilon(u_{ij}) = \delta_{ij}$ is a $*$-homomorphism.
(iv) Haar state: $h : C(\mathrm{Qut}(G)) \to \mathbb{C}$ satisfying $(h \otimes id) \circ \Delta = (id \otimes h) \circ \Delta = h$. For $\mathrm{Qut}(G)$, h is tracial.

Intertwiners:
Take $\mathcal{U}^{\otimes k}$ as a $V(G)^k \times V(G)^k$ matrix with

$$(\mathcal{U}^{\otimes k})_{i_1,\dots,i_k,j_1,\dots,j_k} = u_{i_1 j_1} \cdots u_{i_k j_k}.$$

We set $\mathcal{U}^{\otimes 0} = (1)$.

Definition 4.25. *An (ℓ, k)-intertwiner of Qut(G) is a matrix $T \in \mathbb{C}^{V(G)^\ell \times V(G)^k}$ such that $\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k}$.*

Example 4.26. *(i) A_G is a $(1,1)$-intertwiner by definition.*
(ii) $(M^{1,2})_{i,j,j'} = \begin{cases} 1, & i = j = j', \\ 0, & \textit{otherwise}. \end{cases}$ *. This is a $(1,2)$-intertwiner.*

We set $C_q^G(\ell, k)$ as the set of (ℓ, k)-intertwiners of $\mathrm{Qut}(G)$. Note that

$$C_q^G := \cup_{\ell,k} C_q(\ell, k).$$

Proposition 4.27. *C_q^G is a tensor category with duals, i.e.*
(i) $C_q^G(\ell, k)$ *is a vector space.*
(ii) $T \in C_q^G(\ell, k), T' \in C_q^G(r, s) \Rightarrow T \otimes T' \in C_q^G(\ell + r, k + s)$.
(iii) $T \in C_q^G(\ell, k), T' \in C_q^G(k, r) \Rightarrow TT' \in C_q^G(\ell, r)$.
(iv) $T \in C_q^G(\ell, k) \Rightarrow T^* \in C_q^G(k, \ell)$.
(v) $I \in C_q^G(1, 1)$.
(vi) $\psi = \sum_{i \in V(G)} e_i \otimes e_i \in C_q^G(2, 0)$.

Tannaka-Krein Duality (Woronowicz)
The correspondence between a CMQG $\mathbb{G} \subseteq O_n^+$ and its intertwiners is a one-to-one correspondence between such $\mathbb{G}$ and tensor categories with duals contained in $\cup_{\ell,k} \mathbb{C}^{n^\ell \times n^k}$.

Remark: If $\mathbb{G} \subseteq O_n^+$ is a CMQG, then $C(\mathbb{G})$ is commutative if and only if S (defined as $S(e_i \otimes e_j) = e_j \otimes e_i$) is an intertwiner of $\mathbb{G}$.

Proof. **Exercise!** □

Theorem 4.28 (Chassaniol)**.** *We have*

$$C_q^G = \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*},$$
$$C^G = \langle M^{1,2}, M^{1,0}, A_G, S \rangle_{+,\circ,\otimes,*}.$$

The latter notation is for the intertwiners of Aut(G).

Classical Case:
Let $\mathcal{U}$ be the fundamental representation of $\mathrm{Aut}(G)$. Then

$$\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k} \iff P^{\otimes \ell} T = T P^{\otimes k}, \text{ for all } P \in \mathrm{Aut}(G).$$

Thus, $T \in C^G$ if and only if T is constant on the orbits of the action of $\mathrm{Aut}(G)$ on $V(G)^\ell \times V(G)^k$. Note that

$$C^G(\ell, k) = \text{ span of characteristic matrices of orbits of } \mathrm{Aut}(G) \text{ on } V(G)^\ell \times V(G)^k.$$

In the quantum case, we can also define a notion of orbits of $\mathrm{Qut}(G)$ on $V(G)^\ell \times V(G)^k$ if $\ell + k \leq 2$.

Orbits and Orbitals of Qut(G):
Let $\mathcal{U} = (u_{ij})$ be the fundamental representation of $\mathrm{Qut}(G)$. We define the following relations on $V(G)$ and $V(G) \times V(G)$:

(i) $x \sim_1 y$ if $u_{xy} \neq 0$;
(ii) $(x, x') \sim_2 (y, y')$ if $u_{xy}u_{x'y'} \neq 0$.

Classically, we have:

$$u_{xy} \neq 0 \iff \text{there exists a } P \in \mathrm{Aut}(G) \text{ such that } P_{xy} = 1,$$
$$u_{xy}u_{x'y'} \neq 0 \iff \text{there exists a } P \in \mathrm{Aut}(G) \text{ such that } P_{xy}P_{x'y'} = 1.$$

Lemma 4.29. *Both $\sim_1$ and $\sim_2$ are equivalence relations.*

Proof. We prove it for $\sim_1$. To show reflexivity, recall the counit $\epsilon : C(\mathrm{Qut}(G)) \to \mathbb{C}$ is a $*$-homomorphism and $\epsilon(u_{xy}) = \delta_{xy}$. Thus,

$$\epsilon(u_{xx}) = 1 \Rightarrow u_{xx} \neq 0, \text{ i.e. } x \sim_1 x.$$

For symmetry, we use the antipode $S : C(\mathrm{Qut}(G)) \to C(\mathrm{Qut}(G))^{op}$. Suppose that $x \sim_1 y$, i.e. $u_{xy} \neq 0$. Then

$$S(u_{yx}) = u_{xy} \neq 0 \Rightarrow u_{yx} \neq 0, \text{ i.e. } y \sim_1 x.$$

Finally, for transitivity we use comultiplication $\Delta : C(\mathrm{Qut}(G)) \to C(\mathrm{Qut}(G))^{\otimes 2}$. Suppose that $x \sim_1 y$ and $y \sim_1 z$, i.e. $u_{xy} \neq 0$ and $u_{yz} \neq 0$. Then

$$\begin{aligned}(u_{xy} \otimes u_{yz})\Delta(u_{xz}) &= (u_{xy} \otimes u_{yz}) \sum_w u_{xw} \otimes u_{wz} \\ &= \sum_w u_{xy}u_{xw} \otimes u_{yz}u_{wz} \\ &= u_{xy} \otimes u_{yz} \neq 0.\end{aligned}$$

Thus, $u_{xz} \neq 0$, and so $x \sim_1 z$. This shows $\sim_1$ is an equivalence relation. To show $\sim_2$ is an equivalence relation, **exercise!** □

We now define the orbits and orbitals of $\mathrm{Qut}(G)$ as the equivalence classes of $\sim_1$ and $\sim_2$ respectively.

Note: $x \sim_1 y \iff u_{xy} \neq 0 \iff u_{xy}u_{xy} \neq 0 \iff (x, x) \sim_2 (y, y)$, and $(x, x) \not\sim_2 (y, z)$ if $y \neq z$ since $u_{xy}u_{yz} = 0$ in this case. Thus, the orbits are the orbitals that are contained in the diagonal of $V(G) \times V(G)$.

Coherent Configurations/Algebras:
A coherent configuration on a set X is a partition $\mathcal{R} = \{R_i : i \in I\}$ of $X \times X$ into the relations/classes R_i satisfying the following:

(i) there is a subset $\mathcal{D} \subseteq I$ such that $\{R_d : d \in \mathcal{D}\}$ is a partition of the diagonal $\{(x, x) : x \in X\}$;
(ii) for each R_i, its converse/transpose $\{(y, x) : (x, y) \in R_i\}$ is a relation, say R_i' in $\mathcal{R}$;
(iii) for all $i, j, k \in I$ there exists $p_{ij}^k \in \mathcal{N}$ such that for any $(x, z) \in R_k$:

$$\Big|\{y \in X : (x, y) \in R_i \text{ and } (y, z) \in R_j\}\Big| = p_{ij}^k.$$

We recall that the characteristic matrix of R_i is given as follows:

$$(A^i)_{xy} = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

The span of the matrices A^i is a *coherent algebra*, i.e. a subalgebra $\mathcal{A}$ of $\mathbb{C}^{X\times X}$ such that

(i) $I, J \in \mathcal{A}$;
(ii) $M \in \mathcal{A} \Rightarrow M^* \in \mathcal{A}$;
(iii) $\mathcal{A}$ is closed under Schur/entrywise product.

Conversely, any coherent algebra will have a unique basis of orthogonal (with respect to $\langle A, B\rangle = \mathrm{Tr}(AB^*)$) OI-matrices (minimal Schur idempotents) and these will be the characteristic matrices of some coherent configuration.

Remark: $\mathrm{span}\{A^i : i \in I\}$ = matrices constant on the classes of $\mathcal{R}$. In other words, they are the matrices such that

$$M \in \mathcal{A} \iff M_{xx'} = M_{yy'} \text{ whenever there exists } i \in I \text{ such that } (x, x'), (y, y') \in R_i.$$

Example 4.30. *The following are examples of coherent configurations/algebras.*

- 2*-class scheme:* $R_1 = \{(x, x) : x \in X\}, R_2 = \{(x, y) : x \neq y\}$. *Then* $A^1 = I$, *and* $A^2 = J - I$.
- *Singleton partition:* $\{(x, y)\} \in \mathcal{R}$ *for all* $x, y \in X$. *The characteristic matrices are* $E_{ij} = e_i e_j^*$.
- *Strongly regular graphs.*

Coherent Configurations/Algebras Associated to Graphs
Note: The intersection of two coherent algebras is a coherent algebra.

The *coherent algebra (configuration) of a graph* G is the smallest coherent algebra containing A_G. This can be computed efficiently by the Weisfeiler-Leman algorithm.

The orbital configuration of G is the partition of $V(G) \times V(G)$ into the orbitals of $\mathrm{Aut}(G)$. The corresponding orbital algebra of G is the commutant of $\mathrm{Aut}(G)$:

$$\{M \in \mathbb{C}^{V(G)\times V(G)} : MP = PM \text{ for all } P \in \mathrm{Aut}(G)\}.$$

The Quantum Orbital Algebra:

Lemma 4.31. *Let* $\mathcal{U} = (u_{xy})$ *be an* $X \times Y$ *quantum permutation matrix. For* $M \in \mathbb{C}^{X\times X}$ *and* $N \in \mathbb{C}^{Y\times Y}$,

$$M\mathcal{U} = \mathcal{U}N \iff M_{xx'} = N_{yy'} \text{ whenever } u_{xy}u_{x'y'} \neq 0.$$

Proof. Let $x, x' \in X$ and $y, y' \in Y$. Consider

$$u_{xy}(M\mathcal{U})_{xy'}u_{x'y'} = u_{xy}\left(\sum_{x''} M_{xx''}u_{x''y'}\right)u_{x'y'} = M_{xx'}u_{xy}u_{x'y'},$$
$$u_{xy}(\mathcal{U}N)_{xy'}u_{x'y'} = u_{xy}\left(\sum_{y''} u_{xy''}N_{y''y'}\right)u_{x'y'} = N_{yy'}u_{xy}u_{x'y'}.$$

Thus,

$$M\mathcal{U} = \mathcal{U}N \Rightarrow M_{xx'}u_{xy}u_{x'y'} = N_{yy'}u_{xy}u_{x'y'}$$
$$\Rightarrow M_{xx'} = N_{yy'} \text{ if } u_{xy}u_{x'y'} \neq 0.$$

Conversely, if $M_{xx'} = N_{yy'}$ whenever $u_{xy}u_{x'y'} \neq 0$, then

$$\sum_{x'y} u_{xy}(M\mathcal{U})_{xy'}u_{x'y'} = \sum_{x'y} u_{xy}(\mathcal{U}N)_{xy'}u_{x'y'},$$

which implies $(M\mathcal{U})_{xy'} = (\mathcal{U}N)_{xy'}$. □

Corollary 4.32. *Let $\mathcal{U}$ be the fundamental representation of Qut(G). Then $M\mathcal{U} = \mathcal{U}M$ (i.e., M is a $(1,1)$-intertwiner of Qut(G)) if and only if M is constant on the orbitals of Qut(G).*

Corollary 4.33. *The orbitals of Qut(G) form a coherent configuration, i.e. $C_q^G(1,1)$ is a coherent algebra.*

Proof. $C_q^G(1,1)$ is an algebra and $I, J \in C_q^G(1,1)$ is trivial to verify. If $M\mathcal{U} = \mathcal{U}M$, then $\mathcal{U}^*M^* = M^*\mathcal{U}^*$ and thus

$$\begin{aligned} \mathcal{U}(\mathcal{U}^*M^*)\mathcal{U} &= \mathcal{U}(M^*\mathcal{U}^*)\mathcal{U} \\ \Rightarrow M^*\mathcal{U} &= \mathcal{U}M^*. \end{aligned}$$

Thus, $C_q^G(1,1)$ is self-adjoint. Lastly, the previous corollary shows that $C_q^G(1,1)$ is closed under the Schur product. □

Theorem 4.34. *$\mathcal{U}\psi = \psi\mathcal{U}^{\otimes D}$ if and only if ψ is constant on the orbits of Qut(G). Also, the orbits form an equitable partition.*

Corollary 4.35. *Let $R_1, \ldots, R_n$ be the classes of the coherent configuration of G, and $A^1, \ldots, A^n$ their characteristic matrices. Let $\mathcal{U} = (u_{xy})$ be the fundamental representation of Qut(G). Then $u_{xy}u_{x'y'} = 0$ if (x, x') and (y, y') are not contained in some common class R_i.*

Proposition 4.36 (Babai, Kucera?)**.** *Almost all graphs have their coherent algebra equal to the full matrix algebra.*

Corollary 4.37. *Almost all graphs have trivial quantum automorphism group.*

Theorem 4.38 (Junk, Schmidt, Weber)**.** *Almost all trees have quantum symmetry, i.e. C(Qut(G)) is noncommutative.*

The Haar State:

Lemma 4.39. *Let $O_1, \ldots, O_r$ be the orbits of Qut(G) and let $R_1, \ldots, R_s$ be its orbitals. If $h : C(Qut(G)) \to \mathbb{C}$ is the Haar state of Qut(G), then*

$$h(u_{xy}) = \begin{cases} |O_i|^{-1}, & \text{if } x, y \in O_i, \\ 0, & \text{otherwise.} \end{cases}$$

Additionally,

$$h(u_{xy}u_{x'y'}) = \begin{cases} |R_i|^{-1}, & \text{if } (x, x'), (y, y') \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. **Exercise!** □

Note: $u_{xy} \neq 0 \Rightarrow h(u_{xy}) \neq 0$.

Theorem 4.40. *Let G and H be graphs. Then $G \cong_{qc} H$ if and only if there is a QPM P such that $A_G P = P A_H$.*

Theorem 4.41. *Let G and H be connected graphs. Then $G \cong_{qc} H$ if and only if there exists $g \in V(G)$ and $h \in V(H)$ in the same orbit of $Qut(G \cup H)$.*

Theorem 4.42 (Brannan, Chirvasitu, Eifler, Harris, Paulsen, Su, Wasilewski)**.** *$G \cong_{qc} H$ if and only if there is a QPM P over a $*$-algebra such that $A_G P = P A_H$.*

What is a QPM over a $*$-algebra? This is nothing different than a QPM over a C^*-algebra except we must explicitly require that

$$p_{ij}p_{ik} = 0 \text{ if } j \neq k, \text{ and } p_{ij}p_{\ell j} = 0 \text{ if } i \neq \ell.$$

Isomorphisms of Coherent Algebras:
Suppose that $\mathcal{A}, \mathcal{A}'$ are coherent algebras. A *(weak) isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is a bijective linear map $\phi : \mathcal{A} \to \mathcal{A}'$ such that

(i) $\phi(AB) = \phi(A)\phi(B)$, i.e. ϕ is an algebra isomorphism;
(ii) $\phi(A \cdot B) = \phi(A) \cdot \phi(B)$;
(iii) $\phi(A^*) = \phi(A)^*$;
(iv) $\phi(I) = I$, and $\phi(J) = J$.

If p_{ij}^k for $i, j, k \in I$ and q_{ij}^k for $i, j, k \in I'$ are the intersection numbers of $\mathcal{A}$ and $\mathcal{A}'$ respectively, then the existence of an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is equivalent to the existence of a bijection $\pi : I \to I'$ such that

$$p_{ij}^k = q_{\pi(i)\pi(j)}^{\pi(k)},$$

for all $i, j, k \in I$ and the corresponding isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ is given by $\phi(A^i) = A'^{\pi(i)}$.

Theorem 4.43. *Let $\mathcal{A}_G$ and $\mathcal{A}_H$ be the quantum orbital algebras of G and H, respectively. If $G \cong_{qc} H$, then there is an isomorphism $\phi : \mathcal{A}_G \to \mathcal{A}_H$ such that $\phi(A_G) = \phi(A_H)$.*

Corollary 4.44. *Let $\mathcal{A}_G$ and $\mathcal{A}_H$ be the coherent algebras of G and H respectively. If $G \cong_{qc} H$, then there is an isomorphism $\phi : \mathcal{A}_G \to \mathcal{A}_H$ such that $\phi(A_G) = \phi(A_H)$.*

Corollary 4.45. *If $G \cong_{qc} H$, then G and H are not distinguishable by the (2-dimensional) Weisfeiler-Leman algorithm.*

5. Open problems

5.1. **Open Problems.** The following is a list of all open problems mentioned throughout the course. As might be expected, the open problems range throughout a variety of subtopics within the field.

(i) For a graph G, if $\chi_q^r(G) \leq n$ does this imply $\chi_q^{r+1}(G) \leq n$?
(ii) For a graph G, does

$$\xi_f(G) = \inf \left\{ \frac{\chi_q(G[K_n])}{n} : n \in \mathbb{N} \right\}$$

hold?
(iii) For a graph G, does

$$\xi_f(G) = \inf \left\{ \sqrt[n]{\chi_q(G^{*n})} : n \in \mathbb{N} \right\}$$

hold?
(iv) Does

$$\alpha_q(G) = \min\{\lfloor x \rfloor : G \text{ has a projective packing of value } x\}$$

hold?
(v) For what graphs G does it hold that $G \to_q H \Rightarrow G \to H$ for all graphs H? What about $G = K_4$? What about if G is planar?
(vi) For what graphs H does it hold that $G \to_q H \Rightarrow G \to H$ for all graphs G? E.g. if H is bipartite, does this work? **Current conjecture:** No others work.
(vii) Is $\chi_q(G) = 3$ and $\chi(G) > k$ possible for any k?
(viii) Can we find examples of $G \to_q H$ but $G \not\to H$ where neither G nor H are complete? We also don't want something of the form $G \to_q K_n \to_q H$.

(ix) We know

$$\alpha_p(G) \geq \sup\left\{\frac{1}{d}\alpha_q(dG) : d \in \mathbb{N}\right\},$$
$$\alpha_p(G) \geq \sup\left\{\frac{1}{d}\alpha_q(G[\overline{K_d}] : d \in \mathbb{N}\right\}.$$

Are either of these equality?

(x) Values of various quantum graph parameters on various (families of) graphs? For example, it is known that $\chi_q(K_{n:r}) = n - 2r + 2$ (for Kneser graphs)