HARMONIC AND FOURIER ANALYSIS

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Contents

1. Introduction

These notes were taken in University of Delaware's MATH867 (Harmonic and Fourier Analysis) course, taught by Dr. Mahya Ghandehari in Spring 2022. I typed them based on hand-written notes taken during class each week- the hope was that a typed version would provide a better record in the future and be much more useful. Dr. Ghandehari's lecture notes were self-contained, though we took material from:

- *A Course in Abstract Harmonic Analysis, G.B. Folland*
- *Fourier Analysis: An Introduction, E.M. Stein & R. Shakarchi*

These notes are a work in progress; all mistakes are mine and mine alone (either through mistyping or a misunderstanding of the material). If you have any error corrections, tips, or general comments, please reach out to me at: ghoefer@udel.edu.

2. Classical harmonic and Fourier analysis

2.1. **Beginnings and the Fourier transform on $L^1(\mathbb{T})$.** As a bit of background introduction, we will begin with a discussion on the foundations and motivation for harmonic

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analysis. In brief, the study of harmonic analysis is generally about understanding important objects (typically functions/measures) on a topological group. The main objectives of our study will be:

(i) To identify "elementary components" in this space, which have the simplest behavior (for intuition, think about finding a nice basis for a vector space).
(ii) To find decompositions of my objects into elementary components.
(iii) To reconstruct an object from its decomposition.

Notation: We use the following notation in the course:

- $\mathbb{R}$ denotes the group of real numbers under addition.
- $\mathbb{Z}$ denotes the group of integers.
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denotes the group of complex numbers under multiplication.
- $\mathbb{T}$ denotes the subgroup of $\mathbb{C}^*$ defined via

$$\mathbb{T} := \{e^{ix} : \ x \in [0, 2\pi)\} \equiv \{e^{2\pi ix} : \ x \in [0, 1)\}.$$

We identify $\mathbb{T}$ with $[0, 1)$ using the functions above. We note that under this identification we have

$$e^{2\pi ix}e^{2\pi iy} = e^{2\pi i(x+y)} \leftrightsquigarrow x + y \pmod 1,$$
$$(e^{2\pi ix})^{-1} = e^{-2\pi ix} \leftrightsquigarrow 1 - x.$$

A function $\tilde{f} : \mathbb{T} \to \mathbb{C}$ is in a 1-1 correspondence with $f : \mathbb{R} \to \mathbb{C}$ which is 1-periodic (i.e., $f(x) = \tilde{f}(x \pmod 1)$). As a result, we can transport the concept of continuity/differentiability/etc. to functions on $\mathbb{T}$. We also associate

$$\int_{\mathbb{T}} f(\theta)d\theta = \int_{[0,1)} f(x)dx,$$

where we use the restriction of the Lebesgue integral of $\mathbb{R}$ to $[0, 1)$. This also allows us to transport ideas of measurability, etc. to $\mathbb{T}$.

Recall: For measurable functions $f, g : \mathbb{T} \to \mathbb{C}$, we say that $f = g$ a.e. if the set

$$\{x \in [0, 1) : \ f(x) \neq g(x)\}$$

has Lebesgue measure zero. If $f = g$ a.e., we say that f is equivalent to g, and we denote the equivalence class of f using $[f]$.

Definition 2.1. *We define*

$$L^1(\mathbb{T}) := \left\{[f] : \ f : \mathbb{T} \to \mathbb{C} \text{ Lebesgue measurable, and } \int_{\mathbb{T}} |f|d\theta < \infty\right\}.$$

For $f \in L^1(\mathbb{T})$, we define

$$\|f\|_1 = \int_0^1 |f(x)|dx.$$

Facts (easy to verify):

(i) $L^1(\mathbb{T})$ is a vector space under pointwise addition and pointwise scalar multiplication.
(ii) $(L^1(\mathbb{T}), \|\cdot\|_1)$ is a complete normed space (i.e., a Banach space).
(iii) We have an involution on $L^1(\mathbb{T})$:

$$* : L^1(\mathbb{T}) \to L^1(\mathbb{T}),$$
$$f \mapsto f^*$$

where $f^*(x) = \overline{f(1-x)} = \overline{f(-x)}$ defined pointwise.
Reading assignment! Read more about involutions to refresh yourself on their properties.

Definition 2.2. *We define*

$$L^2(\mathbb{T}) := \left\{ [f] : \ f : \mathbb{T} \to \mathbb{C} \text{ Lebesgue measureable, and } \int_0^1 |f(x)|^2 dx < \infty \right\}.$$

For $f \in L^2(\mathbb{T})$, we define

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}.$$

Fact: $(L^2(\mathbb{T}), \|\cdot\|_2)$ is not only a Banach space, but a Hilbert space as well. The inner product on the space is given via

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x)\overline{g(x)} dx.$$

Elementary components in $L^1(\mathbb{T})$
For every $n \in \mathbb{Z}$, define the function $\chi_n : \mathbb{T} \to \mathbb{C}$ via

$$\chi_n(x) = e^{2\pi i n x}.$$

(Note that here we are associating $\mathbb{T} \cong [0, 1)$).

Facts:

(i) Clearly, χ_n is continuous for each $n \in \mathbb{Z}$.
(ii) We have

$$\|\chi_n\|_1 = \int_0^1 |\chi_n(x)| dx = 1,$$

$$\|\chi_n\|_2 = 1.$$

(iii) For $n \in \mathbb{Z}$, we have $\chi_n \in L^2(\mathbb{T})$ and for any $n, m \in \mathbb{Z}$

$$\langle \chi_n, \chi_m \rangle_{L^2} = \int_0^1 e^{2\pi i n x} e^{-2\pi i m x} dx = \begin{cases} 1, & n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.3. *Suppose $f = \sum_{n=-N}^{N} a_n \chi_n$ is a linear combination of elementary components. By the comments/facts above, we note that we may recover a_n for each $n \in \mathbb{Z}$ via*

$$a_n = \langle f, \chi_n \rangle_{L^2}.$$

Polynomials of these form are called trigonometric polynomials.

Definition 2.4. *For $n \in \mathbb{Z}$, the n^{th} Fourier coefficient of a function f is*

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Fact: If f is Lebesgue measurable, then $f(x)e^{-2\pi inx}$ is Lebesgue measurable as well. Furthermore,

$$\left|\int_0^1 f(x)e^{-2\pi inx}dx\right| \leq \int_0^1 |f(x)|dx = \|f\|_1 < \infty,$$

and so $\hat{f}_n$ is well-defined (and bounded) for each $n \in \mathbb{Z}$. We associate to every $f \in L^1(\mathbb{T})$ the sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$.

Definition 2.5 (The Fourier Transform)**.** *We define the Fourier transform $\mathcal{F}$ as the function*

$$\mathcal{F} : L^1(\mathbb{T}) \to C_b(\mathbb{Z}),$$
$$f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$$

where $C_b(\mathbb{Z})$ denotes the space of bounded continuous functions on $\mathbb{Z}$.

Remark: Clearly, $C_b(\mathbb{Z})$ is a vector space under the regular pointwise operations. Equip $C_b(\mathbb{Z})$ with the norm $\|\cdot\|_{\sup}$, where

$$\|\{a_n\}_{n\in\mathbb{Z}}\|_{\sup} = \sup_{n\in\mathbb{Z}} |a_n|.$$

Then $(C_b(\mathbb{Z}), \|\cdot\|_{\sup})$ is a Banach space.

Facts:

(i) As we have seen, for all $n \in \mathbb{Z}$, $|\hat{f}(n)| \leq \|f\|_1$.

(ii) $(\widehat{f+g})(n) = \hat{f}(n) + \hat{g}(n)$, and thus

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g).$$

(iii) For $\lambda \in \mathbb{C}$ and $f \in L^1(\mathbb{T})$, we have

$$\widehat{\lambda f}(n) = \lambda\hat{f}(n) \text{ for all } n \in \mathbb{Z}.$$

Thus, $\mathcal{F}(\lambda f) = \lambda\mathcal{F}(f)$. This shows $\mathcal{F}$ is a linear operator.

(iv) The Fourier transform is bounded: for $f \in L^1(\mathbb{T})$, we have

$$\|\mathcal{F}(f)\|_{C_b(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |\hat{f}(n)| \leq \|f\|_1.$$

Thus, $\|\mathcal{F}\| \leq 1$. As a result, we know that $\mathcal{F}$ must be a continuous operator.

Corollary 2.6. *If $f_k, f \in L^1(\mathbb{T})$ where $k \in \mathbb{N}$ and $\|f_k - f\|_1 \to 0$ as $k \to \infty$, then $\hat{f}_k(n) \to \hat{f}(n)$ uniformly. Equivalently, we mean $\mathcal{F}(f_k) \to \mathcal{F}(f)$ uniformly, or $\sup_{n\in\mathbb{Z}} |\hat{f}_k(n) - \hat{f}(n)| \to 0$ as $k \to \infty$.*

Looking at $L^1(\mathbb{T})$, we will define a multiplication operation using *convolution*; multiplication in $C_b(\mathbb{Z})$ is defined pointwise. Our goal is to show that $\mathcal{F}$ is a continuous injective $*$-homomorphism between Banach $*$-algebras. Note that when we say a $*$-homomorphism between algebras, we meant that it respects involutions and takes the structure on one space to the structure on the other.

Proposition 2.7. *The Fourier transform $\mathcal{F}$ respects involution; that is,*

$$\hat{f^*}(n) = \overline{\hat{f}(n)} \quad n \in \mathbb{Z}.$$

Proof. By definition,

$$\hat{f^*}(n) = \int_0^1 f^*(x)e^{-2\pi inx}dx = \int_0^1 \overline{f(1-x)}e^{-2\pi inx}dx$$

$$= \int_1^0 \overline{f(z)}e^{-2\pi in(1-z)}d(-z) = \int_0^1 \overline{f(z)}e^{2\pi inz}dz = \overline{\int_0^1 f(z)e^{-2\pi inz}dz},$$

where we use the change of variable $z = 1 - x$. □

Remark: To simplify computations, identify $f \in L^1(\mathbb{T})$ with a 1-periodic function on $\mathbb{R}$. So,

$$\int_0^1 f(x)dx = \int_0^1 f(x+y)dx$$

for all $y \in \mathbb{R}$.

Definition 2.8. *For $f, g \in L^1(\mathbb{T})$, we define the convolution of f and g as*

$$(f * g)(x) = \int_0^1 f(y)g(-y+x)dy.$$

Note: We pause to remark on the motivation for such a definition. Think of the multiplication of two series:

$$\left(\sum_{n\in\mathbb{N}} a_n x^n\right) \cdot \left(\sum_{m\in\mathbb{N}} b_m x^m\right) = \sum_{k\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} a_n b_{-n+k}\right) x^k$$

where we have the stipulation that $-n + k \geq 1$ in the last sum.

Theorem 2.9. *Let $f, g \in L^1(\mathbb{T})$. Then for almost every $x \in \mathbb{T}$, the function*

$$y \mapsto f(y)g(-y+x)$$

is integrable. From this, we have $f * g \in L^1(\mathbb{T})$ with $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. First, since f and g are Lebesgue measurable (as elements in $L^1(\mathbb{T})$), the function

$$(x, y) \mapsto f(y)g(-y+x)$$

is measurable in the product space. Then

$$\int_0^1 \left(\int_0^1 |f(y)||g(-y+x)|dx\right)dy = \int_0^1 \left(|f(y)| \int_0^1 |g(x)|dx\right)dy$$

$$= \|g\|_1 \int_0^1 |f(y)|dy = \|f\|_1 \|g\|_1 < \infty.$$

Note that we use the fact that g is translation invariant in the computation above. By Tonelli's Theorem, we may say that

$$\int\int |f(y)g(-y+x)|d(x,y)$$

exists and is bounded. Then by Fubini's Theorem, we have

$$\int_0^1 \left(\int_0^1 f(y)g(-y+x)dx \right) dy = \int_0^1 \left(\int_0^1 f(y)g(-y+x)dy \right) dx$$

and the integral is bounded. Therefore,

$$\int_0^1 f(y)g(-y+x)dy < \infty$$

almost everywhere. Moreover,

$$\|f*g\|_1 = \int_0^1 \left| \int_0^1 f(y)g(-y+x)dy \right| dx \le \int_0^1 \int_0^1 |f(y)||g(-y+x)|dydx = \|f\|_1 \|g\|_1$$

by our work above. This completes the proof. □

Reading assignment: Read independently about Tonelli and Fubini's theorems.

Properties of convolution
For $f, g, h \in L^1(\mathbb{T})$, we have
(i) $f * g = g * f$;
(ii) $f * (g * h) = (f * g) * h$;
(iii) $f * (g + h) = f * g + f * h$.

Facts:

(i) For $n, m \in \mathbb{Z}$ we have

$$\chi_n * \chi_m = \begin{cases} \chi_n, & n = m, \\ 0, & \text{otherwise.} \end{cases}$$

To see why, write

$$\chi_m * \chi_n(x) = \int e^{2\pi imy} e^{2\pi in(-y+x)} dy = e^{2\pi inx} \langle \chi_m, \chi_n \rangle.$$

(ii) For $m \in \mathbb{Z}$,

$$f * \chi_m = \hat{f}(m)\chi_m.$$

To see why, write

$$(f * \chi_m)(x) = \int_0^1 f(y)e^{2\pi im(-y+x)}dy = \chi_m(x) \int_0^1 f(y)e^{-2\pi imy}dy = \hat{f}(m)\chi_m(x).$$

As this holds for any x, we have the result above.

Open problem: Note that χ_n is an idempotent under convolution. For $f \in L^1(G)$, we say that f is called a projection if $f = f^* = f * f$. So, χ_n is one example of a projection in $L^1(G)$. What are all projections in $L^1(G)$?

Proposition 2.10. *The Fourier transform maps convolution to the product in $C_b(\mathbb{Z})$. That is,*

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

So,

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$$

for all $n \in \mathbb{Z}$.

Proof. **Exercise!** (Hint: use the fact that $\chi_n(x+y) = \chi_n(x)\chi_n(y)$). □

Definition 2.11. *Let $(\mathcal{A}, +, \cdot)$ be a vector space over a field $\mathbb{K}$. Suppose a product $*$ is defined on $\mathcal{A}$, which satisfies for all $x, y, z \in \mathcal{A}$ and $\alpha \in \mathbb{K}$*

(i) $x * (y * z) = (x * y) * z$;
(ii) $x * (y + z) = x * y + x * z$, $(y + z) * x = y * x + z * x$;
(iii) $(\alpha \cdot x) * y = x * (\alpha \cdot y) = \alpha \cdot (x * y)$.

Then $\mathcal{A}$ is called an algebra. If $x * y = y * x$ for all $x, y \in \mathcal{A}$ we say $\mathcal{A}$ is a commutative algebra.

Definition 2.12. *A Banach algebra is a Banach space $\mathcal{A}$ which is also an algebra with a product $*$ such that*

$$\|x * y\| \leq \|x\| \cdot \|y\|$$

for all $x, y \in \mathcal{A}$.

Definition 2.13. *An involution on a Banach algebra $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ defined via*

$$x \mapsto x^*$$

such that for all $x, y \in \mathcal{A}$ and $\alpha \in \mathbb{C}$

(i) $(x + y)^* = x^* + y^*$;
(ii) $(\alpha x)^* = \overline{\alpha} x^*$;
(iii) $(x * y)^* = y^* * x^*$;
(iv) $(x^*)^* = x$.

A Banach algebra with involution is called a Banach $*$-algebra.

Theorem 2.14. *The Fourier transform $\mathcal{F} : L^1(\mathbb{T}) \to C_b(\mathbb{Z})$ is an injective $*$-homomorphism between Banach algebras.*

Proof. (Sketch) We have already seen how $\mathcal{F}$ is a continuous $*$-homomorphism between algebras; all that is left is to show injectivity.

To that end, suppose $f \in L^1(\mathbb{T})$ such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. We will use the idea of summability kernels $\{k_n\}_{n \in \mathbb{N}}$. For $n \in \mathbb{N}$, such a function k_n is defined so that

(i) $\int_0^1 k_n = 1$;
(ii) For all $\delta > 0$,
$$\lim_{n \to \infty} \int_\delta^{1-\delta} |k_n(x)| dx = 0.$$
(iii) We have
$$\int_0^1 |k_n| < \text{ a fixed constant independent of our choice of } n$$
(iv) k_n is continuous.

It can be shown that $k_n \in L^1(\mathbb{T})$, with

$$k_n * f \to f$$

in $L^1(\mathbb{T})$ as $n \to \infty$. So, for our choice of f in $\ker(\mathcal{F})$ we have $\|k_n * f\|_1 \to 0$ as $n \to \infty$. Writing this out explicitly, we find for each $n \in \mathbb{N}$

$$k_n * f = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \chi_j * f = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j) \chi_j = 0.$$

So, $k_n * f = 0$ for all $n \in \mathbb{Z}$, and as $k_n * f \to f$ in $L^1(\mathbb{T})$ this implies that $f = 0$ almost everywhere. This means if $f \in \ker(\mathcal{F})$, then $f = 0$. Thus, $\mathcal{F}$ is injective. □

Note: The summability kernel $\{k_n\}_{n\in\mathbb{N}}$ is an approximate identity for $L^1(\mathbb{T})$.

Theorem 2.15 (Riemann-Lebesgue Lemma)**.** *For every $f \in L^1(\mathbb{T})$,*

$$|\hat{f}(n)| \to 0$$

as $n \to \pm\infty$.

Proof. We first claim that

$$\operatorname{span}\{\chi_n : \ n \in \mathbb{Z}\}$$

is $\|\cdot\|_{\sup}$-dense in $C(\mathbb{T})$- this claim follows from the Stone-Weierstrass Theorem (as $\operatorname{span}\{\chi_n : \ n \in \mathbb{Z}\}$ forms a $*$-closed subalgebra of $C(\mathbb{T})$ which separates points and does not vanish at some point). We also know that $C(\mathbb{T})$ is $\|\cdot\|_1$-dense in $L^1(\mathbb{T})$. As $\|\cdot\|_1 \leq \|\cdot\|_{\sup}$, this implies the space $\operatorname{span}\{\chi_n : \ n \in \mathbb{Z}\}$ is $\|\cdot\|_1$-dense in $L^1(\mathbb{T})$ (using a standard $\epsilon/2$ argument).

Let $\epsilon > 0$ be given. There exists a trigonometric polynomial

$$p = \sum_{k=-N}^{N} a_k \chi_k$$

such that $\|p - f\|_1 < \epsilon$. Then as $\mathcal{F}$ is bounded, we have

$$\|\hat{f} - \hat{p}\|_{\sup} \leq \|f - p\|_1 < \epsilon.$$

Thus, $|\hat{f}(n) - \hat{p}(n)| < \epsilon$ for all $n \in \mathbb{Z}$. If $|n| > N$, we know $\hat{p}(n) = 0$. This means $|\hat{f}(n)| < \epsilon$ for all $|n| > N$, which means $|\hat{f}(n)| \to 0$ as $n \to \pm\infty$. □

Theorem 2.16. *The Fourier transform has dense range.*

Remark: The Fourier transform $\mathcal{F}$ is not surjective. As an example, take the sequence

$$\left\{\frac{\operatorname{sgn}(n)}{\log n}\right\}_{|n|\geq 2}.$$

This is not the Fourier transform of any $f \in L^1(\mathbb{T})$, which can be shown using a proof involving the rate of decay of such a sequence. For proof details, see Chapter 1, Section 4 of Katz-Nelson.

Proof. Let $\mathcal{A} = \mathcal{F}(L^1(\mathbb{T})) \subseteq C_0(\mathbb{Z})$. We know that $\mathcal{A}$ is a $*$-subalgebra of $C_0(\mathbb{Z})$, as $\mathcal{F}$ is a $*$-homomorphism. Also, $\mathcal{A}$ separates points, as

$$\hat{\chi_n} = (0, 0, \ldots, 0, 1, 0, \ldots,).$$

Furthermore, $\mathcal{A}$ vanishes nowhere (again, by looking at $\hat{\chi_n}$ for $n \in \mathbb{Z}$). By the Stone-Weierstrass Theorem, we once again have

$$\overline{\mathcal{A}} = C_0(\mathbb{Z}).$$

□

2.2. **The Fourier transform on $L^2(\mathbb{T})$.** We begin by noting that $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$, as $\|\cdot\|_1 \leq \|\cdot\|_2$ (we can see this via Holder, where

$$\int |f| \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2$$

for $f \in L^2(\mathbb{T})$).

We can therefore look at the restriction of $\mathcal{F}$ to $L^2(\mathbb{T})$, where

$$\mathcal{F} : L^2(\mathbb{T}) \to c_0(\mathbb{Z}).$$

Remark: Recall that L^2 is a Hilbert space, with inner product

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \int_0^1 f(x)\overline{g(x)}dx.$$

Another Hilbert space we will encounter often is the space $\ell^2(\mathbb{Z})$, with inner product

$$\langle \{a_n\}, \{b_n\}\rangle_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} a_n\overline{b_n}$$

and norm

$$\|\{a_n\}\|_{\ell^2(\mathbb{Z})} = \sqrt{\sum_{n\in\mathbb{Z}} |a_n|^2}$$

We claim that $\{\chi_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$, which we prove as follows.

Proof. To show that $\{\chi_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$, first note that $\{\chi_n\}_{n\in\mathbb{Z}}$ is an orthonormal family. We claim that it is complete. Indeed, if $f \in L^2(\mathbb{T})$ such that $\langle f, \chi_n\rangle = 0$ for all $n \in \mathbb{Z}$, then $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. As $\mathcal{F}$ is injective, this means $f = 0$ almost everywhere. Thus, if $f \perp \chi_n$ for all $n \in \mathbb{Z}$, $f = 0$; hence, the family $\{\chi_n\}_{n\in\mathbb{Z}}$ is complete, and thus an orthonormal basis. □

Remark: Recall from the theory of Hilbert spaces that if $\{\varphi_n\}_{n\in\mathbb{Z}}$ is an orthonormal family in a Hilbert space $\mathcal{H}$, the following are equivalent:

(i) $\{\varphi_n\}_{n\in\mathbb{Z}}$ is complete; i.e., $\overline{[\{\varphi_n\}]^\perp} = 0$.

(ii) For all $f \in \mathcal{H}$, we have

$$f = \sum_n \langle f, \varphi_n\rangle\varphi_n$$

which converges in $\mathcal{H}$.

(iii) For all $f \in \mathcal{H}$,

$$\|f\|_{\mathcal{H}}^2 = \sum_n |\langle f, \varphi_n\rangle|^2.$$

Corollary 2.17. *The space $L^2(\mathbb{T})$ is separable.*

Theorem 2.18 (Parseval's Identity)**.** *For all $f, g \in L^2(\mathbb{T})$,*

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \langle f, \chi_n\rangle\overline{\langle g, \chi_n\rangle} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$

Note: For any $f \in L^2(\mathbb{T})$, f is the $\|\cdot\|_2$-limit of $\sum_{k=-N}^{N} \hat{f}(k)\chi_k$.

We omit the proof of the theorem above; we can cobble it together using what we have discussed so far. Alternatively, see the lecture notes for MATH806 for a detailed proof.

Theorem 2.19. *The function $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ defined via*

$$f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$$

is an isometric isomorphism of Hilbert spaces.

Proof. The proof follows from the restriction of $\mathcal{F}$ on $L^1(\mathbb{T})$ to $L^2(\mathbb{T})$, along with Parseval's Identity. □

Theorem 2.20 (Fourier Inversion)**.** *Let $f \in L^1(\mathbb{T})$. If $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$- i.e.,*

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty,$$

then for almost every $x \in \mathbb{T}$, we have

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\chi_n(x).$$

If f is continuous, then the convergence holds everywhere.

Proof. Suppose $\{\hat{f}(n)\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$. Then the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\chi_n$$

converges in $L^2(\mathbb{T})$. Thus, there exists a $g \in L^2(\mathbb{T})$ such that $\sum\limits_{n=-N}^{N} \hat{f}(n)\chi_n \to g$ in $\|\cdot\|_{L^2(\mathbb{T})}$. Since $\|\cdot\|_{L^1(\mathbb{T})} \leq \|\cdot\|_{L^2(\mathbb{T})}$, we also have $\sum\limits_{n=-N}^{N} \hat{f}(n)\chi_n \to g$ in $\|\cdot\|_{L^1(\mathbb{T})}$ as $N \to \infty$. Since $\mathcal{F}$ is continuous, we have $\hat{g}(k) = \hat{f}(k)$; by injectivity of $\mathcal{F}$, this implies $g = f$ almost everywhere. On the other hand, by the Weierstrass M-test, $\sum\limits_{n\in\mathbb{Z}} \hat{f}(n)\chi_n$ converges uniformly to g. Thus, g must be continuous (proving the second statement). □

2.3. **Classic examples of dual groups.** Up until now, there was very little about the inherent properties of the space $\mathbb{T}$ which we used to prove results about $\mathcal{F}$. We used the fact that there was a "good" measure on $\mathbb{T}$ to integrate, and the continuity of $\mathcal{F}$ came from a "nice" topology on $\mathbb{T}$. We aim (ultimately) to see how far we can abstract our space $\mathbb{T}$ while keeping sufficiently nice topologies and measures in order to generalize our statements about $\mathcal{F}$.

Question: What is the relation between $\mathbb{T}$ and $\mathbb{Z}$?

Recall: $\chi_n : [0,1) \to \mathbb{T}$ where

$$\chi_n(x) = e^{2\pi i n x}, \quad n \in \mathbb{Z}.$$

Definition 2.21. *We define*

- *(i)* $\hat{\mathbb{T}} :=$ *{all continuous group homomorphisms* $[0,1) \to \mathbb{T}$*};*
- *(ii)* $\hat{\mathbb{R}} :=$ *{all continuous group homomorphisms* $\mathbb{R} \to \mathbb{T}$*};*
- *(iii)* $\hat{\mathbb{Z}} :=$ *{all continuous group homomorphisms* $\mathbb{Z} \to \mathbb{T}$*};*
- *(iv)* $\hat{\mathbb{Z}_n} :=$ *{all continuous group homomorphisms* $\mathbb{Z}_n \to \mathbb{T}$*}.*

Lemma 2.22. *Let $\varphi : \mathbb{R} \to \mathbb{T}$ be a continuous group homomorphism. Then there exists some $\xi \in \mathbb{R}$ such that*

$$\varphi(x) = e^{i\xi x}.$$

Proof. For such a φ, we have

$$\varphi(0) = \varphi(0+0) = \varphi(0)\varphi(0).$$

This implies $\varphi(0) = 1$. Since φ is continuous, both the real and imaginary parts of φ must be continuous. Thus, there exists a $\delta > 0$ such that $(\mathrm{Re}\varphi)(t) > 1/2$ for all $t \in [0, \delta)$ (as $\varphi(0) = 1$). Let

$$A := \int_0^\delta \varphi(t)dt.$$

We know $A \neq 0$, by basics of complex integration for functions from $\mathbb{R} \to \mathbb{C}$.

For all $x \in \mathbb{R}$, we see

$$\varphi(x) = \frac{1}{A}(A\varphi(x)) = \frac{1}{A}\varphi(x) \int_0^\delta \varphi(t)dt$$
$$= \frac{1}{A} \int_0^\delta \varphi(x+t)dt = \frac{1}{A} \int_x^{x+\delta} \varphi(t)dt.$$

Therefore, $\varphi(x)$ is differentiable with

$$\varphi'(x) = \frac{1}{A}[\varphi(x+\delta) - \varphi(x)] = \frac{1}{A}[\varphi(\delta) - 1]\varphi(x)$$

for every $x \in \mathbb{R}$. Solving this differential equation, we get

$$\varphi(x) = ce^{\eta x},$$

for some $\eta, c \in \mathbb{C}$. As $\varphi(0) = 1$, this forces $c = 1$. Additionally, as $|\varphi(x)| = 1$ (as $\varphi(x) \in \mathbb{T}$), this forces η to not have any real part. Thus,

$$\varphi(x) = e^{i\xi x}$$

for some $\xi \in \mathbb{R}$. This completes the proof. □

Theorem 2.23. *We have*

$$\hat{\mathbb{T}} = \{\chi_n : n \in \mathbb{Z}\}.$$

Proof. To show this, we need to prove two things:

(i) For all $n \in \mathbb{Z}$, χ_n is a continuous group homomorphism.
(ii) Every continuous group homomorphism $\psi : [0,1) \to \mathbb{T}$ is equal to χ_k for some $k \in \mathbb{Z}$.

To that end, we first prove (i). Note that if $x_n \to x$ where $x \in (0,1)$, then χ_n clearly preserves the limit through continuity. Else, if $x_n \to 0$ in $[0,1)$ then for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ where either $|x_n| < \epsilon$ for all $n \geq N$ or $|1 - x_n| < \epsilon$ for all $n \geq N$. This should imply continuity of χ_n everywhere on $[0,1)$; as an **exercise**, write this out formally. Alternatively, think about how $\mathbb{T}$ and $[0,1)$ are homeomorphic, and consider the topology of $\mathbb{T}$ as a subset of $\mathbb{C}$.

To show that χ_n is a group homomorphism, we note

$$\chi_n((x+y) \pmod 1) = e^{2\pi i n(x+y \pmod 1)} = e^{2\pi i n x}e^{2\pi i n y} = \chi_n(x)\chi_n(y).$$

Thus, χ_n is a homomorphism like we claimed, which finishes establishing (i).

To show (ii), let $\psi : [0,1) \to \mathbb{T}$ be a continuous group homomorphism. Let $q : \mathbb{R} \to [0,1)$ be defined via

$$x \mapsto x \pmod 1.$$

It is easy to check that q is also a continuous group homomorphism. Thus, the composition

$$\psi \circ q : \mathbb{R} \to \mathbb{T}$$

is a continuous group homomorphism from $\mathbb{R}$ into $\mathbb{T}$. By Lemma 2.22, we have some $\xi \in \mathbb{R}$ such that

$$(\psi \circ q)(x) = e^{i\xi x}$$

for all $x \in \mathbb{R}$. Let $x = 1$; we see $q(1) = 0$, and hence $\psi(q(1)) = \psi(0) = 1 = e^{i\xi}$. Thus, ξ must be an integer multiple of 2π (using what we know about the complex exponential function); i.e., $\xi = 2\pi k$ for some $k \in \mathbb{Z}$. This means

$$\psi(x) = e^{2\pi kix}$$

for all $x \in [0,1)$, which completes the proof. □

Examples:

(i) By Lemma 2.22,

$$\hat{\mathbb{R}} = \{\chi_r : \ r \in \mathbb{R}\}$$

where $\chi_r : \mathbb{R} \to \mathbb{T}$ is defined via

$$x \mapsto e^{irx}.$$

(ii) $\hat{\mathbb{Z}} = \{\chi_\theta : \ \mathbb{Z} \to \mathbb{T}, n \mapsto \theta^n\}$.

Proof. If $\chi : \mathbb{Z} \to \mathbb{T}$ is a continuous group homomorphism,

$$\chi(n) = \chi(1 + \cdots + 1) = \chi(1)^n, \quad n \geq 0.$$

The same result holds even if $n < 0$, which we can prove in an almost identical manner as above. Thus, χ is defined entirely by where it sends 1. As $\chi(1)$ must be an element of $\mathbb{T}$, we can pick any $\theta \in \mathbb{T}$ so that $\chi(1) = \theta$. This establishes a one-to-one correspondence with functions in $\hat{\mathbb{Z}}$. □

Remark: Let G be either of $\mathbb{T}, \mathbb{R}$, or $\mathbb{Z}$. Let $\psi, \varphi \in \hat{G}$- i.e., $\varphi, \psi : G \to \mathbb{T}$ are continuous group homomorphisms. Then

$$\varphi\psi : G \to \mathbb{T},$$
$$(\varphi\psi)(x) = \varphi(x)\psi(x)$$

is a continuous group homomorphism.

Notation: Let G be either of $\mathbb{R}, \mathbb{T}$, or $\mathbb{Z}$. $\hat{G}$, equipped with pointwise multiplication of functions, is a group (by the previous remark). The identity is

$$\iota : G \to \mathbb{T},$$
$$x \mapsto 1.$$

$\hat{G}$ is called the dual group of G.

Remark: The following are group isomorphisms:

(i)

$$\hat{\mathbb{Z}} \to \mathbb{T},$$
$$\chi_\theta \mapsto \theta.$$

(ii)

$$\hat{\mathbb{R}} \to \mathbb{R},$$
$$\chi_r \mapsto r.$$

(iii)

$$\hat{\mathbb{T}} \to \mathbb{Z},$$
$$\chi_n \mapsto n.$$

For a proof, see Assignment 1.

Remark: We had the Fourier transform on $\mathbb{T}$:

$$\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z}) = c_0(\hat{\mathbb{T}}),$$
$$f \mapsto \hat{f},$$
$$\hat{f}(n) = \int_0^1 f(x\overline{\chi_n}(x)dx.$$

We shall show the Fourier transform on $\mathbb{R}$ is

$$\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R}),$$
$$f \mapsto \hat{f},$$
$$\hat{f}(r) = \int_{\mathbb{R}} f(x)\overline{\chi_r}(x)d\lambda$$

and the Fourier transform on $\mathbb{Z}$ is

$$\mathcal{F} : \ell^1(\mathbb{Z}) \to C(\mathbb{T}),$$
$$f \mapsto \hat{f},$$
$$\hat{f}(\theta) = \sum_{n\in\mathbb{Z}} f(n)\overline{\chi_\theta}(n).$$

Fact: $\theta : \mathbb{R} \to \hat{\mathbb{R}}$ defined via

$$r \mapsto \chi_r$$

is a (group) isomorphism, and both θ, θ^{-1} are continuous (i.e., θ is a homeomorphism).

Exercise: Show that θ^{-1} is continuous.

Under this identification, we typically write

$$\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R}).$$

Theorem 2.24. *For all $f \in L^1(\mathbb{R})$, $\hat{f}$ is continuous and vanishing at ∞.*

Proof. Let $f \in L^1(\mathbb{R})$. We first want to show that $\hat{f}$ is continuous. Suppose $r_n \to r$ in $\mathbb{R}$; we will look at $\hat{f}(r_n)$. We want to show

$$\int_{\mathbb{R}} f(x)e^{-2\pi i r_n x}dx \to \int_{\mathbb{R}} f(x)e^{-2\pi i r x}dx.$$

For each $n \in \mathbb{N}$, define

$$g_n(x) = f(x)e^{-2\pi i r_n x},$$
$$g(x) = f(x)e^{-2\pi i r x}.$$

It should be clear that $g_n \to g$ pointwise on $\mathbb{R}$, and all are dominated by $|f| \in L^1(\mathbb{R})$. Thus, but the Dominated Convergence Theorem, this means

$$\int_{\mathbb{R}} f(x)e^{-2\pi i r_n x}dx \to \int_{\mathbb{R}} f(x)e^{-2\pi i r x}dx.$$

Therefore, $\hat{f}(r_n) \to \hat{f}(r)$. This shows $\hat{f}$ is continuous.

Next, we show that $\hat{f}$ vanishes at ∞- that is, for every $\epsilon > 0$ there exists an $N > 0$ such that for all $r \in \mathbb{R}$ such that $|r| > N$, we have $|\hat{f}(r)| < \epsilon$. To that end, let $\epsilon > 0$ be arbitrary. Note that $C_c(\mathbb{R})$ is $\|\cdot\|_1$-dense in $L^1(\mathbb{R})$ (where $C_c(G)$ denotes the space of all continuous and compactly supported functions on G). We also note that the collection of all step functions are $\|\cdot\|_{\sup}$-dense in $C_c(\mathbb{R})$. So, there exists a function g of the form

$$g = \sum_{k=1}^{N} a_k \chi_{[r_k, s_k]}$$

such that $\|g - f\|_1 < \epsilon$. This implies $|\hat{g}(r) - \hat{f}(r)| < \epsilon$ for all $r \in \mathbb{R}$; specifically, we have

$$\hat{g}(r) = \sum_{k=1}^{N} a_k \int_{r_k}^{s_k} e^{-2\pi i r x}dx = \sum_{k=1}^{N} a_k \left(\frac{e^{-2\pi i r s_k} - e^{-2\pi i r r_k}}{-2\pi i r} \right),$$

from which it is easy to deduce the inequality above (after writing out $\hat{f}(r)$). Additionally, we see

$$|\hat{g}(r)| \leq \frac{1}{2\pi|r|} \left(2 \sum_{k=1}^{N} a_k \right),$$

and so $|\hat{g}(r)| \to 0$ as $r \to \infty$. Thus, choose M such that for all $r \in \mathbb{R}$ where $|r| > M$, we get $|\hat{f}(r)| < 2\epsilon$. This implies $\hat{f}$ vanishes at infinity, completing the proof. □

3. Locally compact groups

3.1. **Initial definitions.**

Definition 3.1. *The pair (X, τ) is a topological space if X is a set, and τ is a collection of subsets of X satisfying:*

(i) $\emptyset, X \in \tau$;
(ii) τ *is closed under arbitrary unions;*
(iii) τ *is closed under taking finite intersections.*

We call members of τ "open sets".

Definition 3.2. *We say (X, τ) is a locally compact topological space if for every $x \in X$ there exists an open U, and K compact such that $x \in U \subseteq K$.*

Definition 3.3. *Let G be a group. Suppose G is a topological space as well. We say G is a topological group if both the inverse map*

$$\iota : G \to G,$$
$$g \mapsto g^{-1}$$

and the multiplication map

$$m : G \times G \to G,$$
$$(x, y) \mapsto xy$$

are continuous.

Notation: For $A, B \subseteq G$ and $x \in G$ define

$$xA := \{xa : a \in A\},$$
$$Ax := \{ax : a \in A\},$$
$$AB := \{ab : a \in A, b \in B\}.$$

Proposition 3.4. *Let G be a topological group.*

(i) If $U \subseteq G$ is open and $x \in G$, then xU, Ux are both open.
(ii) If $U \subseteq G$ is open, then $U^{-1} = \{x^{-1} : x \in U\}$ is open.
(iii) If $A, B \subseteq G$ are compact, then AB is compact.
(iv) If $e \in U$ and U is open, there exists an open V such that
  (i) $V = V^{-1}$ (symmetric);
  (ii) $VV \subseteq U$;
  (iii) $e \in V$.
(v) If $H \subseteq G$ is an open subgroup, then H is closed.
(vi) If $U \subseteq G$ is open and $A \subseteq G$, then UA, AU are open.

Proof. (i) Fix $x \in G$ and define the mappings

$$_xm : G \to G,$$
$$y \mapsto xy$$

and

$$m_x : G \to G,$$
$$y \mapsto yx.$$

By joint continuity of the multiplication map $m : G \times G \to G$, both $_xm$ and m_x are continuous maps (you should verify this!). Take any open $U \subseteq G$, and consider

$$_xm^{-1}(U) = \{x^{-1}y : y \in U\} = x^{-1}U.$$

Similarly,

$$m_x^{-1}(U) = Ux^{-1}.$$

Then by continuity of the two operators, both $x^{-1}U$ and Ux^{-1} are open for any $x \in G$- thus, we are done.

(ii) This follows directly from the continuity of the inverse mapping.

(iii) As m is continuous, and $A \times B \subseteq G \times G$ is compact, the set $m(A \times B) = AB$ must be compact (as the image of a compact set under a continuous function).

(iv) Let U be open, and $e \in U$ be given. Then there should exist neighborhoods W_1, W_2 of e such that $W_1W_2 \subseteq U$ (by continuity of m at (e, e)). Let

$$V = W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}.$$

As $e \in W_i, W_i^{-1}$ for $i = 1, 2$ we see V is open (as the finite intersection of open sets), symmetric, with $VV \subseteq U$ and $e \in V$.

(v) Note that $H^C = G \setminus H = \bigcup_{x \in G \setminus H} xH$. As H was open, we have that H^C is the arbitrary union of open sets- hence, it must also be open. This means H is closed.

(vi) We note that

$$UA = \bigcup_{a \in A} Ua$$

is an arbitrary union of open sets, and therefore must be open (as U is open). The same can be said for AU. □

Definition 3.5. *A topological group G is called a locally compact group if its topology is locally compact.*

Definition 3.6. *A left Haar measure on a locally compact group G is a measure*

$$\lambda : \mathscr{B}(G) \to [0, \infty]$$

(where $\mathscr{B}(G)$ is the Borel σ-algebra of G) satisfying

(i) λ is a Radon measure:

  (i) For every $B \in \mathscr{B}(G)$,

$$\lambda(B) = \inf\{\lambda(U) : \ B \subseteq U, \ U \ open\}.$$

  (ii) For every U open,

$$\lambda(U) = \sup\{\lambda(K) : \ K \subseteq U, \ K \ compact\}.$$

  (iii) $\lambda(K) < \infty$ for all compact sets K.

(ii) λ must be translation invariant; that is, for every $B \in \mathscr{B}(G)$ and $x \in G$, we have

$$\lambda(xB) = \lambda(B).$$

A right Haar measure can be defined similarly.

Examples:

(i) Any discrete group: $(\mathbb{Z}, \text{discrete}), (\mathbb{R}, \text{discrete})$, etc. For any discrete group G, the Haar measure on G is the counting measure:

$$\lambda(E) = |E| = \begin{cases} \text{\# of elements if } E \text{ is finite,} \\ \infty \text{ if } E \text{ is infinite.} \end{cases}$$

(ii) $(\mathbb{R}, \text{Euclidean metric})$ is a locally compact group, and the Haar measure is the Lebesgue measure.

(iii) For $n \in \mathbb{N}$, $\mathrm{GL}_n(\mathbb{R}) = \{n \times n \text{ real invertible matrices}\}$. The topology is given via

$$A_k \to A \iff (A_k)_{ij} \to A_{ij}$$

for all $i, j = 1, \ldots, n$.

Exercise: What is the Haar measure here?

3.2. **Existence of a left Haar measure.**

Definition 3.7. *Let G be a locally compact group. We define*

$$M(G) := \{all\ complex\text{-}valued\ Radon\ measures\ on\ G\},$$

and

$$C_c(G) = \{f : G \to \mathbb{C} \ continuous\ and\ \mathrm{supp}(f) \ is\ compact\}.$$

Theorem 3.8 (Riesz-Markov-Kakutatni Representation Theorem)**.** *There is a one-to-one correspondence*

$$M(G) \to C_c(G)^*$$

given via

$$\mu \mapsto T_\mu,$$

$$T_\mu(f) = \int_G f d\mu.$$

We omit the proof of the theorem.

Remark: Let λ be the left Haar measure on a locally compact group G. We clearly have $\lambda \in M(G)$; so look at $T_\lambda : C_c(G) \to \mathbb{C}$ where

$$T_\lambda(f) = \int_G f d\lambda.$$

We have

$$\int_G f(yx)d\lambda(x) = \int_G f(x)d\lambda(x) \tag{1}$$

for all $y \in G$. (To see why, show it first for a characteristic function and then extend via linearity). We have that property (1) holds if and only if λ is left translation invariant.

Definition 3.9. *Let $f \in C_c(G)$, and $y \in G$. Define $L_y f : G \to \mathbb{C}$ and $R_y f : G \to \mathbb{C}$ via*

$$(L_y f)(x) = f(y^{-1}x),$$
$$(R_y f)(x) = f(xy).$$

Note: For $y, z \in G$ we have

$$L_y L_z = L_{yz}, \quad R_y R_z = R_{yz}.$$

Proposition 3.10. *Let $f \in C_c(G)$ be fixed. The maps*

$$\Psi_L : G \to C_c(G),$$
$$x \mapsto L_x f,$$

and

$$\Psi_R : G \to C_c(G),$$
$$x \mapsto R_x f$$

are continuous at $e \in G$.

Remark: The map Ψ_L is continuous at e if and only if for all $\epsilon > 0$, there exists $e \in V \subseteq G$ open such that for every $x \in V$,

$$\|L_x f - f\|_{\sup} < \epsilon.$$

However, the latter means that for every $y \in G$,

$$|f(x^{-1}y) - f(y)| < \epsilon.$$

As we may assume without loss of generality that V is symmetric, if we let $z = x^{-1}y$ we have $yz^{-1} \in V$; i.e., if z is in the ϵ-neighborhood of y, then the pre-image under a shift is in the ϵ-neighborhood around e. This is essentially an equivalent version of uniform continuity for topological groups.

Proof. Since $f \in C_c(G)$, we know that $K = \text{supp}(f)$ is compact. Take $x \in G$, and let $\epsilon > 0$ be given. As f is continuous at x, there exists an open neighborhood $e \in V_x \subseteq G$ such that for every $y \in V_x$, we have $|f(yx) - f(x)| < \epsilon/2$. Without loss of generality, assume V_x is symmetric and $V_x V_x \subseteq V_x$. Now $\{V_x x\}_{x \in K}$ is an open cover for K- as K is compact, there exists a finite subcover; call it

$$(V_{x_1})x_1, \ldots, (V_{x_n})x_n.$$

Let $V := \cap_{i=1}^n V_{x_i}$- this is clearly open in G. We claim that for all $x \in V$, for all $y \in G$ we have

$$|f(x^{-1}y) - f(y)| < \epsilon.$$

Case I: If $y \in K$, there exists a $j \in [n]$ such that $y \in (V_{x_j})x_j$. We see

$$|f(x^{-1}y) - f(y)| \leq |f(x^{-1}y) - f(x_j)| + |f(x_j) - f(y)|.$$

As $x^{-1} \in V$, we see $x^{-1}y \in V^{-1}(V_{x_j})x_j \subseteq (V_{x_j})x_j$. Thus, $|f(x^{-1}y) - f(x_j)| < \epsilon/2$. Similarly as $y \in (V_{x_j})x_j$ we have $|f(x_j) - f(y)| < \epsilon/2$. These together show

$$|f(x^{-1}y) - f(y)| < \epsilon.$$

Case II: If $x^{-1}y \in K$, the proof is similar.

Case III: If $x^{-1}y, y \notin K$ then $f(x^{-1}y) = f(y) = 0$. In such a case, there is nothing to prove.

As the statement holds in all three cases, by the remark above we complete the proof. □

Theorem 3.11. *Every locally compact group G has a Haar measure.*

Proof. (Sketch) Let G be a locally compact group. By the Riesz Representation Theorem, we are looking for some linear map $T : C_c(G) \to \mathbb{C}$ such that

$$T(L_x f) = T(f)$$

for all $f \in C_c(G)$, and all $x \in G$. Define for any $f, \varphi \in C_c(G)^+$ the value

$$I(f : \varphi) = \inf\{\sum_{i=1}^{n} c_i : \ f \leq \sum_{i=1}^{n} c_i L_{x_i}\varphi, \ c_i \in \mathbb{R} \text{ for some choice of } x_i \in G\}.$$

We note that

$$I(cf : \varphi) = cI(f : \varphi), \quad c > 0.$$

Similarly,

$$I(f + g : \varphi) \leq I(f : \varphi) + I(g : \varphi),$$
$$I(f : \varphi) \leq I(f : \psi)I(\psi : \varphi),$$
$$I(L_x f : \varphi) = I(f : \varphi)$$

for all $x \in G$.

Now, we normalize. For a fixed $f_0 \in C_c^+(G)$, define

$$I_\varphi(f) = \frac{I(f : \varphi)}{I(f_0 : \varphi)}.$$

Then for a fixed φ, I_φ is positive homogeneous, sub-additive, translation invariant, and

$$\frac{1}{I(f_0 : f)} \leq I_\varphi(f) \leq I(f : f_0).$$

This implies we have a bound independent of φ.

Lemma 3.12. *For $f_1, f_2 \in C_c^+(G)$ fixed, and for $\epsilon > 0$ there exists an open set $e \in V$ such that for every $\varphi \in C_c^+(G)$, if $\operatorname{supp}(\varphi) \subseteq V$ we have*

$$I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \epsilon.$$

We will build a "good" space: for all $f \in C_c^+(G)$, let

$$X_f = [(f_0 : f)^{-1}, (f : f_0)] \subseteq \mathbb{R}.$$

We let $X = \prod_{f \in C_c^+(G)} X_f$. We note that points of X can give us functions $\theta : C_c^+(G) \to \mathbb{R}$ where

$$f \mapsto x_f, \quad \{x_f\}_{f \in C_c^+(G)}.$$

We have that $\theta(f) \in X_f$, and additionally $I_\varphi \in X$ for all $\varphi \in C_c^+(G)$ (as we have shown). We also know that X is compact (by Tychonoff's Theorem), and it is Hausdorff. For every open set V with $e \in V$, define

$$K_V = \overline{\{I_\varphi : \text{ supp}(\varphi) \subseteq V\}} \subseteq X.$$

Note that K_V is compact, as a closed subset of a compact Hausdorff space. It is also easy to show that

$$K_{\cap_{i=1}^n V_i} \subseteq K_{V_1} \cap K_{V_2} \cap \cdots \cap K_{V_n}.$$

This implies that the sets K_V satisfy the Finite Intersection Property, and thus

$$\bigcap_{\substack{e \in V \subseteq G \\ V \text{ open}}} K_V \neq \emptyset$$

by compactness of X. Let I be in the intersection of K_V's. For a sketch of the remainder of the proof, note that as $I \in K_V$ for all open $V \subseteq G$ which are neighborhoods of $e \in G$, it must be "near" some I_φ from some V. Using the lemma above, we can therefore show linearity of I via approximations- i.e., we end up getting

$$I(f+g) = I(f) + I(g)$$

for $f, g \in C_c^+(G)$. It is easy to check that the other properties listed for I_φ must also hold for I. Therefore, our functional behaves how we'd like on $C_c^+(G)$. If we then take any $f \in C_c(G)$, we know there exists $f^+, f^- \in C_c^+(G)$ such that $f = f^+ - f^-$. Defining $I : C_c(G) \to \mathbb{C}$ (abusing notation here) via

$$I(f) = I(f^+) - I(f^-),$$

we have our desired map. Applying the Riesz-Markov-Kakutani Representation Theorem, this implies the existence of a Haar measure on G. □

Remarks:

(i) We have

$$\int_G (L_x f)(y) d\lambda(y) = \int_G f(y) d\lambda(y), \quad x \in G.$$

(ii) We have

$$\int_G f(xy) d\lambda(y) = \int_G f(y) d\lambda(y)$$

for $x \in G$, where λ is the Haar measure in both.

Theorem 3.13. *If λ, μ are left Haar measures of a locally compact group G, then there exists $c > 0$ such that $\lambda = c\mu$.*

We omit the proof of the theorem.

Remark: Let λ be the left Haar measure of a locally compact group G. If $\emptyset \neq U \subseteq G$ is open, then $\lambda(U) > 0$.

Proof. Suppose there exists $U \subseteq G$, with $U \neq \emptyset$ open such that $\lambda(U) = 0$. Then for all compact $K \subseteq G$, we must have $\lambda(K) = 0$. Indeed, as $\{xU\}_{x \in G}$ is an open cover of K, there exists $x_1, \ldots, x_n$ such that

$$K \subseteq (x_1 U) \cup \cdots \cup (x_n U).$$

Then

$$\lambda(K) \leq \lambda(x_1 U) + \cdots + \lambda(x_n U) = 0,$$

as λ is a left-translation invariant measure. This implies $\lambda(G)$, as λ must be inner regular. However, this contradicts the fact that the Haar measure λ is non-zero. Thus, such a U cannot exist. $\square$

Corollary 3.14. *If $f \in C_c^+(G)$, then $\int\limits_G f d\lambda > 0$.*

Proof. This follows from the fact that

$$\int\limits_G f d\lambda > \frac{\|f\|_{\sup}}{2} \lambda\left(f^{-1}\left(\left(\frac{\|f\|_{\sup}}{2}, \infty\right)\right)\right) > 0,$$

as $f^{-1}\left(\left(\frac{\|f\|_{\sup}}{2}, \infty\right)\right)$ is open. $\square$

Remark: Let λ be a left Haar measure of a locally compact group G. Then $\tilde{\lambda}$ is a right Haar measure of G, where $\tilde{\lambda}(E) = \lambda(E^{-1})$. This holds, as

$$\tilde{\lambda}(Ey) = \lambda((Ey)^{-1}) = \lambda(y^{-1}E^{-1}) = \lambda(E^{-1}) = \tilde{\lambda}(E).$$

Examples:

(i) If $G = \mathbb{R}$, then dx (where we integrate with respect to the Lebesgue measure) is the Haar measure.

(ii) If $G = \mathbb{R} \setminus \{0\}$, then $\frac{dx}{|x|}$ is the Haar measure. We can see this, using the following:

$$\int\limits_{\mathbb{R}\setminus\{0\}} f(yx) \frac{dx}{|x|} = \int\limits_{\mathbb{R}\setminus\{0\}} f(x) \frac{d(y^{-1}x)}{|y^{-1}x|} = \int\limits_{\mathbb{R}\setminus\{0\}} f(x) \frac{dx}{|x|}$$

where $y \neq 0$, and dx is the Lebesgue measure on $\mathbb{R}$.

(iii) If $G = \mathrm{GL}_n(\mathbb{R})$, the Haar measure is $dX = \frac{dx}{|\det x|^n}$ where $dx = \prod_{i,j} dx_{ij}$.

Exercise: Show that for all $f \in C_c(G)$,

$$\int\limits_{\mathrm{GL}_n(\mathbb{R})} f(Y \cdot X) d\lambda(X) = \int\limits_{\mathrm{GL}_n(\mathbb{R})} f(X) d\lambda(X)$$

using the fact that

$$\int f\left(A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) dx_1 \cdots dx_n = \int f\left(\begin{bmatrix} x_1 \\ \vdots \\ v_n \end{bmatrix}\right) \frac{dx_1 \cdots dx_n}{|\det A|}.$$

(iv) For matrices of the form

$$T_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

we identify any element in T_3 with a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$. We note

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+a & z+ay+c \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{bmatrix}$$

on the "group" side. On the manifold side, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+a \\ y+b \\ z+ay+c \end{bmatrix}.$$

If we let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we have

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + B.$$

Note here that our matrix with elements a, b, c is fixed.

(v) For the matrices of the form

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

the Haar measure is: **exercise!**

3.3. **Brief measure theory review and revisiting examples.** In what follows, let X be any fixed space, and $\mathcal{M}$ be a chosen σ-algebra on X.

Definition 3.15. *A map $\mu : \mathcal{M} \to [0, \infty]$ is called a measure if $\mu(\emptyset) = 0$ and μ is countably additive- i.e.,*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for every $E_n \in \mathcal{M}$ where $E_n \cap E_m = \emptyset$ for $m \neq n$.

Definition 3.16. *Let $(X, \mathcal{M})$ be a measurable space. Elements of $\mathcal{M}$ are called measurable sets. If a measure μ exists on $(X, \mathcal{M})$, we say $(X, \mathcal{M}, \mu)$ is a measure space.*

Definition 3.17. *Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is called a measurable function if*

$$f^{-1}((\alpha, \infty)) \in \mathcal{M}$$

for all $\alpha \in \mathbb{R}$.

Examples: Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Characteristic functions: for $E \in \mathcal{M}$, we define

$$\chi_E : X \to \mathbb{R},$$

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The function χ_E is measurable for all $E \in \mathcal{M}$.

(ii) Simple functions: for $E_1, \dots, E_n \in \mathcal{M}$ and $a_1, \dots, a_n \in \mathbb{R}$ let

$$\varphi = \sum_{k=1}^{n} a_k E_k.$$

Then φ is a measurable function.

Theorem 3.18 (Simple Approximation Theorem)**.** *Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \mathbb{R}$ be a measurable function. Then there exists a sequence of simple functions $\{\varphi_n\}$ such that $\varphi_n \to f$ pointwise as $n \to \infty$, and $|\varphi_n| \leq |f|$. Moreover, if f is pointwise positive, then we may choose the simple functions such that*

$$0 \leq \varphi_n \leq \varphi_{n+1}.$$

We omit the proof here- it should be exactly the same as we have seen in the context of the Lebesgue measure on $\mathbb{R}$.

We may also establish integration in exactly the same way we establish Lebesgue integration on $\mathbb{R}$. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) For simple functions $\varphi = \sum\limits_{k=1}^{n} a_k \chi_{E_k}$, we let

$$\int\limits_X \varphi d\mu = \sum_{k=1}^{n} a_k \mu(E_k).$$

(ii) For positive-valued functions $f : X \to \mathbb{R}$, we take

$$\int\limits_X f d\mu = \sup_{\substack{0 \leq \varphi \leq f \\ \varphi \text{ simple}}} \int\limits_X \varphi d\mu.$$

(iii) For general measurable functions $f : X \to \mathbb{R}$, we split $f = f^+ - f^-$ (i.e. we look at its positive and negative parts, where f^+, f^- are both positive-valued functions). We let

$$\int\limits_X f d\mu = \int\limits_X f^+ d\mu - \int\limits_X f^- d\mu.$$

Many of the important theorems from Lebesgue integration hold here: e.g. Fatou's Lemma, the Monotone Convergence Theorem, the Dominated Convergence Theorem, Fubini's Theorem, Tonelli's Theorem, etc...

Our setting is on $(G, \mathsf{B}(G), \lambda)$ where G is a locally compact group, $\mathsf{B}(G)$ is the Borel σ-algebra on G, and λ is the left Haar measure.

Recall: Let G, λ be as before. We have said

$$\int\limits_G f(zx) d\lambda(x) = \int\limits_G f(x) d\lambda(x)$$

for any $z \in G$ and all $f \in C_c(G)$ (note that it is sufficient to state this for functions in $C_c(G)$ only instead of the space of all measurable functions, as $C_c(G)$ is dense in this space- thus, it should hold on the entire space). We could easily show this by first establishing the result for simple functions and building up; that it holds for simple functions follows immediately from the translation-invariance of the Haar measure.

Example: Every continuous function $G \to \mathbb{R}$ is measurable; the proof is exactly the same as we've seen before. Moreover, $\int\limits_G f d\lambda < \infty$ as f is continuous and compactly supported, where $\lambda(K) < \infty$ for any compact K (as we work with the Haar measure).

We will return to considering some examples introduced earlier.

Proposition 3.19. *Let G be a locally compact group. Suppose the manifold of G is an open subset of $\mathbb{R}^N$. Suppose also that $xy = A(x)y + b(x)$ where A is an $N \times N$ matrix dependent on x, and $b(x)$ is a vector in $\mathbb{R}^N$. Then the left Haar measure in this space is*

$$d\lambda = \frac{dx_1 dx_2 \cdots dx_N}{|\det(A(x))|}.$$

That is, suppose $f : G \to \mathbb{R}$ is continuous with compact support. Then

$$\int\limits_G f(x) d\lambda(x) = \int\limits_{\mathbb{R}^N} \frac{f(x)}{|\det A(x)|} dx,$$

where $dx = dx_1 dx_2 \cdots dx_N$.

Note: Note that if we consider $f(x)$ in $\mathbb{R}^N$, while technically it should lie in a subset of $\mathbb{R}^N$ we can extend it to all of $\mathbb{R}^N$ by letting f be 0 outside the manifold associated to G. This allows us to integrate over the whole region $\mathbb{R}^N$.

Proof. Let $z \in G$ be arbitrary but fixed. Define the function g via $g(x) = f(zx)$ for $x \in G$. We see

$$\int\limits_G f(zx) d\lambda(x) = \int\limits_G g(x)\lambda(x) = \int\limits_{\mathbb{R}^N} \frac{g(x)}{|\det A(x)|} dx = \int\limits_{\mathbb{R}^N} \frac{f(zx)}{|\det A(x)|} dx$$
$$= \int\limits_{\mathbb{R}^N} \frac{f(A(z)x + b(z))}{|\det A(x)|} dx.$$

Via a change of variables, we claim we may write

$$\int\limits_{\mathbb{R}^N} \frac{f(A(z)x + b(z))}{|\det A(x)|} dx = \int\limits_{\mathbb{R}^N} \frac{f(x)}{|\det A(x)|} dx.$$

To see why- **exercise!** □

Examples:

(i) The Heisenberg group of dimension 3: define

$$\mathbb{H}_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We have seen that for two elements of $\mathbb{H}_3$,

$$\begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x + x' & x'y + z + z' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{bmatrix}.$$

Associating G to its manifold in $\mathbb{R}^3$, the product looks like

$$\begin{bmatrix} x + x' \\ y + y' \\ x'y + z + z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x' & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}.$$

Note that we may express this using

$$A\left(\begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x' & 1 \end{bmatrix}, \quad b\left(\begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}.$$

Then by the previous proposition, we have

$$\int\limits_{\mathbb{H}_3} f d\lambda = \int\limits_{\mathbb{R}^3} f\left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right) dx dy dz.$$

(ii) Recall the group G of affine transformations defined by

$$G := \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0, b \in \mathbb{R} \right\}.$$

We say that G is the group of affine transformations, as the collection of affine transformations on $\mathbb{R}^2$ under composition forms a group which is isomorphic to G. The manifold associated to G lies inside $\mathbb{R}_{>0} \times \mathbb{R}$. It can be easily found that

$$A\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},$$

and thus

$$\int_G f d\lambda = \int_{\mathbb{R}^2} f\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) \frac{dadb}{a^2}.$$

These give the continuous wavelet transform(s).

(iii) Recall again the group $\mathrm{GL}_n(\mathbb{R})$- this is an open subset of $\mathbb{R}^{n^2}$, as $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map. Multiplying matrices $Y \cdot X$, we write

$$Y \cdot X = Y \begin{bmatrix} c_1| & c_2| & \cdots & |c_n \end{bmatrix} = \begin{bmatrix} Y & & & 0 \\ & Y & & \\ & & \ddots & \\ 0 & & & Y \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Clearly, the determinant of the previous map is $(\det(Y))^n$.

(iv) Let G be a discrete group. Let λ be a left Haar measure on G. We know
- $\lambda(\{e_G\}) > 0$ as $\{e_G\}$ is open in G.
- $\lambda(\{x_1, \ldots, x_n\}) = \sum_{k=1}^{n} \lambda(\{x_k\}) = n\lambda(\{e_G\})$, as λ is a translation invariant measure.

Let $c = \lambda(\{e_G\})$. It is easy to see that

$$\lambda(E) = \begin{cases} c|E|, & \text{if } E \text{ is finite,} \\ \infty, & \text{if not.} \end{cases}$$

Conventions:

(i) If G is discrete, we take $\lambda(\{e_G\}) = 1$.
(ii) If G is compact, we take $\lambda(G) = 1$. Note that this means the Haar measure on compact groups acts as a probability measure.
(iii) For $\mathbb{R}^n$, we set $\lambda([0,1]^n) = 1$.

3.4. **The modular function.** Fix a locally compact group G, and a left Haar measure λ on G. Take $x \in G$ arbitrary but fixed. Define the new measure

$$\lambda_x : \mathscr{B}(G) \to [0, \infty],$$
$$\lambda_x(E) = \lambda(Ex).$$

It is easy to verify that λ_x is a Radon measure (**exercise!**). We also see that λ_x is left-invariant, as for any $E \in \mathscr{B}(G)$ and $y \in G$ we have

$$\lambda_x(yE) = \lambda(yEx) = \lambda(Ex) = \lambda_x(E)$$

as λ is left-invariant. By the uniqueness of the Haar measure, this implies there exists $\Delta(x) > 0$ such that

$$\Delta(x)\lambda = \lambda_x.$$

Definition 3.20. *Define a function* $\Delta : G \to \mathbb{R}^{>0}$ *via*
$$x \mapsto \lambda_x.$$
The function Δ *is called the modular function.*

Proposition 3.21. *We have*
(i) Δ *is a group homomorphism.*
(ii) Δ *is a continuous map.*
(iii) For all $f \in C_c(G)$,
$$\int_G (R_y f)(x) d\lambda(x) = \int_G f(xy) d\lambda(x) = \Delta(y^{-1}) \int_G f(x) d\lambda(x).$$

Proof. i) We first want to show that
$$\Delta(xy) = \Delta(x)\Delta(y)$$
for all $x, y \in G$. Let $E \subseteq G$ be an open set. We see
$$\begin{aligned}\Delta(xy)\lambda(E) &= \lambda_{xy}(E) = \lambda(Exy) = \lambda_y(Ex) = \Delta(y)\lambda(Ex) \\ &= \Delta_y \lambda_x(E) = \Delta(y)\Delta(x)\lambda(E) = \Delta(x)\Delta(y)\lambda(E).\end{aligned}$$
As $\lambda(E) \neq 0$, this implies $\Delta(xy) = \Delta(x)\Delta(y)$. As our choice of $x, y \in G$ were arbitrary, this establishes the claim.

ii) Fix some $f \in C_c^+(G)$, with $f \neq 0$. Recall that the map sending $G \to C_c(G)$ via
$$y \mapsto R_y f$$
is continuous, and the map sending $C_c(G) \to \mathbb{R}$ via
$$R_y f \mapsto \int_G R_y f d\lambda$$
is also continuous. Thus, the map $G \to \mathbb{R}$ defined via
$$y \mapsto \int_G R_y f(x) d\lambda(x)$$
is continuous. Assuming (iii), we know
$$\Delta(y^{-1}) = \frac{\int_G (R_y f)(x) d\lambda(x)}{\int_G f(x) d\lambda(x)}$$
is continuous. Finally, as the map $y \mapsto y^{-1}$ is continuous on G, this implies Δ must be as well.

iii) If $f = \chi_E$ for some $E \in \mathscr{B}(G)$, then
$$(R_y f)(x) = f(xy) = 1 \iff xy \in E \iff x \in \chi_{Ey^{-1}}.$$
We then see
$$\begin{aligned}\int_G (R_y f)(x) d\lambda(x) &= \int_G \chi_{Ey^{-1}}(x) d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) \\ &= \Delta(y^{-1}) \int_G f d\lambda.\end{aligned}$$
By linearity, we can easily extend the result above to the span of all characteristic functions (i.e., the simple functions). Then by the Simple Approximation Theorem (3.18), it should hold for general f. □

Note: This means we have a change of variables

$$d\lambda(xy) = \Delta(y)d\lambda(x).$$

Definition 3.22. *If $\Delta = 1$ on G, we say that G is unimodular.*

Remark: If G is unimodular, then the left Haar measure is also the right Haar measure.

Examples:

(i) Abelian groups are unimodular.
(ii) Discrete groups are unimodular.
(iii) Compact groups are unimodular.

Proposition 3.23. *Let G be a compact group. Then $\Delta = 1$.*

Proof. As $\Delta : G \to (0,\infty)$ is a continuous group homomorphism, we know that $\Delta(G)$ is compact and a group. This implies $\Delta(G) = \{1\}$; if there exists a $y \in \Delta(G)$ where $y > 1$, our group becomes unbounded (taking powers of that element). Similarly, if $y < 1$, it tends to zero- a contradiction in both cases. □

Next, we discuss some examples of non-unimodular groups.

Examples:

(i) The "$ax + b$" group: we let

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a > 0, \ b \in \mathbb{R} \right\}.$$

The left Haar measure here is

$$d\lambda\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = \frac{dadb}{a^2}.$$

To show that λ is not a right Haar measure- **exercise!** (Find Δ).

(ii) Semi-direct products: let $A \ltimes H$, where $H \leq \mathrm{Aut}(A)$ for some group A.

Recall: For a locally compact group G with left Haar measure λ, we associate the right Haar measure ρ with λ where

$$\rho : \mathscr{B}(G) \to [0,\infty],$$
$$\rho(E) = \lambda(E^{-1}).$$

Theorem 3.24. *Let ρ, λ and G be as above. Then ρ and λ are strongly equivalent measures; i.e.,*

$$d\rho(x) = \Delta(x^{-1})d\lambda(x).$$

Thus,

$$\int_G f(x)d\rho(x) = \int_G f(x)\Delta(x^{-1})d\lambda(x).$$

Proof. First, we will show that $\Delta(x^{-1})\lambda(x)$ is a right Haar measure. To that end, we prove that the functional $T : C_c(G) \to \mathbb{R}$ defined via

$$f \mapsto \int_G f(x)\Delta(x^{-1})d\lambda(x)$$

is right-invariant. For arbitrary but fixed $f \in C_c(G)$ and $y \in G$, we see

$$\begin{aligned} T(R_y f) &= \int_G (R_y f)(x)\Delta(x^{-1})\lambda(x) = \int_G f(xy)\Delta(x^{-1})d\lambda(x) \\ &= \Delta(y)\int_G \frac{f(xy)}{\Delta(xy)}d\lambda(x) = \Delta(y)\Delta(y^{-1})\int_G f(x)\Delta(x^{-1})d\lambda(x) \\ &= \int_G f(x)\Delta(x^{-1})d\lambda(x) = T(f). \end{aligned}$$

This shows right-invariance of $\Delta(x^{-1})d\lambda(x)$.

By the uniqueness of the right Haar measure, there exists a $c > 0$ such that

$$\Delta(x^{-1})d\lambda(x) = cd\rho(x).$$

Suppose $c \neq 1$- so $|c-1| > 0$. Since Δ is continuous, there exists a symmetric neighborhood U of e_G for which

$$|\Delta(x) - 1| < \frac{|c-1|}{2}$$

for all $x \in U$. As $\rho(U) = \lambda(U^{-1}) > 0$, we see

$$\begin{aligned} |c-1|\lambda(U) &= |c\lambda(U) - \lambda(U)| = |c\rho(U) - \lambda(U)| \\ &= \left| c\int_G \chi_U(x)d\rho(x) - \int_G \chi_U(x)d\lambda(x) \right| = \left| \int_G \chi_U(x)(\Delta(x^{-1}) - 1)d\lambda(x) \right| \\ &\leq \int_U |\Delta(x^{-1}) - 1|d\lambda(x) \leq \frac{|c-1|}{2}\lambda(U). \end{aligned}$$

This shows $|c-1|\lambda(U) \leq \frac{|c-1|}{2}\lambda(U)$; however, as $\lambda(U) > 0$, this is not possible. Thus, as we have reached a contradiction, we conclude that $c = 1$- so $d\rho(x) = \Delta(x^{-1})d\lambda(x)$. □

Recall: We have the following relations (for "change of variables") with the left Haar measure:

$$\begin{aligned} d\lambda(xz) &= d\lambda(z), \quad x \in G, \\ d\lambda(zx) &= \Delta(x)d\lambda(z), \quad x \in G, \\ d\rho(x) &= \Delta(x^{-1})d\lambda(x), \quad x \in G, \rho \text{ the right Haar measure.} \end{aligned}$$

Suppose we take $\emptyset \neq U \subseteq G$ open. We have

$$\int_G \chi_{U^{-1}}(x)d\lambda(x) = \lambda(U^{-1}) = \rho(U) = \int_G \chi_U(x)d\rho(x) = \int_G \frac{\chi_U(x)}{\Delta(x)}d\lambda(x).$$

Noting that

$$\chi_{U^{-1}}(x) = 1 \iff x \in U^{-1} \iff x^{-1} \in U \iff \chi_U(x^{-1}) = 1,$$

the integral equation above becomes

$$\int_G \chi_{U^{-1}}(x)d\lambda(x) = \int_G \chi_U(x^{-1})d\lambda(x) = \int_G \chi_U(x)d\lambda(x^{-1})$$

by a change of variables.

Corollary 3.25. *For any $f \in C_c(G)$,*

$$\int_G f(x)d\lambda(x^{-1}) = \int_G \frac{f(x)}{\Delta(x)}d\lambda(x).$$

Proof. See the discussion immediately preceding this. ☐

Note: We also have $d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x)$.

Recall: In what follows, we extend some important definitions to any locally compact group G.

(i) $C(G) = \{f : G \to \mathbb{C} : \ f \text{ continuous}\}$.
(ii) $f : G \to \mathbb{C}$ is compactly supported if $\operatorname{supp}(f)$ is compact.
(iii) $f : G \to \mathbb{C}$ vanishes at infinity if for all $\epsilon > 0$, there exists a compact set $K \subseteq G$ such that $|f(x)| < \epsilon$ for $x \in G \setminus K$. We denote this space as $C_0(G)$.
(iv) $C_c(G) = \{f : G \to \mathbb{C} : \ \operatorname{supp}(f) \text{ is compact}\}$.

Facts:

(i) $C_c(G)$ is $\|\cdots\|_{\sup}$-dense in $C_0(G)$.
(ii) $C_c(G)$ is $\|\cdot\|_p$-dense in L^p for $1 \leq p < \infty$.

We now move to a discussion on generalizing L^p spaces.

Example: $L^1(G)$ is a Banach $*$-algebra, which we show as follows.

(i) $(L^1(G), \|\cdot\|_1)$ is a Banach space under pointwise addition and pointwise scalar multiplication.
(ii) Multiplication is convolution: for $[f], [g] \in L^1(G)$,

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\lambda(y).$$

By Fubini's Theorem, the above integral is absolutely convergent for a.e. $x \in G$. Additionally,

$$\|f * g\|_1 = \int_G |f * g(x)|d\lambda(x) = \int_G \left| \int_G f(y)g(y^{-1}x)d\lambda(y) \right| d\lambda(x)$$
$$\leq \int_G \int_G |f(y)||g(y^{-1}x)|d\lambda(y)d\lambda(x) = \int_G |f(y)| \left(\int_G |g(y^{-1}x)|d\lambda(x) \right) d\lambda(y)$$
$$= \|f\|_1\|g\|_1.$$

This holds precisely because of Fubini's Theorem and the left-invariance of the Haar measure. This shows that $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ for $f, g \in L^1(G)$, and thus $L^1(G)$ is a Banach algebra.

(iii) Involution on $L^1(G)$ is defined via

$$f^*(x) = \frac{\overline{f(x^{-1})}}{\Delta(x)}$$

for $x \in G$. With this definition, we have $\|f^*\|_1 = \|f\|_1$. To see why, note

$$\int_G \frac{|\overline{f(x^{-1})}|}{\Delta(x)} d\lambda(x) = \int_G \frac{|f(x)|}{\Delta(x^{-1})} d\lambda(x^{-1}) = \int_G \frac{|f(x)|}{\Delta(x^{-1})} \Delta(x^{-1})d\lambda(x)$$
$$= \int_G |f(x)|d\lambda(x).$$

Facts:

(i) If G is an abelian locally compact group, then $L^1(G)$ is commutative (i.e. $f * g = g * f$ for $f, g \in L^1(G)$).

(ii) $L^1(G)$ is unital if and only if G is discrete- we need the function defined via

$$e_G \mapsto 1,$$
$$g \mapsto 0$$

for all $g \in G \setminus \{e_G\}$ to have non-zero measure at e_G, which happens only if G is discrete.

Example: For $G = \mathbb{Z}$, λ is the counting measure. If $f, g \in L^1(G)$ (where we write them as $f = \{a_n\}, g = \{b_n\}$ specifying $f(n) = a_n$) then

$$(f * g)(n) = \sum_{m \in \mathbb{Z}} f(m)g(m-n).$$

Definition 3.26. *Le t$\mathcal{H}$ be a Hilbert space. An operator $T : \mathcal{H} \to \mathcal{H}$ is called unitary if $T^*T = TT^* = I$. Equivalently, if T is surjective and an isometry. We let $\mathcal{U}(H)$ denote the set of all unitary operators on $\mathcal{H}$.*

Theorem 3.27. *Let G be a locally compact group with left Haar measure λ. Define $L : G \to \mathcal{U}(L^2(G))$ via*

$$x \mapsto L_x$$

where

$$L_x : L^2(G) \to L^2(G),$$
$$(L_x f)(z) = f(x^{-1}z)$$

and $\rho : G \to \mathcal{U}(L^2(G))$ via

$$x \mapsto \sqrt{\Delta(x)} R_x$$

where

$$R_x : L^2(G) \to L^2(G),$$
$$(R_x f)(z) = f(zx).$$

Then

(i) L, ρ are group homomorphisms.
(ii) They are well-defined.
(iii) For any fixed $f \in L^2(G)$, the following maps are continuous:

$$L^f : G \to L^2(G) \qquad \rho^f : G \to L^2(G),$$
$$x \mapsto L_x f \qquad x \mapsto \rho(x) f.$$

Note: We have $(L_x \cdot L_y)(f) = L_x(L_y f)$, and for any $z \in G$ we see

$$(L_x \cdot L_y)(f)(z) = L_x(L_y f)(z) = (L_y f)(x^{-1}z) = f(y^{-1}x^{-1}z)$$
$$= f((xy)^{-1}z) = (L_{xy})(f)(z).$$

This means $L_{xy} = L_x \cdot L_y$.

Proof. That L, ρ are group homomorphisms follows directly from our comment above. With this, we know

$$(L_x)^{-1} = L_{x^{-1}}$$

for any $x \in G$, and thus L is surjective (using a similar argument, so is ρ). Let $f \in L^2(G)$. We have

$$\|\rho(x) f\|_2 = \int_G |[\rho(x) f](z)|^2 d\lambda(z) = \int_G |f(zx)|^2 \Delta(x) d\lambda(z).$$

Using the change of variable $zx \mapsto z$, then $z \mapsto zx^{-1}$ with $d\lambda(z) = \Delta(x)^{-1} d\lambda(z)$. Thus,

$$\int_G |f(zx)|^2 \Delta(x) d\lambda(z) = \int_G |f(z)|^2 \Delta(x)\Delta(x)^{-1} d\lambda(z) = \int_G |f(z)|^2 d\lambda(z) = \|f\|_2^2.$$

This shows ρ is an isometry; a similar argument may be used for L as well. Thus, both are well defined (i.e., unitary operators on $L^2(G)$).

To show continuity, we will prove it for ρ; the proof for L is similar. Let $f \in L^2(G)$ be arbitrary but fixed. We want to show that for $\epsilon > 0$, there exists a neighborhood $e_G \in U$ such that if $y^{-1}x \in U$, then $\|\rho(x)f - \rho(y)f\|_2 < \epsilon$. Since $C_c(G)$ is $\|\cdot\|_2$-dense in $L^2(G)$, there exists $\varphi \in C_c(G)$ such that $\|f - \varphi\|_2 < \epsilon/4$. Let $K = \text{supp}(f)$, and fix some symmetric open neighborhood V with $e_G \in V$. For every $z \in V$, $\text{supp}(\rho(z)\varphi) \subseteq KV$. Suppose also (without loss of generality) that $\overline{V}$ is compact. Then $\text{supp}(\rho(z)\varphi) \subseteq K\overline{V}$ where $K\overline{V}$ is compact.

We know the map $G \to C_c(G)$ defined via

$$x \mapsto R_x f$$

is continuous (from a previous proposition). We have

$$\|\rho(x)f - \rho(y)f\|_2 \le \|\rho(x)f - \rho(x)\varphi\|_2 + \|\rho(x)\varphi - \rho(y)\varphi\|_2 + \|\rho(y)\varphi - \rho(y)f\|_2.$$

As ρ is an isometry, then $\|\rho(x)f - \rho(x)\varphi\|_2 = \|f - \varphi\|_2$; similarly, $\|\rho(y)f - \rho(y)\varphi\|_2 = \|f - \varphi\|_2$. Thus,

$$\begin{aligned} \|\rho(x)f - \rho(y)f\|_2 &\le \frac{\epsilon}{4} + \|\rho(x)\varphi - \rho(y)\varphi\|_2 + \frac{\epsilon}{4} \\ &= \|\rho(x)\varphi - \rho(y)\varphi\|_2 + \frac{\epsilon}{2}. \end{aligned}$$

We note

$$\|\rho(x)\varphi - \rho(y)\varphi\|_2 = \|\rho(y^{-1}x)\varphi - \varphi\|_2 \le \|\rho(y^{-1}x)\varphi - \varphi\|_{\sup} \sqrt{\lambda(K\overline{V})}.$$

By continuity between G and $C_c(G)$, there exists a $e_G \in U$ such that

$$\|R(y^{-1}x)\varphi - \varphi\|_{\sup} < \frac{\epsilon}{2M\sqrt{\lambda(K\overline{V})}},$$

for all $y^{-1}x \in U$ where $M = \sup \Delta|_{\mathcal{E}}$ with $\mathcal{E} = (K\overline{V})^{-1}(K\overline{V})$. From this, we get

$$\|\rho(x)f - \rho(y)f\|_2 \le \frac{\epsilon}{2} + \frac{\epsilon}{2M\sqrt{\lambda(K\overline{V})}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

Definition 3.28. *Let G be a locally compact group, and let $\mathcal{H}$ be a Hilbert space. A map $\pi : G \to \mathcal{U}(H)$ is called a unitary representation if*

$$\pi(xy) = \pi(x)\pi(y)$$

for all $x, y \in G$. We say a unitary representation π is continuous if for every $\xi, \eta \in \mathcal{H}$ the map $G \to \mathbb{C}$ defined via

$$x \mapsto \langle \pi(x)\xi, \eta \rangle_{\mathcal{H}}$$

is continuous.

Notes:

(i) We restrict our focus to continuous unitary representations.

(ii) Maps of the form

$$x \mapsto \langle \pi(x)\xi, \eta \rangle_{\mathcal{H}}$$

are called coefficient functions associated with π for each fixed $\xi, \eta \in \mathcal{H}$.

Notation: The dimension of a representation π is just the dimension of the Hilbert space $\mathcal{H}_\pi$ (i.e., the Hilbert space associated to π).

Example: Look at any finite dimensional representation

$$\pi : G \to \mathcal{U}(\mathbb{C}^{d_\pi}).$$

Note that $\mathcal{U}(\mathbb{C}^{d_\pi})$ is the space of $d_\pi \times d_\pi$ unitary matrices. Fix the standard orthonormal basis for $\mathbb{C}^{d_\pi}$, and consider $\pi_{e_i,e_j} : G \to \mathbb{C}$. The map is defined via

$$x \mapsto \langle \pi(x)e_i, e_j\rangle.$$

Note that $\langle \pi(x)e_i, e_j\rangle$ is just the ji^{th}-entry in the matrix $\pi(x)$.

Example: Let G be a locally compact group, and $\mathcal{H} = \mathbb{C}$. Then

$$\mathcal{U}(\mathbb{C}) = \{[\alpha]_{1\times 1} : \ \alpha \in \mathbb{C} \text{ with } |\alpha| = 1\} = \mathbb{T}.$$

So, one-dimensional representations are continuous group homomorphisms $G \to \mathbb{T}$. The trivial representation is given by

$$\iota : G \to \mathcal{U}(\mathbb{C}) := \mathbb{T},$$
$$\iota(g) = 1.$$

Example: Fix $n \in \mathbb{N}$, and consider $G = S_n$. This group has only 2 one-dimensional representations:

(i) The trivial representation, and
(ii) $\tau : S_n \to \mathbb{T}$ the sign representation where

$$\tau(g) = \begin{cases} 1, & \text{if } g \text{ is even,} \\ -1 & \text{if } g \text{ is odd.} \end{cases}$$

4. Fourier analysis on locally compact abelian groups

4.1. **Definitions and introductory analysis.** Let G be a locally compact Abelian group, and let λ be a Haar measure.

Definition 4.1. *A character of G is a 1-dimensional continuous representation $\chi : G \to \mathbb{T}$.*

If we let

$$\hat{G} = \{\text{all characters of } G\}$$

and equip it with pointwise multiplication, $\hat{G}$ becomes an abelian group. We call $\hat{G}$ the dual group of G. Equip the group $\hat{G}$ with the weak*-topology of $L^\infty(G)$. Then $\hat{G}$ becomes a locally compact group.

Fact: The topology of $\hat{G}$ is the same as the topology of uniform convergence on compact subsets, with basic open sets

$$O_{\chi_0,K,\epsilon}$$

for $\chi_0 \in \hat{G}, K \subseteq G$ compact, and $\epsilon > 0$ where

$$O_{\chi_0,K,\epsilon} = \{\chi \in \hat{G} : \ |\chi(g) - \chi_0(g)| < \epsilon \text{ for all } g \in K\}.$$

Note: This only holds, as the norm of $|\chi(g)| = 1$ for all $g \in G$.

Lemma 4.2. *If G is a locally compact abelian group, then $L^1(G)$ is commutative.*

Note: We call $L^1(G)$ the group algebra of G.

Proof. For any $f, g \in L^1(G)$, for almost every $x \in G$,

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy = \int_G f(xy)g(y^{-1})dy$$
$$= \int_G f(xy^{-1})g(y)dy = \int_G g(y)f(y^{-1}x)dy = (g * f)(x)$$

using the fact that $\Delta(x) = 1$ for all $x \in G$. Thus, $f * g = g * f$ in $L^1(G)$, and so $L^1(G)$ is commutative. □

Recall: The spectrum of $L^1(G)$ is

$$\sigma(L^1(G)) = \Big\{\phi : L^1(G) \to \mathbb{C} : \ \phi \text{ is nonzero, multiplicative, and a linear functional}\Big\}.$$

We know $\sigma(L^1(G))$ is locally compact (by functional analysis, as we endow $\sigma(L^1(G))$ with the weak*-topology.

Proposition 4.3. *For any abelian locally compact group G, there exists a one-to-one correspondence between characters in $\hat{G}$ and elements of $\sigma(L^1(G))$.*

Proof. Take $\chi \in \hat{G}$. We want to construct a nonzero multiplicative linear functional $T : L^1(G) \to \mathbb{C}$. For $f \in L^1(G)$, define a map T via

$$T(f) = \int_G f(x)\chi(x)dx.$$

It is clear that T is a linear map, and $\|T\| \leq 1$ as

$$|T(f)| = \left|\int_G f(x)\chi(x)dx\right| \leq \int_G |f(x)|dx = \|f\|_1.$$

To show multiplicativity, take $f, g \in L^1(G)$. We see

$$T(f * g) = \int_G \int_G f(y)g(y^{-1}x)\chi(x)dydx = \int_G \int_G f(y)g(y^{-1}x)\chi(x)dxdy$$
$$= \int_G f(y) \int_G g(y^{-1}x)\chi(x)dxdy = T(f)T(g),$$

using the change of variable $x \mapsto yx$ and liberal application of Fubini-Tonelli. To show that T is non-zero, as χ is continuous at e_G there exists a precompact set U with $e_G \in U \subseteq G$ such that $|\chi(x) - 1| < 1/3$ for all $x \in U$. Take $f = \chi_U$; clearly, $f \in L^1(G)$ with

$$T(f) = \int_G \chi_U(x)\chi(x)dx = \int_U \chi(x)dx \neq 0.$$

Thus, for any $\chi \in \hat{G}$, we can construct a non-zero multiplicative linear functional on $L^1(G)$.

Conversely, suppose we have $T : L^1(G) \to \mathbb{C}$ which is non-zero, multiplicative, and linear. We want to show that there exists a character χ such that

$$T(f) = \int_G f(x)\chi(x)dx.$$

As T is linear on $L^1(G)$, there exists a $\varphi \in L^\infty(G)$ such that

$$T(f) = \int_G f(x)\varphi(x)dx$$

(by the Riesz-Representation theorem). As T is non-zero, there exists an $f_0 \in L^1(G) \cap L^2(G)$ such that $T(f_0) \neq 0$. For each $f \in L^1(G)$,

$$T(f_0)T(f) = T(f_0 * f) = \int_G (f_0 * f)(x)\varphi(x)dx$$
$$= \int_G \int_G f_0(y)f(y^{-1}x)\varphi(x)dydx.$$

Using the change of variable $y \mapsto xy^{-1}$, then $y^{-1} = yx^{-1}$ and so (by unimodularity)

$$\int_G \int_G f_0(y)f(y^{-1}x)\varphi(x)dydx = \int_G \left(\int_G f_0(xy^{-1})\varphi(x)dx\right) f(y)dy.$$

Note that if we really want to be careful, we need to first start with $f \in C_c(G)$ and establish the result there, and then approximate any function in $L^1(G)$ with compactly supported functions; we do not include these details, and take them for granted in this proof (essentially). The above integral simplifies to

$$\int_G \left(\int_G f_0(xy^{-1})\varphi(x)dx\right) f(y)dy = \int_G \left(\int_G f_0(y^{-1}x)\varphi(x)dx\right) f(y)dy = \int_G (L_y f_0) f(y)dy.$$

Since $T(f_0) \neq 0$, in the latter equation we may divide everything above by $T(f_0)$, replacing the function $\varphi(x)$ by the function $\psi : G \to \mathbb{C}$ defined via

$$\psi(x) = \frac{T(L_x f_0)}{T(f_0)}.$$

Since $G \to L^2(G)$ via $x \mapsto L_x f_0$ is continuous (as we have shown previously), and T is continuous (as it is bounded), we see that ψ must be continuous as well.

We want to show that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$. We note

$$\psi(xy)T(f_0) = T(L_{xy}f_0) = \int_G (L_{xy}f_0)(z)\psi(z)dz = \int_G (L_x(L_y f_0))(z)\psi(z)dz$$
$$= \int_G (L_y f_0)(z)\psi(xz)dz = \int_G f_0(z)\psi(xyz)dz.$$

Then

$$\int_G \psi(xy)\psi(z)f_0(z)dz = \int_G \psi(xyz)f_0(z)dz,$$

and so $\psi(xy)\psi(z) = \psi(xyz)$. This suggests ψ is indeed a homomorphism.

As $\mathrm{rng}(\psi) \subseteq \mathbb{T}$, where ψ is a homomorphism, we know $\psi(x^n) = (\psi(x))^n$ for all $n \in \mathbb{Z}$. But as $\psi \in L^\infty(G)$, this implies $|\psi(x)| = 1$ for all $x \in G$ (otherwise, we'd have a "blow-up" in terms of absolute value behavior). □

Theorem 4.4. *When G is a locally compact abelian group, then $\hat{G}$ is also a locally compact abelian group.*

Proof. By the previous proposition, we know $\hat{G} \cong \sigma(L^1(G))$. But the latter space is locally compact- hence, so is $\hat{G}$. □

Examples:

(i) $\hat{\mathbb{R}} = \left\{\chi_r : \mathbb{R} \to \mathbb{T},\ x \mapsto e^{2\pi i x r} \mid r \in \mathbb{R}\right\}$.
(ii) $\hat{\mathbb{T}} \cong \mathbb{Z}$.
(iii) $\hat{\mathbb{Z}} \cong \mathbb{T}$.

Proposition 4.5. *If G is a compact abelian group, then $\hat{G} \subseteq L^{\infty}(G) \subseteq L^p(G)$ for $1 \leq p \leq \infty$. Also, $\hat{G}$ forms an orthonormal set in $L^2(G)$.*

Note: By convention, when we deal with a compact abelian group G, we normalize the measure so that $\lambda(G) = 1$.

Proof. Take $\chi \in \hat{G}$- then

$$\langle \chi, \chi \rangle = \int_G |\chi|^2 d\lambda = 1.$$

For distinct $\chi_1, \chi_2 \in \hat{G}$, we want to show that if $\chi_1 \neq \chi_2$ then $\langle \chi_1, \chi_2 \rangle = 0$. Since χ_1, χ_2 are distinct and continuous, there exists $x_0 \in G$ such that $\chi_1(x_0) \neq \chi_2(x_0)$. Then $\chi_1(x_0)\overline{\chi_2(x_0)} \neq 1$. We see

$$\chi_1(x_0)\overline{\chi_2(x_0)} \int_G \chi_1(z)\chi_2(z)dz = \int_G \chi_1(x_0 z)\overline{\chi_2(x_0 z)}dz.$$

With a change of variable, and using unimodularity of the group we get

$$\int_G \chi_1(x_0 z)\overline{\chi_2(x_0 z)}dz = \int_G \chi_1(z')\overline{\chi_2(z')}dz' = \langle \chi_1, \chi_2 \rangle.$$

This shows $\chi_1(x_0)\overline{\chi_2(x_0)}\langle \chi_1, \chi_2 \rangle = \langle \chi_1, \chi_2 \rangle$. However, as $\chi_1(x_0)\overline{\chi_2(x_0)} \neq 1$, this forces $\langle \chi_1, \chi_2 \rangle - 0$. Thus, they form an orthonormal set. □

4.2. **Statements of important theorems (in the abelian case).**
Recall that in the case where G is a locally compact abelian group, the Fourier transform $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ is given by

$$f \mapsto \hat{f},$$

where

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\lambda(x).$$

In what follows, we will be stating several important theorems and making some remarks on results, without providing proofs (at this time). We will most likely revisit these theorems at a later date, in order to prove the results we are using at various times.

Note: Involution in $L^1(G)$ is given by

$$f^*(x) = \overline{f(x^{-1})}$$

for $f \in L^1(G)$, while involution in $C_0(\hat{G})$ is given by

$$\varphi^*(\chi) = \overline{\varphi(\chi)}$$

for $\varphi \in C_0(\hat{G})$.

Theorem 4.6. *The Fourier transform $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ is a norm-decreasing $*$-homomorphism. Furthermore, the image of $\mathcal{F}$ in $C_0(\hat{G})$ is dense.*

Remark: As mentioned, we will not prove the theorem above (as we essentially already have before in specific cases). However, we do note that the fact that $\mathcal{F}$ maps into $C_0(\hat{G})$ follows from Gelfand theory.

Theorem 4.7 (Plancherel Formula). *For a fixed Haar measure λ on G, we may choose a Haar measure on $\hat{G}$ such that the Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary map*

$$\mathcal{F} : L^2(G) \to L^2(\hat{G}).$$

The extension is unique by density of $L^1(G) \cap L^2(G)$ in $L^2(G)$, and the mapping is given via

$$\langle f, g \rangle_{L^2(G)} = \langle \hat{f}, \hat{g} \rangle_{L^2(\hat{G})}.$$

Theorem 4.8 (Pontryagin Duality). *The map $\Phi : G \to \hat{\hat{G}}$ given by*

$$\Phi(g)(\chi) = \chi(g)$$

for $g \in G$ is injective and surjective.

Theorem 4.9 (Fourier Inversion Theorem). *If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$ then $\hat{\hat{f}}(x^{-1} = f(x)$ for almost every x, where we equip $\hat{G}$ with the dual Haar measure discussed in the theorem of Plancherel (i.e., the measure which allows $\mathcal{F}$ to extend to a unitary map). To write it more explictly, we have*

$$f(x) = \int\limits_{\hat{G}} \hat{f}(\chi)\overline{x^{-1}(\chi)} d\chi,$$

where $d\chi$ is the dual Haar measure on $\hat{G}$.

Example: Let $G = \mathbb{T}$, and $f \in L^1(\mathbb{T})$. Then $\hat{G} = \mathbb{Z}$, with $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$. If we also suppose $f \in L^2(G)$, we know $\{\chi_n\}_{n \in \mathbb{Z}}$ form an orthonormal basis- so

$$\hat{f}(n) = \langle \chi_n, f \rangle$$

with

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) \chi_n.$$

Note: Note that the inclusion of x^{-1} is what ensures the second equality is an expansion in terms of the χ_n's, and not $\overline{\chi_n}$'s (i.e., we recover the Fourier transform on $L^1(G)$).

Corollary 4.10. *$\mathcal{F} : L^1(G) \to C_0(\hat{G})$ is injective.*

Remarks:

(i) $\mathcal{F}$ extends to $M(G)$, where $M(G)$ is the space of bounded Radon measures on G. It is injective on $M(G)$ as well.

(ii) We can think of $L^1(G)$ as being contained in $M(G)$ via

$$f \mapsto f d\lambda.$$

In fact, $L^1(G)$ is an ideal in $M(G)$.

Theorem 4.11 (Hausdorff-Young Inequality). *Let $1 \leq p \leq 2$, and let q be conjugate to p. If $f \in L^p(G)$, then $\hat{f} \in L^q(G)$ with*

$$\|\hat{f}\|_q \leq \|f\|_p.$$

Note: For $p = 1$, we have $q = \infty$ and so we get the result above using the typical Fourier transform above from $L^1(G)$ to $C_0(\hat{G})$. If $p = 2$, then $q = 2$ and the result follows by Parseval. If $1 < p < 2$, we must involve interpolation theorems.

Recall: The weak*-topology on X^* is given as follows: for $\psi_0 \in X^*$,

$$N_{\psi_0, x_1, \epsilon_1} = \{\varphi \in X^* : \; |\varphi(x_1) - \psi_0(x_1)| < \epsilon_1\}$$

for $x_1 \in X, \epsilon_1 > 0$.

Theorem 4.12. *Let G be a locally compact abelian group.*

(i) If G is discrete, then $\hat{G}$ is compact.
(ii) If G is compact, then $\hat{G}$ is discrete.

Proof. For (i), begin by recalling that $\hat{G} \cong \sigma(L^1(G))$. Since G is discrete, the algebra $L^1(G)$ is unital with unit $\delta_e : G \to \mathbb{C}$ defined by

$$\delta_e(g) = \begin{cases} 1, & g = e, \\ 0, & \text{otherwise.} \end{cases}$$

As $L^1(G)$ is unital, this means the spectrum $\sigma(L^1(G))$ must be compact- thus, $\hat{G}$ is compact.

To prove (ii), begin by supposing G is compact. We have shown that in this case, for $\chi_1, \chi_2 \in \hat{G}$ we have

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1, & \chi_1 = \chi_2, \\ 0, & \chi_1 \neq \chi_2. \end{cases}$$

Let $\iota : G \to \mathbb{T}$ be the trivial character- i.e., $\iota(g) = 1$ for all $g \in G$. We know $\iota \in L^1(G)$, as G is compact; indeed,

$$\int_G \iota(x) d\lambda(x) = \int_G d\lambda(x) = \lambda(G) < \infty$$

for compact G where λ is a Radon measure. We want to show that $\{\iota\} \subseteq \hat{G}$ is open. We claim that

$$\{\iota\} = \{\chi \in \hat{G} : \left| \int_G f\chi - \int_G f\iota \right| < 1/2\}$$

for a fixed $f \in L^1(G)$. To show this, pick $f = \iota \in L^1(G)$. Then $\int_G f\iota = 1$, while $\int_G f\chi = \int_G \chi$. As the constant function 1 is orthogonal to all other characters for a compact group G, we have

$$\left| \int_G \chi - \int_G 1 \right| < \frac{1}{2}.$$

Thus, $\{\iota\}$ is as we claimed above- and so $\{\iota\}$ is an open neighborhood. This implies every singleton is open in the topology of $\hat{G}$, and so $\hat{G}$ is discrete. □

Theorem 4.13. *If G, H are locally compact abelian groups, then*

$$\widehat{G \times H} \cong \hat{G} \times \hat{H}.$$

Proof. On the homework. □

Corollary 4.14. *If $G_1, \ldots, G_n$ are locally compact abelian groups,*

$$G_1 \widehat{\times \cdots \times} G_n \cong \hat{G}_1 \times \cdots \times \hat{G}_n.$$

Theorem 4.15. *If we consider $\{G_\alpha\}_{\alpha\in I}$ where G_α is a compact abelian group for each $\alpha \in I$, then*

$$\Pi_{\alpha\in I}\hat{G}_\alpha = \bigoplus_{\alpha\in I}\hat{G}_\alpha.$$

5. Representation theory

5.1. **Basics.** In what follows, G is assumed only to be a locally compact group with a fixed left Haar measure λ.

Recall: A continuous unitary representation of G is a group homomorphism

$$\pi : G \to \mathcal{U}(H),$$

where $\mathcal{H}$ is some Hilbert space, and $\mathcal{U}(H)$ is the collection of unitary operators on $\mathcal{H}$. Note that when we say π is continuous, we mean all coefficient functions $\pi_{\xi,\eta} : G \to \mathbb{C}$ defined via

$$x \mapsto \langle \pi(x)\xi, \eta\rangle$$

is continuous for all $\xi, \eta \in \mathcal{H}$.

Recall: We recall some of the different topologies we may endow the space $\mathcal{B}(H)$ with:

(i) Operator topology: we say $T_n \to_{\text{op}} T$ if and only if $\|T_n - T\|_{\text{op}} \to 0$ as $n \to \infty$.
(ii) Strong operator topology: we say $T_n \to_{\text{SOT}} T$ if and only if $T_n\xi \to T\xi$ for all $\xi \in \mathcal{H}$.
(iii) Weak operator topology: we say $T_n \to_{\text{WOT}} T$ if and only if

$$\langle T_n\xi, \eta\rangle \to \langle T\xi, \eta\rangle$$

for all $\xi, \eta \in \mathcal{H}$.

Example: Let $S : \ell^2 \to \ell^2$, where S is the shift operator. Then S^n converges to 0 in the WOT but not in norm: indeed,

$$S^n(x_1, x_2, \ldots,) = (0, \ldots, 0, x_1, x_2, \cdots)$$

which clearly tends to the constant 0 sequence as $n \to \infty$. However, $\|S^n\| = 1$ for all $n \in \mathbb{N}$, and thus $\|S^n - 0\| \not\to 0$ as $n \to \infty$.

Recall: Let $\mathcal{H} = L^2(G)$. We have the standard representations:

$$L : G \to \mathcal{U}(L^2(G)) \qquad \rho : G \to \mathcal{U}(L^2(G)),$$
$$x \mapsto L_x \qquad x \mapsto \rho_x,$$
$$(L_xf)(z) = f(x^{-1}z), \qquad (\rho_xf)(z) = \sqrt{\Delta(x)}f(zx).$$

We have shown before that all coefficient functions for L, ρ are continuous. However, L and ρ are not continuous when $\mathcal{U}(L^2(G))$ is equipped with the $\|\cdot\|_{\text{op}}$-topology.

Proof. We will show that if $x \in G$ with $x \neq e_G$, then $\|L_x - L_{e_G}\|_{\text{op}} > 1$. To show this, it is sufficient to show that there exists an $f \in L^2(G)$ with $\|f\|_2 = 1$ such that $\|L_xf - f\|_2 > 1$. As $x \neq e_G$ and G is Hausdorff, there exists neighborhoods V, W with $e_G \in V, x \in W$ and $V \cap W = \emptyset$. Let $Z = (x^{-1}W) \cap V$ be a precompact neighborhood of e_G. Then $Z \cap xZ \subseteq V \cap W = \emptyset$. If we let $f = \chi_Z/\sqrt{\lambda(z)}$, it is clear that $f \in C_c(G)$ with $\|f\|_2 = 1$. Furthermore, $\text{supp}(L_xf) \subseteq \text{supp}(f) \subseteq xZ$. From this, we have

$$\|f - L_xf\|_2^2 = \|f\|_2^2 + \|L_xf\|_2^2 > 1.$$

As this exists for all $x \in G$, we see $\|L_x - L_{e_G}\| > 1$. The proof for the right regular representation follows in a similar manner. □

Note: In the above, we first fix $x \in G$; with the WOT topology, we fix f, g and vary across all $x \in G$ instead.

Definition 5.1. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ be two representations of G. A map $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is intertwining from π to σ if:*

(i) T is linear and bounded.
(ii) $T\pi(x) = \sigma(x)T$ for all $x \in G$.

We say $\mathcal{C}(\pi, \sigma) = \{$all intertwining operators from π to $\sigma\}$.

Note: The set $\mathcal{C}(\pi, \sigma)$ is actually a linear space (which is not difficult to show).

Definition 5.2. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ be two representations of G. We say π and σ are unitarily equivalent if there exists a $U \in \mathcal{C}(\pi, \sigma)$ which is unitary.*

Proposition 5.3. *The left and right regular representation are unitarily equivalent.*

Proof. Let $U : L^2(G) \to L^2(G)$ be defined via

$$f \mapsto \frac{\check{f}}{\sqrt{\Delta}},$$

where $\check{f}(x) = f(x^{-1})$. For $f \in L^2(G)$, we compute

$$\begin{aligned}\|Uf\|_2^2 = \int_G |Uf(x)|^2 d\lambda(x) = \int_G \frac{|f(x^{-1})|^2}{\Delta(x)} d\lambda(x) = \int_G \frac{|f(x)|^2}{\Delta(x^{-1})} d\lambda(x^{-1}) \\ = \int_G \frac{|f(x)|^2}{\Delta(x^{-1})} \Delta(x^{-1}) d\lambda(x) = \|f\|_2^2,\end{aligned}$$

via a change of variables. This shows that U is an isometry. It is also easy to see that $U^2 = I$; thus, U is invertible, and hence a unitary map (as a surjective isometry). We wish to show that U intertwines L_x, ρ_x. For $f \in L^2(G)$, let $z \in G$ be arbitrary. We have

$$L_x Uf(z) = Uf(x^{-1}z) = \frac{f(z^{-1}x)}{\sqrt{\Delta(x^{-1}z)}} = \frac{f(z^{-1}x)}{\sqrt{\Delta(z)}}\sqrt{\Delta(x)} = \frac{(\rho_x f)(z^{-1})}{\sqrt{\Delta(z)}} = U(\rho_x f)(z).$$

As this holds for all $z \in G$ (as our choice of z was arbitrary), we see $L_x Uf = U\rho_x f$- i.e., they are unitarily equivalent as claimed. □

Definition 5.4. *Let $\pi : G \to \mathcal{U}(H)$. A closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is called π-invariant if $\pi(x)(\mathcal{M}) \subseteq \mathcal{M}$ for all $x \in G$. In more explicit terms, $\pi(x)\xi \in \mathcal{M}$ for all $\xi \in \mathcal{M}$.*

Note: As a result, we can restrict π to an invariant subspace $\mathcal{M} \neq \{0\}$ and obtain a mapping

$$\begin{aligned}\pi^{\mathcal{M}} : G \to \mathcal{U}(\mathcal{M}), \\ x \mapsto \pi(x)|_{\mathcal{M}}.\end{aligned}$$

Such a restriction is called a sub-representation of π. We know that such a sub-representation exists for $\{0\}$ and $\mathcal{H}$ itself, as these are π-invariant subspaces which always exist.

Definition 5.5. *A representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is called irreducible if there do not exist any non-trivial π-invariant subspaces. If it is not irreducible, we say π is reducible.*

Example: $L : \mathbb{T} \to \mathcal{U}(L^2(\mathbb{T}))$ is not irreducible.

Proof. We want to show that there exists a non-trivial closed subspace $\mathcal{M} \subseteq L^2(\mathbb{T})$ such that for all $f \in \mathcal{M}$, $L_x f \in \mathcal{M}$ for all $x \in \mathbb{T}$. Start by taking $f \in L^2(\mathbb{T})$ and $x \in \mathbb{T}$. Applying the Fourier transform, we see

$$\begin{aligned}\widehat{(L_x f)}(n) = \int_{\mathbb{T}} (L_x f)(t) e^{-2\pi i n t} dt = \int_{\mathbb{T}} f(-x + t) e^{-2\pi i n t} dt \\ = \int_{\mathbb{T}} f(t) e^{-2\pi i n(x+t)} dt,\end{aligned}$$

via a change of variable (where $t \mapsto x + t$). Then

$$\int_{\mathbb{T}} f(t)e^{-2\pi i n(x+t)} dt = e^{-2\pi i n x} \int_{\mathbb{T}} f(t)e^{-2\pi i n t} dt = \chi_{-n}(x)\hat{f}(n).$$

If we take $A \subset \mathbb{Z}$, with $A \neq \emptyset$, define

$$\mathcal{M} = \{f \in L^2(\mathbb{T}) : \operatorname{supp}(\hat{f}) \subseteq A\}.$$

It is clear that $\mathcal{M}$ is a subspace, and by the formula above it is invariant under L. We also claim it is closed; indeed, suppose $\{f_n\} \subseteq \mathcal{M}$, with $f_n \to_{\|\cdot\|_2} f$ for $f \in L^2(\mathbb{T})$. As $\mathbb{T}$ is compact, $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$- so $f_n \to_{\|\cdot\|_1} f$ as well. Thus, $\hat{f}_n \to \hat{f}$ pointwise. As $\hat{f}_n(k) = 0$ for all $k \in \mathbb{Z} \setminus A$, this implies $\hat{f}(k) = 0$ for all $k \in \mathbb{Z} \setminus A$ as well. We also claim $\mathcal{M}$ is non-trivial; this follows immediately from the fact that A is not all of $\mathbb{Z}$. Thus, $\mathcal{M}$ is a closed non-trivial L-invariant subspace, and hence L is not irreducible. □

Remark: If $A \subseteq \mathbb{Z}$ is a singleton (say $A = \{m\}$), then

$$\mathcal{M}_m = \{f \in L^2(\mathbb{T}) : \hat{f}(k) = 0 \text{ for all } k \neq m\} = \mathbb{C}\chi_m.$$

Question: What sub-representation is $L^{\mathcal{M}_m} : \mathbb{T} \to \mathcal{U}(\mathbb{C})$? (The answer will come later).

Recall: Suppose G is a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a continuous unitary representation. If $\mathcal{M} \subseteq \mathcal{H}_\pi$ is closed, we say $\mathcal{M}$ is π-invariant if $\pi(x)\xi \in \mathcal{M}$ for all $x \in G$ and $\xi \in \mathcal{M}$.

Lemma 5.6. *If $\mathcal{M}$ is a closed π-invariant subspace, then $\mathcal{M}^\perp$ is also a closed π-invariant subspace.*

Proof. Recall that $\mathcal{M}^\perp = \{\eta \in \mathcal{H} : \langle \eta, \xi \rangle = 0 \text{ for all } \xi \in \mathcal{M}\}$. By basic theory on Hilbert spaces, we know $\mathcal{M}^\perp$ is closed. We want to show that for all $x \in G$, $\pi(x)\eta \in \mathcal{M}^\perp$ if $\eta \in \mathcal{M}^\perp$. To that end, take an arbitrary $\eta \in \mathcal{M}^\perp$ and $\xi \in \mathcal{M}$; as

$$\langle \pi(x)\eta, \xi \rangle = \langle \eta, \pi(x)^*\xi \rangle = \langle \eta, \pi(x^{-1})\xi \rangle = 0,$$

(by π-invariance of $\mathcal{M}$), this shows $\pi(x)\eta \in \mathcal{M}^\perp$ using that $\xi \in \mathcal{M}$ was arbitrary. Then as our choice of $\eta \in \mathcal{M}^\perp$ was also arbitrary, this shows π-invariance of $\mathcal{M}^\perp$. □

Direct sums of Hilbert spaces

Let $\{\mathcal{H}_\alpha\}_{\alpha \in I}$ be a collection of Hilbert spaces. We write

$$\oplus_{\alpha \in I}\mathcal{H}_\alpha = \{(\nu_\alpha)_{\alpha \in I} : \nu_\alpha \in \mathcal{H}_\alpha, \sum_{\alpha \in I} \|\nu_\alpha\|^2 < \infty\}.$$

We can put an inner product on $\oplus_{\alpha \in I}\mathcal{H}_\alpha$ by letting

$$\langle (\nu_\alpha)_{\alpha \in I}, (u_\alpha)_{\alpha \in I} \rangle = \sum_{\alpha \in I} \langle \nu_\alpha, u_\alpha \rangle.$$

Under this inner product, $\oplus_{\alpha \in I}\mathcal{H}_\alpha$ becomes a Hilbert space. We may then define direct sums of unitary representations.

Definition 5.7. *Let $\{\pi_\alpha\}_{\alpha \in I}$ be a collection of representations*

$$\pi_\alpha : G \to \mathcal{U}(\mathcal{H}_{\pi_\alpha}).$$

Define the unitary representation $\oplus_{\alpha \in I}\pi_\alpha : G \to \mathcal{U}\Big(\oplus_{\alpha \in I} \mathcal{H}_\alpha\Big)$ via

$$\Big(\oplus_{\alpha \in I} \pi_\alpha\Big)(x)\Big((\nu_\alpha)_{\alpha \in I}\Big) = \Big(\pi_\alpha(x)\nu_\alpha\Big)_{\alpha \in I}.$$

Proposition 5.8. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation and $\mathcal{M}$ be a non-trivial π-invariant closed subspace. Then*

$$\pi = \pi^{\mathcal{M}} \oplus \pi^{\mathcal{M}^\perp}.$$

Proof. Look at the unitary operator $U : \mathcal{M} \oplus \mathcal{M}^\perp \to \mathcal{H}_\pi$ defined via

$$(\xi, \eta) \mapsto \xi + \eta,$$

and the operator $U^{-1} : \mathcal{H}_\pi \to \mathcal{M} \oplus \mathcal{M}^\perp$ defined via

$$u \mapsto (\xi_u, \eta_u)$$

where $u = \xi_u + \eta_u$ is the unique decomposition of $u \in \mathcal{H}_\pi$ (guaranteed by the fact that $\mathcal{H}_\pi$ is a Hilbert space and $\mathcal{M}$ is closed) with $\xi_u \in \mathcal{M}, \eta_u \in \mathcal{M}^\perp$. It is now easy to verify that

$$U(\pi^{\mathcal{M}} \oplus \pi^{\mathcal{M}^\perp})(x) = \pi(x)U$$

for all $x \in G$. □

Definition 5.9. *Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace. We let $P_{\mathcal{M}}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.*

Proposition 5.10. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a representation, and $\mathcal{M} \subseteq \mathcal{H}_\pi$ be a closed subspace. Then*

$$\mathcal{M} \textit{ is } \pi-\textit{invariant} \iff P_{\mathcal{M}} \in \mathcal{C}(\pi).$$

Proof. Recall that $T \in \mathcal{C}(\pi)$ means that $T\pi(x) = \pi(x)T$ for all $x \in G$. First, suppose that $\mathcal{M}$ is π-invariant. For all $x \in G$, we have $\pi(x)\xi \in \mathcal{M}$ when $\xi \in \mathcal{M}$. We want to show that $\pi(x)P_{\mathcal{M}} = P_{\mathcal{M}}\pi(x)$ for $x \in G$. If we take $u \in \mathcal{H}_\pi$, decomposing $u = \xi_u + \eta_u$ in $\mathcal{H}_\pi$ we note

$$\begin{aligned}\pi(x)P_{\mathcal{M}}u = \pi(x)\xi_u &= P_{\mathcal{M}}(\pi(x)\xi_u + \pi(x)\eta_u) \\ &= P_{\mathcal{M}}\pi(x)u.\end{aligned}$$

The other direction is similar. □

Lemma 5.11 (Schur's Lemma)**.** *Let G be a locally compact group, and π, π_1, π_2 be unitary representations.*

(i) π is irreducible if and only if $\mathcal{C}(\pi) = \mathbb{C}I_{\mathcal{H}_\pi}$.
(ii) Suppose π_1, π_2 are irreducible. Then $\pi_1 \cong \pi_2$ (unitary equivalence) if and only if $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional. Furthermore, $\pi_1 \ncong \pi_2$ if and only if $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.

Proof. For (i), we will prove that π is reducible if and only if $\mathcal{C}(\pi)$ contains some $T \notin \mathbb{C}I_{\mathcal{H}_\pi}$. If we first suppose π is reducible, there exists a non-trivial closed π-invariant subspace $\mathcal{M} \subseteq \mathcal{H}_\pi$. If we let $T = P_{\mathcal{M}}$, we know (by the previous proposition) that $P_{\mathcal{M}} \in \mathcal{C}(\pi)$. Clearly, $P_{\mathcal{M}} \neq \lambda I_{\mathcal{H}_\pi}$ for any $\lambda \in \mathbb{C}$; therefore, $P_{\mathcal{M}} \notin \mathbb{C}I_{\mathcal{H}_\pi}$.

Now, suppose there exists a $T \in \mathcal{C}(\pi)$ such that $T \notin \mathbb{C}I_{\mathcal{H}_\pi}$. Define

$$A = \frac{T + T^*}{2}, \quad B = \frac{T - T^*}{2i}.$$

Then both A, B are self-adjoint operators. We claim $A, B \in \mathcal{C}(\pi)$: indeed, as

$$T\pi(x) = \pi(x)T$$

for all $x \in G$, if we take the $*$ of both sides we find

$$T^*\pi(x)^* = \pi(x)^*T^* \Rightarrow T^*\pi(x^{-1}) = \pi(x^{-1})T^*$$

for all $x \in G$. Thus, $T^* \in \mathcal{C}(\pi)$; as $\mathcal{C}(\pi)$ is a linear space, this implies $A, B \in \mathcal{C}(\pi)$ as well. While A or B might be a scalar multiple of the identity, we know that *both* cannot be a scalar multiple of the identity at the same time (as otherwise this implies T is a scalar multiple of the identity as well- a direct contradiction). Without loss of generality, assume $A \notin \mathbb{C}I_{\mathcal{H}_\pi}$, with $A^* = A$. As A is self-adjoint, we know $\sigma(A) \subseteq \mathbb{R}$, and $\sigma(A)$ is not a singleton. Take $S \subset \sigma(A)$ non-empty. Let $f : \sigma(A) \to \mathbb{R}$ be defined via $f = \chi_S$. Using the Borel functional calculus, we can then define the operator $P = f(A)$. We have that P is an orthogonal projection, with $P \in \mathcal{C}(\pi)$. Additionally, $P \notin \mathbb{C}I_{\mathcal{H}_\pi}$. So, by the previous argument above we see that π must be reducible (as the range of P is a closed π-invariant non-trivial subspace).

For (ii), suppose $T \in \mathcal{C}(\pi_1, \pi_2)$. Note that $T^* \in \mathcal{C}(\pi_2, \pi_1)$- so $T^*T \in \mathcal{C}(\pi_1)$ with $TT^* \in \mathcal{C}(\pi_2)$. The rest of the proof can be shown by applying part (i)- **exercise!** □

5.2. **Weak*-topology review.** Recall that if X is a Banach space, then X^* denotes the dual space of X; this is the collection of all bounded linear functionals on X. The weak*-topology on X^* is defined in the following way: for $f \in X^*$, take a finite number of points $x_1, \ldots, x_n \in X$ and $\epsilon_1, \ldots, \epsilon_n > 0$. Let

$$N_{\substack{x_1,\ldots,x_n \\ \epsilon_1,\ldots,\epsilon_n}}(f) = \{g \in X^* : \ |g(x_i) - f(x_i)| < \epsilon_i, \ i = 1, \ldots, n\}.$$

For $n = 1$, then $N(f)$ becomes an intersection of other neighborhoods of the form above. Every weak*-open set in X^* is a union of such neighborhoods.

Remark: What does it mean for $f_n \to f$ in the weak*-topology? By definition, we have $f_n \to f$ in the weak*-topology if for every $x \in X$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ $f_n \in N_{x,\epsilon}(f)$. This means that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. So, $f_n(x) \to f(x)$ (i.e., f_n converges pointwise to f).

Considering our homework problem, we see $f_n \to f$ in the weak*-topology of L^∞ if and only if for all $\varphi \in L^1$, $\int f_n\varphi \to \int f\varphi$.

Recall:

Lemma 5.12 (The Riemann-Lebesgue lemma)**.** *For all $f \in L^1(\mathbb{T})$, $\hat{f}(n) \to 0$ as $n \to \infty$.*

This means

$$\int f(x)\chi_{-n}(x) \to 0$$

as $n \to \infty$. But by comparing the definition, this means $\chi_{-n} \to 0$ in the weak*-topology (even though pointwise, χ_{-n} does not converge to 0).

Corollary 5.13. *Let G be a locally compact abelian group. Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be an irreducible representation. Then π is one-dimensional.*

Proof. For each $g \in G$, we want to show that $\pi(g) \in \mathcal{C}(\pi)$- that is, for ally $y \in G$, $\pi(y)\pi(g) = \pi(g)\pi(y)$. But this follows directly from the fact that π is a homomorphism, and G is abelian. So for all $g \in G$, $\pi(g) \in \mathcal{C}(\pi)$. As π is irreducible, by Schur's lemma we know $\pi(g) = \lambda_g I_{\mathcal{H}}$. Take some $\xi \neq 0$ with $\xi \in \mathcal{H}$, and let $\mathcal{M} = \mathbb{C}\xi$. Then $\mathcal{M}$ is (clearly) π-invariant. As π is irreducible, this force $\mathcal{H} = \mathbb{C}\xi$, and so π must be one-dimensional. □

Recall: If G is abelian, then

$$\begin{aligned}\hat{G} &= \{\chi : G \to \mathbb{T} \text{ continuous homomorphisms}\} \\ &= \{\text{irreducible unitary representations of } G\}.\end{aligned}$$

For non-abelian groups, we define

$$\hat{G} = \{[\pi] : \ \pi : G \to \mathcal{U}(\mathcal{H}) \text{ irreducible unitary representations}\}.$$

6. Fourier analysis on compact groups

In what follows, we assume G is a compact group, λ is a left/right Haar measure (as our group is unimodular), and $\lambda(G) = 1$.

Examples:

(i) All finite groups.
(ii) The torus $\mathbb{T}$.
(iii) The group
$$\mathrm{SO}(3) = \{T \in \mathrm{GL}_3(\mathbb{R}) : \det(T) = 1,\ TT^t = T^tT = I\}.$$
(iv) The group
$$\mathrm{SU}(2) = \{T \in \mathrm{GL}_2(\mathbb{C}) : \det(T) = 1,\ T \text{ unitary}\}.$$

Definition 6.1. *Let G be compact, and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a continuous and unitary representation. Fix some arbitrary unit vector $\xi \in \mathcal{H}$. For all $x \in G$, define the rank 1-projection $\pi(x)\xi \otimes \pi(x)\xi : \mathcal{H} \to \mathcal{H}$ onto $\pi(x)\xi$ via*
$$\eta \mapsto \langle \eta, \pi(x)\xi\rangle \pi(x)\xi.$$

Definition 6.2. *Let G, π, ξ be as above. Define the operator*
$$T := \int_G \pi(x)\xi \otimes \pi(x)\xi d\lambda(x)$$
where T is interpreted in the weak sense; that is, for all $\eta_1, \eta_2 \in \mathcal{H}$ we know
$$\langle T\eta_1, \eta_2\rangle = \int_G \langle (\pi(x)\xi \otimes \pi(x)\xi)\eta_1, \eta_2\rangle d\lambda(x)$$
$$= \int_G \langle \eta_1, \pi(x)\xi\rangle \langle \pi(x)\xi, \eta_2\rangle d\lambda(x).$$

Lemma 6.3. *Let G, π, and ξ be as in the definitions above. Then the map T is non-zero, positive, compact, and $T \in \mathcal{C}(\pi)$.*

Proof. We first will show that T is positive. This follows directly from choosing any $\eta \in \mathcal{H}$, and noting that
$$\langle T\eta, \eta\rangle = \int_G |\pi(x)\xi, \eta\rangle|^2 dx \geq 0.$$

To show that T is non-zero, consider the vector $\xi \in \mathcal{H}$. We have
$$\langle T\xi, \xi\rangle = \int_G |\langle \pi(x)\xi, \xi\rangle|^2 dx.$$
We know $\langle \pi(x)\xi, \xi\rangle$ is a continuous function (as it is a coefficient function for a continuous unitary representation). Additionally, we have $|\langle \pi(x)\xi, \xi\rangle|^2 = \|\xi\|^2 = 1$. This means that the function must be positive on an open neighborhood around e_G- hence, $\langle T\xi, \xi\rangle > 0$. This implies $T \neq 0$.

To show that T is compact, recall that the function $G \to \mathcal{H}$ mapping $x \mapsto \pi(x)\xi$ is continuous. As G is compact, the mapping must be uniformly continuous. Therefore, for any $\epsilon > 0$, there exists an open neighborhood $V \subseteq G$ such that $e_G \in V$; furthermore, for all $x, y \in G$ if $y^{-1}x \in V$, then $\|\pi(x)\xi - \pi(y)\xi\| < \epsilon/2$. Suppose we take arbitrary $\epsilon > 0$. As G is compact, there exist points $x_1, \ldots, x_n \in G$ such that
$$G \subseteq x_1V \cup \cdots \cup x_nV.$$

Without loss of generality, we can "disjointify" our sets and assume

$$G = E_1 \sqcup \cdots \sqcup E_n$$

(here we potentially abuse notation by assuming we have n sets exactly, instead of some potentially smaller index k) such that for all $y \in E_i$, $\|\pi(x_i\xi - \pi(y)\xi\| < \epsilon/2$.

Define the operator

$$T_{\mathcal{E}} = \sum_{i=1}^{n} \int_{i} \pi(x_i)\xi \otimes \pi(x_i)\xi d\lambda(x)$$

$$= \sum_{i=1}^{n} \lambda(E_i)(\pi(x_i)\xi \otimes \pi(x_i)\xi).$$

As each $\pi(x_i)\xi \otimes \pi(x_i)\xi$ is finite rank for $i \in [n]$, and we can easily show

$$\|T_{\mathcal{E}} - T\| < \epsilon$$

this shows T must be compact (as the norm limit of a sequence of finite rank operators).

Finally, we show that $T \in \mathcal{C}(\pi)$. We want to show that for all $y \in G$, $T\pi(y) = \pi(y)T$- that is, for all $\eta \in \mathcal{H}$ we have $T\pi(y)\eta = \pi(y)T\eta$. We note that for $\eta, \eta' \in \mathcal{H}$

$$\int_G \langle \pi(y)\eta, \pi(x)\xi\rangle\langle\pi(x)\xi, \eta'\rangle d\lambda(x) = \int_G \langle \eta, \pi(y^{-1}x)\xi\rangle\langle\pi(x)\xi, \eta'\rangle d\lambda(x)$$

$$= \int_G \langle \eta, \pi(x)\xi\rangle\langle\pi(x)\xi, \pi(y)^*\eta'\rangle d\lambda(x)$$

using the change of variable $y^{-1}x \mapsto x$. The latter integral is nothing but

$$\langle T\eta, \pi(y)^*\eta'\rangle = \langle \pi(y)T\eta, \eta'\rangle.$$

As this holds for all $\eta, \eta' \in \mathcal{H}$, this implies $T\pi(y) = \pi(y)T$. Therefore, $T \in \mathcal{C}(\pi)$. ☐

Corollary 6.4. *Let G be compact, and π an irreducible representation of G. Then π is finite dimensional.*

Proof. As π is irreducible, and $T \in \mathcal{C}(\pi)$ then by Schur's lemma $T = \lambda I_{\mathcal{H}}$ for some non-zero $\lambda \in \mathbb{C}$. As T is compact, $I_{\mathcal{H}}$ must also be compact; however, this holds if and only if $\mathcal{H}$ is finite dimensional. Thus, π must be finite-dimensional. ☐

Remark: Both the Heisenberg group and $\mathrm{SL}_2(\mathbb{R})$ have *many* infinite-dimensional irreducible representations. So, the previous statement is not trivial in the least.

Theorem 6.5. *Let G be a compact group. Then*

(i) Every irreducible representation of G is finite dimensional.

(ii) Every unitary representation of G is unitarily equivalent to a direct sum of irreducible representations of G. That is, for all $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ there exists a unitary U such that

$$U^*\pi(x)U = \bigoplus_{\text{some } \sigma\in\hat{G}} \sigma(x)$$

for all $x \in G$.

Proof. We have shown (i) already, and so we focus only on (ii). Let $\sigma : G \to \mathcal{U}(\mathcal{H})$ be an arbitrary unitary continuous representation. Let T be the operator defined and analyzed in the previous lemmas (note we are implicitly using the compactness of G here, to discuss T). Since T is a compact, non-zero and self-adjoint operator (as T is positive), by the Spectral Theorem there exists a non-zero c which is an eigenvalue of T. Let $\mathcal{M}_0$ denote the eigenspace corresponding to c; then $\dim(\mathcal{M}_0) < \infty$- this follows from our spectral decomposition of T, as if $(\lambda_i)_{i=1}^{\infty}$ is the sequence of eigenvalues of T we must have $\lambda_i \to 0$

as $i \to \infty$. We claim $\mathcal{M}_0$ is σ-invariant. To see why, take $\eta \in \mathcal{M}_0$ and let $x \in G$ be arbitrary. We want to show that $\sigma(x)\eta \in \mathcal{M}_0$. Since T intertwines σ, we have

$$T\sigma(x)\eta = \sigma(x)T\eta = c\sigma(x)\eta.$$

Thus, $\sigma(x)\eta \in \mathcal{M}_0$ as we claim. Clearly, $\mathcal{M}_0$ is a closed and σ-invariant subspace of $\mathcal{H}$.

Look at the restriction $\sigma^{\mathcal{M}_0}$; if $\sigma^{\mathcal{M}_0}$ is not irreducible, there exists a non-trivial subspace $\sigma^{\mathcal{M}_0}$-invariant $\mathcal{M}_1 \subseteq \mathcal{M}_0$ where $0 < \dim(\mathcal{M}_1) < \dim(\mathcal{M}_0)$. Proceeding by induction, we finally find a subspace $\mathcal{M}'$ such that $\mathcal{M}'$ is closed, σ-invariant, and $\sigma^{\mathcal{M}'}$ is irreducible; that this process terminates follows directly from the fact that $\dim(\mathcal{M}_0) < \infty$. This shows the existence of a subspace $\mathcal{M}'$ which is closed, σ-invariant, and $\sigma^{\mathcal{M}'}$ is irreducible. We then may say that $(\mathcal{M}')^\perp$ is σ-invariant, and so we can repeat the previous argument but for $(\mathcal{M}')^\perp$ instead. This yields a subspace $\mathcal{M}''$ which is closed, inside $(\mathcal{M}')^\perp$ and $\sigma^{(\mathcal{M}')^\perp}$ invariant. Note that $\mathcal{M}'' \perp \mathcal{M}'$.

If we let $\mathcal{F}$ be the collection of all subsets of pairwise orthogonal irreducible subspaces of $\mathcal{H}$, and endow $\mathcal{F}$ with a partial order under set inclusion, we can use a standard argument in order to apply Zorn's Lemma guaranteeing the existence of a maximal element

$$\{\mathcal{M}_\alpha : \alpha \in I\}$$

where $\mathcal{M}_\alpha \perp \mathcal{M}_\beta$ for all $\alpha, \beta \in I$, $\mathcal{M}_\alpha$ is σ-invariant and $\sigma^{\mathcal{M}_\alpha}$ is irreducible for all $\alpha \in I$.

We want to show $\bigoplus_{\alpha \in I} \mathcal{M}_\alpha = \mathcal{H}$. Suppose not- then let $\mathcal{N} = \left(\bigoplus_{\alpha \in I} \mathcal{M}_\alpha\right)^\perp$. The space $\mathcal{N}$ is a non-trivial σ-invariant subspace of $\mathcal{H}$; looking at $\sigma^{\mathcal{N}}$, we can use our first result to find a subspace $\mathcal{N}_0 \subseteq \mathcal{N}$ such that $\mathcal{N}_0$ is $\sigma^{\mathcal{N}}$ (and hence σ)-invariant, and $\sigma^{\mathcal{N}_0}$ is irreducible. Taking

$$\{\mathcal{M}_\alpha : \alpha \in I\} \cup \mathcal{N}_0,$$

we have a contradiction (of the maximality of our element guaranteed by Zorn's Lemma). Thus,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{M}_\alpha.$$

This shows $\sigma = \bigoplus_{\alpha \in I} \sigma^{\mathcal{M}_\alpha}$ is a direct sum decomposition of σ into irreducible representations. □

Definition 6.6. *A representation $\sigma : G \to \mathcal{U}(\mathcal{H})$ is called cyclic if for all $\xi \in \mathcal{H}$, where $\xi \neq 0$ we have*

$$\overline{\operatorname{span}\{\sigma(x)\xi : x \in G\}} = \mathcal{H}.$$

Note: In practice, for compact group G

$$\hat{G} = \{[\pi] : \pi \text{ irreducible}\}$$

and for any representation σ,

$$\sigma \cong \bigoplus_{\substack{\pi \in A \\ A \subseteq \hat{G}}} m_\pi \cdot \pi$$

where m_π is the multiplicity of π in σ, and

$$m_\pi \cdot \pi = \bigoplus_{i=1}^{m} \pi.$$

Note: In Assignment 3, for a specific

$$\pi : G \to \mathcal{U}(L^2(\mathbb{R})),$$
$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a \neq 0, \ b \in \mathbb{R} \right\}$$

then π is irreducible. If L is the left regular representation of G,

$$L = \bigoplus_{i=1}^{\infty} \pi.$$

This observation gives us a wealth of applications to wavelet analysis.

Proposition 6.7. *Let $\sigma : G \to \mathcal{U}(\mathcal{H})$ be a continuous unitary representation. Let $\mathcal{M}_1, \mathcal{M}_2$ be σ-invariant closed subspaces of $\mathcal{H}$ such that $\sigma^{\mathcal{M}_1}, \sigma^{\mathcal{M}_2}$ are irreducible and $\sigma^{\mathcal{M}_1} \ncong \sigma^{\mathcal{M}_2}$. Then $\mathcal{M}_1 \perp \mathcal{M}_2$.*

Proof. Define $P_1 : \mathcal{H} \to \mathcal{M}_1$ as the orthogonal projection onto $\mathcal{M}_1$. We know $P_1 \in \mathcal{C}(\sigma, \sigma^{\mathcal{M}_1})$- so $P_1\sigma(x) = \sigma(x)P_1$ for all $x \in G$. Define $P = P_1|_{\mathcal{M}_2}$. We want to show $P = 0$, which would imply $\mathcal{M}_2 \subseteq \ker(P_1)$ and so $\mathcal{M}_2 \subseteq \mathcal{M}_1^{\perp}$.

To that end, we claim $P \in \mathcal{C}(\sigma^{\mathcal{M}_2}, \sigma^{\mathcal{M}_1})$. We want to show that $P\sigma^{\mathcal{M}_2}(x) = \sigma^{\mathcal{M}_1}(x)P$ for all $x \in G$. Let $x \in G$ and $\eta \in \mathcal{M}_2$ be arbitrary. We note

$$P\sigma^{\mathcal{M}_2}(x)\eta = P\sigma(x)\eta = P_1\sigma(x)\eta$$

as $\sigma(x)\eta \in \mathcal{M}_2$. Similarly,

$$\sigma^{\mathcal{M}_1} P\eta = \sigma(x)P\eta = \sigma(x)P_1\eta.$$

As $P_1 \in \mathcal{C}(\sigma, \sigma^{\mathcal{M}_1})$, we have $\sigma(x)P_1\eta = P_1\sigma(x)\eta$. So $P \in \mathcal{C}(\sigma^{\mathcal{M}_2}, \sigma^{\mathcal{M}_1})$. By Schur's Lemma, this forces $P = 0$. Therefore, $\mathcal{M}_2 \subseteq \ker(P_1)$, and hence $\mathcal{M}_2 \perp \mathcal{M}_1$. $\square$

Note: The proof of the previous proposition works for any locally compact group.

Definition 6.8. *Let $\sigma : G \to \mathcal{U}(\mathcal{H})$ be a continuous unitary representation, and let $\pi \in \hat{G}$. Define*

$$\mathcal{M}_\pi = \overline{\text{span}\{\mathcal{M} : \ \mathcal{M} \subseteq \mathcal{H} \textit{ is a closed } \sigma - \textit{invariant subspace with } \sigma^{\mathcal{M}} \cong \pi\}}.$$

Then $\mathcal{M}_\pi$ is a σ-invariant subspace.

Corollary 6.9. *If $\pi_1, \pi_2 \in \hat{G}$ with $\pi_1 \ncong \pi_2$, with $\sigma : G \to \mathcal{U}(\mathcal{H})$ as above then*

$$\mathcal{M}_{\pi_1} \perp \mathcal{M}_{\pi_2}.$$

Proposition 6.10. *We have*

$$\text{mult}(\sigma, \pi) = \dim(\mathcal{C}(\sigma, \pi)).$$

6.1. **Decomposition of the left regular representation.** In what follows, we let G be compact, λ the left/right Haar measure on G (by unimodularity) and as before, we may assume $\lambda(G) = 1$.

Recall: The left regular representation $L : G \to \mathcal{U}(L^2(G))$ was defined via

$$x \mapsto L_x,$$

where

$$L_x : L^2(G) \to L^2(G),$$
$$(L_x f)(z) = f(x^{-1}z).$$

Q: Fix an arbitrary irreducible representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$. Does there exist a subspace $\mathcal{M} \subseteq L^2(G)$ such that

(i) $\mathcal{M}$ is closed.
(ii) $\mathcal{M}$ is L-invariant.
(iii) $L^{\mathcal{M}} \cong \pi$?

Recall that we may take arbitrary vectors $\xi_0, \eta_0 \in \mathcal{H}_\pi$ to define $\pi_{\xi_0,\eta_0} : G \to \mathbb{C}$ via

$$\pi_{\xi_0,\eta_0}(x) = \langle \pi(x)\xi_0, \eta_0 \rangle.$$

We know $\pi_{\xi_0,\eta_0} \in C(G) \subseteq L^2(G)$ (where the latter inclusion follows from compactness of G). For some $x \in G$,

$$(L_x \pi_{\xi_0,\eta_0})(z) = \pi_{\xi_0,\eta_0}(x^{-1}z) = \pi_{\xi_0,\pi(x)\eta_0}(z).$$

Thus, L_x shifts coefficient functions to coefficient functions.

Suggestion: As we are hoping to construct some sort of subspace $\mathcal{M}$ which satisfies the properties above, let

$$\mathcal{M} = \{\pi_{\xi_0,\eta} : \ \eta \in \mathcal{H}_\pi\}.$$

It is easy to show that:

(i) $\mathcal{M}$ is a subspace of $L^2(G)$; indeed, we have

$$\pi_{\xi_0,\eta} + \lambda \pi_{\xi_0,\eta'} = \pi_{\xi_0,\eta+\lambda\eta'}.$$

(ii) $\dim(\mathcal{M}) \leq \dim(\mathcal{H}_\pi)$. Indeed, let $\{e_1, \ldots, e_{d_\pi}\}$ be a basis for $\mathcal{H}_\pi$; we know that $\mathcal{H}_\pi$ is finite dimensional, as π is a continuous unitary representation of a compact group. For $\eta \in \mathcal{H}_\pi$, we have

$$\eta = \sum_{i=1}^{d_\pi} \alpha_i e_i.$$

Then

$$\pi_{\xi_0,\eta} = \sum_{i=1}^{d_\pi} \alpha_i \pi_{\xi_0,e_i}.$$

Q: Clearly, $\{\pi_{\xi_0,e_i} : \ i = 1, \ldots, d_\pi\}$ is a spanning set; however, is it linearly independent in $L^2(G)$?

(iii) $\mathcal{M}$ is L-invariant.

We want to see if $L^{\mathcal{M}} \cong \pi$. To that end, consider $T : \mathcal{H}_\pi \to \mathcal{M}$ defined via

$$\eta \mapsto \pi_{\xi_0,\eta}.$$

By writing out the definition directly, we can check that T is a conjugate linear map. For $x \in G$, we see

$$L^{\mathcal{M}}(x)T\eta = L^{\mathcal{M}}(\pi_{\xi_0,\eta}) = L(x)(\pi_{\xi_0,\eta}) = L_x(\pi_{\xi_0,\eta}) = \pi_{\xi_0,\pi(x)\eta} = T\pi(x)\eta.$$

As this holds for all $\eta \in \mathcal{H}_\pi$, we see $T \in \mathcal{C}(\pi, L^{\mathcal{M}})$. If we can show that $\overline{T}$ is unitary, we are done; to do so, it will be sufficient to show that T is a non-zero multiple of a unitary.

Notation: If G is compact and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a continuous unitary representation, we denote

$$\mathcal{E}_\pi = \operatorname{span}\{\pi_{\xi,\eta} : \ \xi, \eta \in \mathcal{H}_\pi\}.$$

(i) $\mathcal{E}_\pi$ is a subspace of $C(G)$ (and hence $L^p(G)$ for $1 \leq p \leq \infty$ by compactness of G).
(ii) If $\pi \cong \pi'$, then $\mathcal{E}_\pi = \mathcal{E}_{\pi'}$.

Proof. Let $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ be the unitary intertwining map. Then

$$\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle = \langle U\pi(x)\xi, U\eta \rangle = \langle \pi'(x)U\xi, U\eta \rangle = \pi'_{U\xi,U\eta}.$$

□

(iii) $\mathcal{E}_\pi$ is both L and ρ-invariant (as a subspace).

Before discussing one of the most fundamental theorems for harmonic analysis on compact groups, we introduce a lemma which will aid us in proofs going forward.

Lemma 6.11. *Suppose G is compact, $\pi : G \to \mathcal{U}(\mathcal{H}_\pi), \pi' : G \to \mathcal{U}(\mathcal{H}_{\pi'})$ are irreducible (continuous and unitary) representations. Let $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ be any linear operator. Define $\tilde{T} : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ via*

$$\tilde{T}(x) = \int_G \pi'(x^{-1})T\pi(x)d\lambda(x),$$

where we understand the definition of $\tilde{T}$ in the weak sense- i.e.,

$$\langle \tilde{T}\xi, \eta\rangle = \int_G \langle \pi'(x^{-1})T\pi(x)\xi, \eta\rangle d\lambda(x).$$

Then $\tilde{T} \in \mathcal{C}(\pi, \pi')$.

Proof. For all $x \in G$, and $\xi \in \mathcal{H}_\pi, \xi' \in \mathcal{H}_{\pi'}$ we see

$$\begin{aligned}\langle \tilde{T}\pi(x)\xi, \xi'\rangle &= \int_G \langle \pi'(y^{-1})T\pi(y)\pi(x)\xi, \xi'\rangle d\lambda(y) = \int_G \langle \pi'(xy^{-1})T\pi(y)\xi, \xi'\rangle d\lambda(y) \\ &= \int_G \langle \pi'(y^{-1})T\pi(y)\xi, \pi'(x)^*\xi'\rangle d\lambda(y) = \langle \tilde{T}\xi, \pi'(x)^*\xi'\rangle \\ &= \langle \pi'(x)\tilde{T}\xi, \xi'\rangle.\end{aligned}$$

Note that in the above string of equalities, we used the change of variable $y \mapsto yx^{-1}$ and relied on the unimodularity of our group. □

Theorem 6.12 (Schur Orthogonality Relations)**.** *Let G be compact, $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, and $\pi' : G \to \mathcal{U}(\mathcal{H}_{\pi'})$ where π, π' are continuous unitary irreducible representations. For all $\xi, \eta \in \mathcal{H}_\pi$ and $\xi', \eta' \in \mathcal{H}_{\pi'}$:*

(i) If $\pi \ncong \pi'$, then $\langle \pi_{\xi,\eta}, \pi_{\xi',\eta'}\rangle_{L^2(G)} = 0$. Equivalently, $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.

(ii) If $\pi = \pi'$, then

$$\langle \pi_{\xi,\eta}, \pi_{\xi',\eta'}\rangle_{L^2(G)} = \frac{1}{d_\pi}\langle \xi, \xi'\rangle\langle \eta, \eta'\rangle.$$

Proof. To show (i), let $\xi \in \mathcal{H}_\pi, \xi' \in \mathcal{H}_{\pi'}$ be arbitrary. Define $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ via

$$\eta \mapsto \langle \eta, \xi\rangle\xi'/$$

By Lemma 6.11, $\tilde{T} \in \mathcal{C}(\pi, \pi')$ where $\tilde{T}$ is defined as above. If $\pi \ncong \pi'$, we know $\tilde{T} = 0$ by Schur's lemma. This means that for all $\eta \in \mathcal{H}_\pi, \eta' \in \mathcal{H}_{\pi'}$, we have

$$\begin{aligned}\langle \tilde{T}\eta, \eta'\rangle &= \int_G \langle \pi'(x^{-1})T\pi(x)\eta, \eta'\rangle d\lambda(x) = \int_G \langle \pi(x)\eta, \xi\rangle\langle \xi', \pi'(x)\eta'\rangle d\lambda(x) \\ &= \int_G \pi_{\eta,\xi}(x)\overline{\pi'_{\eta',\xi'}(x)}d\lambda(x) = 0.\end{aligned}$$

Therefore, $\langle \pi_{\xi,\eta}, \pi'_{\xi',\eta'}\rangle_{L^2(G)} = 0$.

We next wish to prove statement (ii). Suppose $\pi = \pi'$, and let $\xi = e_i, \eta = e_j, \xi' = e'_i, \eta' = e'_j \in \mathcal{H}_\pi$. In this case, $\tilde{T} = \lambda I$ for some $\lambda \in \mathbb{C}$ (again, by Schur's lemma). We see

$$\lambda\delta_{jj'} = \langle \tilde{T}e_i, e_j\rangle = \int_G \pi_{e_j,e_i}(x)\overline{\pi_{e'_j,e'_i}(x)}d\lambda(x).$$

Looking at $\tilde{T}$, we have

$$\lambda d_\pi = \mathrm{Tr}(\lambda I) = \mathrm{Tr}(\tilde{T}) = \sum_{k=1}^{d_\pi} \langle \tilde{T} e_k, e_k \rangle$$
$$= \int_G \mathrm{Tr}(\pi(x^{-1}) T \pi(x)) d\lambda(x) = \mathrm{Tr}(T) = \delta_{i,i'}.$$

This implies

$$\frac{1}{d_\pi} \delta_{i,i'} \delta_{j,j'} = \langle \tilde{T} e_j, e'_j \rangle = \int_G \pi_{e_j, e_i}(x) \overline{\pi_{e'_j, e'_i}(x)} d\lambda(x).$$

□

Corollary 6.13. *For G a compact group, let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a continuous unitary irreducible representation. Let $T : \{e_1, \ldots, e_{d_\pi}\}$ be an orthonormal basis for $\mathcal{H}_\pi$. Then*

$$\{\sqrt{d_\pi} \pi_{e_i, e_j} : 1 \le i, j \le d_\pi\}$$

is an orthonormal basis for $\mathcal{E}_\pi$.

Proof. Let $\mathcal{B} = \{\sqrt{d_\pi} \pi_{e_i, e_j} : 1 \le i, j \le d_\pi\}$. We first want to show that $\mathcal{B}$ is a spanning set for $\mathcal{E}_\pi$. To that end, take an arbitrary $\pi_{\xi,\eta} \in \mathcal{E}_\pi$. As $\{e_1, \ldots, e_{d_\pi}\}$ is an orthonormal basis for $\mathcal{H}_\pi$, we may write

$$\xi = \sum_{i=1}^{d_\pi} \alpha_i e_i,$$
$$\eta = \sum_{i=1}^{d_\pi} \beta_i e_i,$$

for $\alpha_i, \beta_i \in \mathbb{C}$. Then

$$\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle = \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} \alpha_i \overline{\beta_j} \langle \pi(x) e_i, e_j \rangle.$$

This implies

$$\pi_{\xi,\eta} = \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} \frac{\alpha_i \overline{\beta_j}}{\sqrt{d_\pi}} (\sqrt{d_\pi} \pi_{e_i, e_j}),$$

and so spanning follows.

Next, to show that $\mathcal{B}$ is an orthonormal set note that

$$\langle \sqrt{d_\pi} \pi_{e_i, e_j}, \sqrt{d_\pi} \pi_{e_\ell, e_k} \rangle_{L^2(G)} = d_\pi \left(\frac{1}{d_\pi} \right) \langle e_i, e_\ell \rangle \langle e_k, e_j \rangle$$
$$= \delta_{i\ell} \delta_{kj}.$$

Clearly, this implies orthonormality. □

Theorem 6.14. *Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be an irreducible representation, and $\{e_1, \ldots, e_{d_\pi}\}$ an orthonormal basis for $\mathcal{H}_\pi$. For fixed $1 \le i \le d_\pi$, define*

$$R_{\pi,i} = \mathrm{span}\{\overline{\pi_{e_i, e_j}} : 1 \le j \le d_\pi\},$$
$$C_{\pi,i} = \mathrm{span}\{\pi_{e_j, e_i} : 1 \le j \le d_\pi\}.$$

Then $R_{\pi,i}$ is L-invariant, $C_{\pi,i}$ is ρ-invariant, and

$$L^{R_{\pi,i}} \cong \pi, \quad \rho^{C_{\pi,i}} \cong \pi.$$

Proof. In what follows, we use the notation π_{ij} for π_{e_i,e_j}. Let $\overline{\pi_{ij}} \in R_{\pi,i}, x \in G$ be arbitrary. We want to show that

$$L_x\overline{\pi_{ij}} \in R_{\pi,i}.$$

Through direct computation, we have

$$L_x\overline{\pi_{ij}}(z) = \overline{\pi_{ij}}(x^{-1}z) - \overline{\langle \pi(x^{-1}z)e_i, e_j\rangle}$$
$$= \overline{\langle \pi(z)e_i, \pi(x)e_j\rangle} = \overline{\left\langle \pi(z)e_i, \sum_{k=1}^{d_\pi} \alpha_k e_k \right\rangle} = \left(\sum_{k=1}^{d_\pi} \overline{\alpha_k \pi_{ik}}\right)(z).$$

Thus, as $z, x \in G$ were arbitrary this shows $R_{\pi,i}$ is L-invariant. Through a similar calculation, we have $C_{\pi,i}$ is ρ-invariant as well.

Next, we wish to show that there exists a unitary map $U : \mathcal{H}_\pi \to R_{\pi,i}$. Let $U : \mathcal{H}_\pi \to R_{\pi,i}$ be defined on the basis vectors via

$$e_j \mapsto \sqrt{d_\pi}\overline{\pi_{e_i,e_j}}.$$

Extend U to $\mathcal{H}_\pi$ by linearity. It is clear that U is surjective, by definition of $R_{\pi,i}$. We also claim it is an isometry: indeed,

$$\langle U\xi, U\eta\rangle = \langle \sqrt{d_\pi}\overline{\pi_{e_i,\xi}}, \sqrt{d_\pi}\overline{\pi_{e_i,\eta}}\rangle_{L^2(G)} = \langle \sqrt{d_\pi}\pi_{e_i,\eta}, \sqrt{d_\pi}\pi_{e_i,\xi}\rangle_{L^2(G)}$$
$$= \langle \xi, \eta\rangle.$$

Therefore, as U is a surjective isometry, we see must be unitary.

To finish, we show that $L_xU = U\pi(x)$ for all $x \in G$. To that end, we note

$$(L_xU\xi)(z) = U\xi(x^{-1}z) = \sqrt{d_\pi}\overline{\pi_{e_i,\xi}(x^{-1}z)} = (U\pi(x)\xi)(z)$$

for all z. As this shows $L^{R_{\pi,i}} \cong \pi$, we conclude the proof. □

Recall: We defined $\mathcal{E}_\pi = \text{span}\{\pi_{ij} : 1 \leq i, j \leq d_\pi\}$. For $i = 1, \ldots, d_\pi$ look at $C_{\pi,i}$. This is ρ-invariant (by the previous theorem), with $\rho^{C_{\pi,i}} \cong \pi$. This implies

$$\rho^{\mathcal{E}_\pi} \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{d_\pi \text{ times}}.$$

Let $U_2 : \mathcal{E}_\pi \to \mathcal{H}_\pi \oplus \cdots \oplus \mathcal{H}_\pi$ be defined via

$$\pi_{ij} \mapsto (0, 0, \ldots, 0, e_i, 0, \ldots, 0, 0)$$

where e_i is in the j^{th} term.

Definition 6.15. *Let* $\mathcal{E} = \oplus_{\pi\in\hat{G}}\mathcal{E}_\pi \subseteq C(G) \subseteq L^2(G)$.

Theorem 6.16. $\mathcal{E}$ *is an algebra.*

Proof. We see

$$(\pi_{ij}\pi'_{k\ell})(x) = \langle (\pi \otimes \pi')(x)(e_i \otimes e_k), e_j \otimes e_\ell\rangle.$$

As $(\pi \otimes \pi')$ is unitary (as the tensor product of unitary representations), and any unitary representation of a compact group can be written as a direct sum of irreducible representations, the coefficient functions of $\pi \otimes \pi'$ become a sum of coefficients functions of the irreducible components. This implies $(\pi_{ij}\pi'_{k\ell}) \in \mathcal{E}$. □

Without much introduction, we include the following theorem (which will be used in a later proposition). For more background on functions of positive type and a better introduction, see Folland.

Theorem 6.17 (Gelfand-Raikov)**.** *If G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if x, y are distinct points of G, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$.*

Proof. If $x \neq y$, there exists a function $f \in C_c(G)$ such that $f(x) \neq f(y)$, and we may assume f is a linear combination of functions of positive type (in $\mathcal{P}_1$). Then there exists a linear combination g of extreme points in $\mathcal{P}_1$ which approximates f on the compact set $\{x, y\}$ close enough such that $g(x) \neq g(y)$. This means there must exist an extreme point $\phi \in \mathcal{P}_1$ such that $\phi(x) \neq \phi(y)$. The associated representation π_ϕ is irreducible (which can be shown), and satisfies

$$\langle \pi_\phi(x)\epsilon, \epsilon\rangle = \phi(x) \neq \phi(y) = \langle \pi_\phi(y)\epsilon, \epsilon\rangle.$$

Hence, $\pi_\phi(x) \neq \pi_\phi(y)$. □

Proposition 6.18. *$\mathcal{E}$ is dense in $(C(G), \|\cdot\|_{\sup})$, and in $(L^p(G), \|\cdot\|_p)$ for $1 \leq p < \infty$.*

Proof. (Sketch) We use the Stone-Weierstrass theorem to show the result above for the $\|\cdot\|_{\sup}$-norm; the result for L^p will follow from this, as G is compact. That $\mathcal{E}$ is an algebra follows from the previous theorem, and $\mathcal{E}$ contains the constant functions (for example: look at the coefficient functions for the trivial representation). Also, $\mathcal{E}$ is closed under taking complex conjugates, which can be verified through direct computation. That $\mathcal{E}$ separates points follows from the Gelfand-Raikov theorem. Then by the Stone-Weierstrass theorem, $\overline{\mathcal{E}} = C(G)$ in the $\|\cdot\|_{\sup}$-norm. □

Theorem 6.19 (Peter-Weyl Theorem)**.** *Let G be a compact group. Then*

$$L^2(G) = \ell^2 - \oplus_{\pi\in\hat{G}}\mathcal{E}_\pi.$$

That is,

$$\bigcup_{\pi\in\hat{G}} \{\sqrt{d_\pi}\pi_{ij} : 1 \leq i, j \leq d_\pi\}$$

is a basis for $L^2(G)$. Moreover,

$$\rho \cong \oplus_{\pi\in\hat{G}}(d_\pi \cdot \pi).$$

Equivalently, there exists a unitary $U : \oplus_{\pi\in\hat{G}} \oplus_{i=1}^{d_\pi} C_{\pi,i} \to L^2(G)$ such that for all $x \in \hat{G}$,

$$U^*\rho(x)U = \oplus_{\pi\in\hat{G}}(d_\pi \cdot \pi(x)).$$

Note: Here

$$d_\pi \cdot \pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{d_\pi \text{ times}}.$$

As a consequence of Peter-Weyl, for all $f \in L^2(G)$ we have

$$f = \sum_{\pi\in\hat{G}} \sum_{i,j=1}^{d_\pi} \langle f, \sqrt{d_\pi}\pi_{ij}\rangle(\sqrt{d_\pi}\pi_{ij}) = \sum_{\pi\in\hat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \pi_{ij}\rangle\pi_{ij}. \tag{2}$$

Recall: $\langle f, \pi_{ij}\rangle = \int_G f(x)\overline{\pi_{ij}(x)}d\lambda(x)$.

We also have

$$\|f\|_2^2 = \sum_{\pi\in\hat{G}} \sum_{i,j=1}^{d_\pi} d_\pi |\langle f, \pi_{ij}\rangle|^2.$$

Note: Such an expansion depends on our fixed choice of an orthonormal basis; we would like to get an expression that is independent of the choice of basis.

Definition 6.20. *Let G be compact, $f \in L^1(G)$, and $\pi \in \hat{G}$. The Fourier transform of f at π is defined as*

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^* d\lambda(x),$$

where we interpret it weakly. That is,

$$\langle \hat{f}(\pi)e_i, e_j\rangle = \int_G f(x)\langle \pi(x^{-1})e_i, e_j\rangle d\lambda(x)$$

for all $i, j \leq d_\pi$.

Note: The integral above is the ji^{th} term of the matrix $\hat{f}(\pi)$; so

$$\hat{f}(\pi)_{ji} = \int_G f(x)\overline{\pi_{ji}(x)} d\lambda(x) = \langle f, \pi_{ji}\rangle.$$

However, this only holds if $f \in L^2(G)$.

Suppose we consider the right hand side of Equation 2 at an arbitrary point: we see

$$\begin{aligned}\sum_{\pi\in\hat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \langle f, \pi_{ij}\rangle \pi_{ij}(x) &= \sum_{\pi\in\hat{G}} d_\pi \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} (\hat{f}(\pi))_{ij}(\pi(x))_{ji} \\ &= \sum_{\pi\in\hat{G}} d_\pi \sum_{i=1}^{d_\pi} (\hat{f}(\pi)\pi(x))_{ii} \\ &= \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi)\pi(x)).\end{aligned}$$

So $f(\cdot) = \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi)\pi(\cdot))$ (where equality holds in $L^2(G)$- i.e., almost everywhere). This is known as the *Inverse Fourier Transform.*

Theorem 6.21 (Parseval)**.** *For all $f \in L^2(G)$, we have*

$$\|f\|_2^2 = \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi)^*\hat{f}(\pi)) = \sum_{\pi\in\hat{G}} d_\pi \|\hat{f}(\pi)\|_{\text{HS}}^2.$$

Proof. We have

$$\|f\|_2^2 = \sum_{\pi\in\hat{G}} \sum_{i,j=1}^{d_\pi} d_\pi |\langle f, \pi_{ij}\rangle|^2 = \sum_{\pi\in\hat{G}} d_\pi \sum_{i,j=1}^{d_\pi} |\hat{f}(\pi)_{ij}|^2.$$

For a square matrix A, as $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\overline{a_{ij}}$ where $\overline{a_{ij}}$ is the ji^{th} entry in A^*, we see

$$\sum_{\pi\in\hat{G}} d_\pi \sum_{i,j=1}^{d_\pi} |\hat{f}(\pi)_{ij}|^2 = \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi)\hat{f}(\pi)^*).$$

□

6.2. **Analysis on compact groups: the bigger picture.** Suppose G is compact, and recall that $\hat{G}$ is the collection of all equivalence classes of irreducible representations of G. For a function $f \in L^1(G)$, the Fourier transform of f sends

$$f \mapsto (f(\pi))_{\pi\in\hat{G}}.$$

Equip $\hat{G}$ with a measure μ, defined as $\mu(\{\pi\}) = d_\pi$. This point mass measure is known as the *Plancherel measure.* For $f \in L^2(G)$, by Parseval's Identity we know

$$f \mapsto (\hat{f}(\pi))_{\pi \in \hat{G}},$$
$$(\hat{f}(\pi))_{\pi \in \hat{G}} \in \ell^2 - \oplus M_{d_\pi}$$

where M_{d_π} is endowed with the Hilbert-Schmidt norm. The norm of an individual element in this direct sum of normed spaces is then

$$\|(\hat{f}(\pi))_{\pi \in \hat{G}}\| = \sum_{\pi \in \hat{G}} d_\pi \|\hat{f}(\pi)\|^2_{\text{HS}}.$$

So Parseval's Identity gives us an "isometry" (in a very loose sense). Note that the previous sum can be written as

$$\int_{\hat{G}} \|\hat{f}(\pi)\|^2_{\text{HS}} d\mu(\pi),$$

as G is compact. This point of view generalizes to many non-compact groups (but with much more effort).

Question: Let $f \in L^2(G)$. What is the projection of f onto $\mathcal{E}_\pi$, for $\pi \in \hat{G}$?

Recall: For a fixed representation $\pi \in \hat{G}$, $\mathcal{E}_\pi = \text{span}\{\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}$.

Let $P_\pi f$ denote the projection of f onto $\mathcal{E}_\pi$. To find $P_\pi f$, we look at

$$\sum_{i,j=1}^{d_\pi} \langle f, \sqrt{d_\pi}\pi_{ij}\rangle(\sqrt{d_\pi}\pi_{ij})$$
$$= \sum_{i,j=1}^{d_\pi} d_\pi \langle f, \pi_{ij}\rangle \pi_{ij}.$$

However, we recall that at a point x, the above is $(P_\pi f)(x) = d_\pi \operatorname{tr}(\hat{f}(\pi)\pi(x))$. Then

$$\operatorname{tr}(\hat{f}(\pi)\pi(x)) = \sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\pi(x)e_i, e_i\rangle = \sum_{i=1}^{d_\pi} \int_G f(z)\langle \pi(z)^*\pi(x)e_i, e_j\rangle d\lambda(z)$$
$$= \int_G f(z)\Big(\sum_{i=1}^{d_\pi} \langle \pi(z^{-1}x)e_i, e_i\rangle\Big) d\lambda(z) = (f * \chi_\pi)(x),$$

where $\chi_\pi(z^{-1}x) = \sum_{i=1}^{d_\pi} \langle \pi(z^{-1}x)e_i, e_i\rangle$.

Definition 6.22. *For a point x,*

$$\chi_\pi(x) = \sum_{i=1}^{d_\pi} \langle \pi(x)e_i, e_i\rangle = \operatorname{tr}(\pi(x)).$$

Recall: We showed that for any $f \in L^1(G)$, $\operatorname{tr}(\hat{f}(\pi)\pi(x)) = f * \chi_\pi(x)$ where χ_π is a character of compact G, and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a finite dimensional representation. We have the following consequences:

(i) If $f \in L^2(G)$, then

$$f = \sum_{\pi \in \hat{G}} d_\pi f * \chi_\pi$$

where equality is in the sense of $L^2(G)$.

(ii) Let $f \in L^2(G)$. Then $d_\pi f * \chi_\pi$ is the projection of f onto $\mathcal{E}_\pi$.

Facts (about characters):

(i) If $\pi \cong \sigma$ (unitarily equivalent), then $\chi_\pi = \chi_\sigma$.

Proof. Let $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be a unitary operator such that

$$U\pi(x) = \sigma(x)U$$

for all $x \in G$. Then for any $x \in G$,

$$\chi_\pi(x) = \mathrm{tr}(\pi(x)) = \mathrm{tr}(U^*U\pi(x)) = \mathrm{tr}(U\pi(x)U^*) = \mathrm{tr}(\sigma(x)) = \chi_\sigma(x).$$

□

(ii) Let $\pi, \sigma \in \hat{G}$ where $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$. We note that

$$\chi_\pi(x) = \mathrm{tr}(\pi(x)) = \sum_{i=1}^{d_\pi} \langle \pi(x)e_i, e_i\rangle$$

where $\{e_1, \ldots, e_{d_\pi}\}$ is an orthonormal basis for $\mathcal{H}_\pi$. Therefore, $\chi_\pi = \sum_{i=1}^{d_\pi} \pi_{ii}$, implying $\chi_\pi \in \mathcal{E}_\pi$. Similarly, $\chi_\sigma \in \mathcal{E}_\sigma$.

If we suppose $\sigma \ncong \pi$, then $\mathcal{E}_\pi \perp \mathcal{E}_\sigma$- hence, $\chi_\pi \perp \chi_\sigma$. This means $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthogonal subset of $L^2(G)$.

(iii) Suppose $\pi \in \hat{G}$. We see

$$\begin{aligned}\|\chi_\pi\|_2^2 &= \int_G |\chi_\pi(x)|^2 dx = \int_G \chi_\pi(x)\overline{\chi_\pi(x)} dx \\ &= \int_G \Big(\sum_{i=1}^{d_\pi} \pi_{ii}(x)\Big)\Big(\sum_{j=1}^{d_\pi} \overline{\pi_{jj}(x)}\Big) dx \\ &= \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \int_G \pi_{ii}(x)\overline{\pi_{jj}(x)} dx = \sum_{i=1}^{d_\pi} \langle \pi_{ii}, \pi_{ii}\rangle = \sum_{i=1}^{d_\pi} \frac{1}{d_\pi} = 1.\end{aligned}$$

Therefore, $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal subset of $L^2(G)$.

Proposition 6.23. *Suppose $\pi, \sigma \in \hat{G}$. Then*

$$\chi_\pi * \chi_\sigma = \begin{cases} 0, & \pi \ncong \sigma, \\ \frac{1}{d_\pi}\chi_\pi, & \pi \cong \sigma. \end{cases}$$

Remark: Note that $f \perp g$ in $L^2(G)$ does not always imply $f * g = 0$; indeed, let $f = \chi_{[0,1]}, g = \chi_{[1,2]} \in L^2(\mathbb{R})$. Then $f \perp g$, but $f * g \neq 0$.

Proof. First, suppose $\pi \ncong \sigma$; we know that $\chi_\pi * \chi_\sigma$ is the projection of χ_π onto $\mathcal{E}_\sigma$. However, $\chi_\pi \perp \mathcal{E}_\sigma$, as $\mathcal{E}_\sigma \perp \mathcal{E}_\pi$. Thus, $\chi_\pi * \chi_\sigma = 0$.

Now assume $\pi \cong \sigma$. For a point x, we see

$$\begin{aligned}\chi_\pi * \chi_\pi(x) &= \int_G \chi_\pi(y)\chi_\pi(y^{-1}x)dy \\ &= \int_G \left(\sum_{i=1}^{d_\pi} \pi_{ii}(y)\right)\left(\sum_{j=1}^{d_\pi} \langle \pi(y^{-1}x)e_j, e_j\rangle\right)dy \\ &= \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \int_G \langle \pi(y)e_i, e_i\rangle \overline{\langle \pi(y)e_j, \pi(x)e_j\rangle} dy \\ &= \sum_{i=1}^{d_\pi}\sum_{j=1}^{d_\pi} \frac{1}{d_\pi}\langle e_i, e_j\rangle\langle \pi(x)e_j, e_i\rangle = \frac{1}{d_\pi}\sum_{i=1}^{d_\pi} \pi_{ii}(x) \\ &= \frac{1}{d_\pi}\operatorname{tr}(\pi(x)) = \frac{1}{d_\pi}\chi_\pi(x).\end{aligned}$$

This implies the result above. □

Algebraic properties of characters:

(i) Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be finite dimensional. We see

$$\chi_\pi(gh) = \operatorname{tr}(\pi(gh)) = \operatorname{tr}(\pi(g)\pi(h)) = \operatorname{tr}(\pi(h)\pi(g)) = \operatorname{tr}(\pi(hg)) = \chi_\pi(hg)$$

for any $h, g \in G$. Equivalently,

$$\chi_\pi(g^{-1}hg) = \chi_\pi(h)$$

for all $h, g \in G$. This means χ_π is constant on each conjugacy class of G- i.e., χ_π is a class function.

Goal: Our goal for the remainder of this section is to show that $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal basis for the subspace of class functions in $L^2(G)$.

Definition 6.24. *A functional $f \in C(G)$ is called a class function (or a central function) if for every $x, y \in G$,*

$$f(xy) = f(yx).$$

Equivalently, f is constant on each conjugacy class of our group. If $f \in L^p(G)$, we say f is a class function (or central) if $f(xy) = f(yx)$ almost everywhere for all $x, y \in G$.

Remark: When G is compact, $1 \le p \le \infty$, then $L^p(G)$ is a Banach algebra (under pointwise addition and convolution).

We denote

$$ZL^p(G) = \{f \in L^p(G) : \ f \text{ is a class function.}\}$$

Then $ZL^p(G)$ is the center of $L^p(G)$; that is, for all $f_1, f_2 \in ZL^p(G)$ we have

$$f_1 * f_2 = f_2 * f_1.$$

Proof. We have

$$\begin{aligned}(f_1 * f_2)(x) &= \int_G f_1(y)f_2(y^{-1}x)dy = \int_G f_1(xy^{-1})f_2(y)dy \\ &= \int_G f_1(y^{-1}x)f_2(y)dy = (f_2 * f_1)(x)\end{aligned}$$

using the variable change $y \mapsto xy^{-1}$, unimodularity of G, and the fact that $f_1, f_2 \in ZL^p(G)$. □

Theorem 6.25. *Let G be compact. Then $\{\chi_\pi : \pi \in \hat{G}\}$ is an orthonormal basis for $ZL^2(G)$.*

Proof. We know that $\{\chi_\pi : \pi \in \hat{G}\} \subseteq ZL^2(G)$ is an orthonormal set. We (therefore) only need to show that for any $f \in ZL^2(G)$,

$$f = \sum_{\pi \in \hat{G}} \langle f, \chi_\pi \rangle \chi_\pi.$$

By the inverse Fourier transform, as $f \in L^2(G)$ we know

$$f(x) = \sum_{\pi \in \hat{G}} d_\pi \operatorname{tr}(\hat{f}(\pi)\pi(x))$$

where the latter equality is interpreted in the sense of $L^2(G)$. We cliam that if $f \in L^2(G)$ is a class function, then

$$\operatorname{tr}(\hat{f}(\pi)\pi(x)) = \frac{1}{d_\pi} \langle f, \chi_\pi \rangle \chi_\pi.$$

To that end, let $x \in G$ be arbitrary. We have

$$\begin{aligned} \langle \hat{f}(\pi)\pi(x)\xi, \eta \rangle &= \langle \hat{f}(\pi)(\pi(x)\xi), \eta \rangle \\ &= \int_G f(y) \langle \pi(y^{-1})(\pi(x)\xi), \eta \rangle dy \end{aligned}$$

by definition of $\hat{f}(\pi)$ and interpreting in the weak sense. Then using change of variable $y^{-1} \mapsto xy^{-1}$, we see

$$\begin{aligned} \int_G f(y) \langle \pi(y^{-1})(\pi(x)\xi), \eta \rangle dy = \int_G f(y) \langle \pi(y^{-1}x)\xi, \eta \rangle dy &= \int_G f(xy^{-1}) \langle \pi(y)\xi, \eta \rangle dy \\ = \int_G f(y^{-1}x) \langle \pi(y)\xi, \eta \rangle dy &= \int_G f(y) \langle \pi(xy^{-1})\xi, \eta \rangle dy \\ = \int_G f(y) \langle \pi(y^{-1})\xi, \pi(x)^* \eta \rangle dy &= \langle \hat{f}(\pi)\xi, \pi(x)^*\eta \rangle \\ &= \langle \pi(x)\hat{f}(\pi)\xi, \eta \rangle. \end{aligned}$$

Thus, $\hat{f}(\pi)\pi(x) = \pi(x)\hat{f}(\pi)$ for all $x \in G$. Then as $\pi \in \hat{G}$, by Schur's Lemma we know $\hat{f}(\pi) = c_\pi I_{d_\pi}$ for some constant $c_\pi \in \mathbb{C}$. So,

$$d_\pi \operatorname{tr}(\hat{f}(\pi)\pi(x)) = d_\pi c_\pi \operatorname{tr}(\pi(x)) = d_\pi c_\pi \chi_\pi(x).$$

We want to show that $d_\pi c_\pi = \langle f, \chi_\pi \rangle$. It suffices to calculate c_π; to do so, we wish to show that $\operatorname{tr}(\hat{f}(\pi)) = \langle f, \chi_\pi \rangle$. **Exercise!** □

6.3. **A few examples and wavelet theory.**

Example: We begin by considering the group SU(2), where

$$\mathrm{SU}(2) = \left\{ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C},\ TT^* = I,\ \det(T) = 1 \right\}.$$

It is easy to see that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SU}(2) \iff |a|^2 + |b|^2 = 1,\ |c|^2 + |d|^2 = 1,\ a\bar{c} + b\bar{d} = 0,\ ad - bc = 1.$$

We try and solve the equations above to determine some relation involving $a, b, c, d \in \mathbb{C}$. First, we note that $(a, b) \ni \mathbb{C}^2$ must be a unit vector. Without loss of generality, assume $a \neq 0$ (as the case when $b \neq 0$ is similar). We have

$$\overline{c} = -\frac{b\overline{d}}{a},$$

so

$$c\Big(-\frac{b\overline{d}}{a}\Big) + d\overline{d} = 1 \Rightarrow \frac{\overline{d}(ad - bc)}{a} = 1 \Rightarrow \overline{d} = a,\ \overline{c} = -b.$$

Therefore, for $T \in \mathrm{SU}(2)$ as above we have

$$T = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix},$$

with $|a|^2 + |b|^2 = 1$. Thus,

$$\mathrm{SU}(2) = \Bigg\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} :\ a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \Bigg\}.$$

As a set, we identify SU(2) with the unit sphere $S^3 \subseteq \mathbb{C}^2$ (as a has two components, b has two components, and we impose the restriction $|a|^2 + |b|^2 = 1$ we are dealing with a three-dimensional space). We use the "loose" identification

$$\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \in \mathrm{SU}(2) \leftrightsquigarrow (a, b) \in S^3.$$

We equip S^3 with the surface measure $d\sigma$, normalized so that $\sigma(S^3) = 1$.

How do we describe the representations of SU(2)? Begin by letting

$$\mathcal{P} = \{p(z, w) = \sum_{\text{finite}} c_{jk} z^j w^k :\ c_{jk} \in \mathbb{C}\}.$$

That is, $p(z, w) \in \mathcal{P}$ is a complex polynomial in variables z and w. Equip $\mathcal{P}$ with the following inner product: for $P, Q \in \mathcal{P}$ let

$$\langle P, Q \rangle_{\mathcal{P}} = \int_{S^3} P\overline{Q} d\sigma.$$

For each $m \geq 0$, define

$$\mathcal{P}_m = \Bigg\{ \sum_{j=0}^{m} c_j z^j w^{m-j} :\ c_j \in \mathbb{C} \Bigg\}.$$

By definition, $\mathcal{P}_m$ is the space of all homogeneous polynomials of degree m. These are subspaces of $\mathcal{P}$, for each $m \geq 0$.

Fact: The set

$$\{z^j w^k :\ j, k = 0, 1, \dots, \}$$

is orthogonal in $\mathcal{P}$. Indeed, we have

$$\langle z^p w^r, z^q w^s \rangle_{\mathcal{P}} = \begin{cases} 0, & p \neq q \text{ or } r \neq s, \\ \frac{p!r!}{(p+r+1)!}, & p = q \text{ and } r = s. \end{cases}$$

As a consequence, we see that our subspaces $\mathcal{P}_m$ are orthogonal in $\mathcal{P}$.

We define a representation on SU(2) as follows:

$$\pi : \mathrm{SU}(2) \to \mathcal{B}(\mathcal{P}),$$

$$\Big(\pi\Big(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \Big) Q)(z, w) = Q(\overline{a}z - \overline{b}w, \overline{b}z + \overline{a}w).$$

We check that π is indeed a representation:

$$\pi\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a}\end{bmatrix}\begin{bmatrix} a' & b' \\ -\overline{b'} & \overline{a'}\end{bmatrix}\right) = \pi\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a}\end{bmatrix}\right)\pi\left(\begin{bmatrix} a' & b' \\ -\overline{b'} & \overline{a'}\end{bmatrix}\right)$$

(as we have computed before). For every fixed $m = 0, 1, \ldots$, the subspace $\mathcal{P}_m$ is invariant for π.

As each space $\mathcal{P}_m$ is finite-dimensional, it is complete (as a Hilbert space). The restriction of π to $\mathcal{P}_m$ is then well-defined, and denoted π_m.

Remarks:

(i) π_m is a continuous unitary representation of SU(2) for each $m \in \mathbb{N}$.
(ii) π_m is irreducible for every $m \in \mathbb{N}$.
(iii) $\dim(\pi_m) = \dim(\mathcal{P}_m) = m + 1$.
(iv) We have

$$\hat{\mathrm{SU}(2)} = \{[\pi_m] : \ m = 0, 1, \ldots\}.$$

Note: It should be clear that $\pi_m \ncong \pi_{m'}$ for $m \neq m'$, as they have different dimensions.

Suppose we fix an orthonormal basis for $\mathcal{P}_m$- specifically, use the basis

$$\left\{\sqrt{\frac{(m+1)!}{i!(m-i)!}}z^i w^{m-i} : \ i = 0, \ldots, m\right\}.$$

Let e_i denote the corresponding basis vector. We consider the coefficient function of π_m associated with e_k, e_j:

$$\pi_m^{kj}\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a}\end{bmatrix}\right) = \left\langle \pi_m\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a}\end{bmatrix}\right)e_k, e_j\right\rangle.$$

We note that the above is a polynomial in $a, b\overline{a}, \overline{b}$ and homogeneous of degree $m-k$ in (a, b), degree k in $(\overline{a}, \overline{b})$. Also, π_m^{kj} is a harmonic function for each choice of k, j (in that they satisfy Laplace's equations).

By the Peter-Weyl decomposition of SU(2), we have

$$L^2(\mathrm{SU}(2)) = \bigoplus_{m=0}^{\infty} \mathcal{E}_{\pi_m}.$$

This is the same decomposition of a function on the sphere into spherical harmonics. (Possible connection to spin of a particle?)

Example: Now, we'll consider a non-compact and non-abelian locally compact group. Let

$$G = \left\{\begin{bmatrix} a & b \\ 0 & 1\end{bmatrix} : \ a \in \mathbb{R}^+, \ b \in \mathbb{R}\right\}$$

be the "connected" $ax + b$ group, and

$$\left\{\begin{bmatrix} a & b \\ 0 & 1\end{bmatrix} : \ a \in \mathbb{R} \setminus \{0\}, \ b \in \mathbb{R}\right\}$$

be the "full" $ax + b$ group.

Definition 6.26. *For a group G with $H, N \subseteq G$ we define*

$$N \rtimes H = \{(n, h) : \ n \in N, \ h \in H\}$$

where the group operation is given by

$$(n_1, h_1)(n_2, h_2) = (n_1 + h_1 n_2, h_1 h_2).$$

Using the Mackey machinery, we can describe a full list of non-equivalent irreducible representations of the connected $ax+b$ group G. We break them into the following cases:

(i) One-dimensional representations: for $\lambda \in \mathbb{R}$, let $\pi_\lambda : G \to \mathbb{T}$ be defined via

$$\pi_\lambda\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = a^{i\lambda} = e^{i\ln(a)\lambda}.$$

(ii) There are precisely two infinite-dimensional representations: first, define $\tilde{\sigma_+} : G \to \mathcal{U}(L^2(\mathbb{R}^+, dt/t))$ be defined via

$$\tilde{\sigma_+}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)f(s) = e^{2\pi i(b/s)}f(a^{-1}s).$$

This comes directly out of Mackey's "recipe" for the irreducible representations. Define the unitary map

$$U : L^2(\mathbb{R}^+, dt/t) \to L^2((0,\infty), dt),$$
$$f \mapsto (Uf)(t) = f(t^{-1})/\sqrt{t}.$$

It is easy to check that

$$\|f\|^2_{L^2(\mathbb{R}^+, dt/t)} = \int_{\mathbb{R}^+} |f(t)|^2 \frac{dt}{t} = \|Uf\|^2_{L^2((0,\infty),dt)}.$$

Let $\sigma_+ = U\tilde{\sigma_+}U^*$ where

$$\sigma_+ : G \to \mathcal{U}(L^2((0,\infty), dt))$$

is defined via

$$\sigma_+\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)f(t) = \sqrt{a}e^{2\pi ibt}f(at).$$

Then σ_+ is one of our two infinite-dimensional representations. We can also define

$$\sigma_- : G \to \mathcal{U}(L^2((-\infty,0), dt),$$
$$\sigma_-\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)f(t) = \sqrt{a}e^{2\pi ibt}f(at).$$

If we apply the Fourier transform to $\sigma_+ \oplus \sigma_-$, we arrive at the representation seen in Problem 4 of Homework 3.

Note: It is known that

$$L = \infty \cdot \sigma_+ \oplus \infty \cdot \sigma_-,$$

where L denotes the left-regular representation (as usual).

We continue with our discussion of the connected $ax+b$ group. We have previously defined the unitary transformation $U : L^2(\mathbb{R}^{>0}, dt/t) \to L^2((0,\infty), dt)$ where

$$f \mapsto Uf,$$
$$(Uf)(t) = \frac{f(t^{-1})}{\sqrt{t}}.$$

If we wanted to remain on the space $L^2(\mathbb{R}^{>0}, dt/t)$ we could instead define the unitary transformation $U : L^2(\mathbb{R}^{>0}, dt/t) \to L^2(\mathbb{R}^{>0}, dt/t)$ where

$$U\xi(t) = \xi(t^{-1}).$$

This is indeed unitary, as

$$\|U\xi\|_2^2 = \int_{\mathbb{R}^{>0}} |U\xi(t)|^2 dt/t = \int_{\mathbb{R}^{>0}} |\xi(t^{-1})|^2 dt/t = \|\xi\|_2^2.$$

Also, $U^* = U^{-1} = U$. Using this U, we can define a "new" representation $\pi^+(\cdot)$ via

$$\pi^+(\cdot) = U\tilde{\sigma_+}(\cdot)U^*,$$

where

$$\pi^+\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)\xi(t) = U\tilde{\sigma_+}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)U^*\xi(t)$$
$$= \tilde{\sigma_+}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)(U\xi)(t^{-1}) = e^{2\pi ibt}U\xi(a^{-1}t^{-1})$$
$$= e^{2\pi ibs}\xi(at).$$

From here on out, we pick π^+ as one of our two infinite-dimensional representations of the connected group; as shown above, we define $\pi^+ : G \to \mathcal{U}(L^2(\mathbb{R}^{>0}, dt/t))$ via

$$\pi^+\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)\xi(t) = e^{2\pi ibt}\xi(at).$$

In an almost identical manner, we can transform the other infinite-dimensional irreducible representation to get $\pi^- : G \to \mathcal{U}(L^2(\mathbb{R}^{>0}, dt/t))$ where

$$\pi^-\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)\xi(t) = e^{-2\pi ibt}\xi(at).$$

Fact: π^+, π^- are irreducible subrepresentations of the left regular representation L.

We'll focus our discussion on π^+ for the moment. Let $\xi, \eta \in L^2(\mathbb{R}^{>0})$ be arbitrary but fixed. The coefficient function associated with ξ and η is a function $\pi^+_{\xi,\eta} : G \to \mathbb{C}$ defined via

$$\pi^+_{\xi,\eta}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = \left\langle \pi^+\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)\xi, \eta \right\rangle_{\mathcal{H}}.$$

We wish to compute explicitly what the inner product above should be- however, before we do so we'll need to introduce a few definitions.

Definition 6.27.

(i) For $a > 0$, we let $\xi_a : \mathbb{R}^{>0} \to \mathbb{C}$ be defined via

$$\xi_a(t) = \xi(at).$$

(ii) We let $k : \mathrm{Dom}(k) \to L^2(\mathbb{R}^{>0})$ be defined via

$$\xi \mapsto (k\xi),$$
$$(k\xi)(t) = t\xi(t).$$

Note: We may virtually consider $k : L^2(\mathbb{R}^{>0}) \to L^2(\mathbb{R}^{>0})$ instead of just on its domain, as the domain of k is dense in $L^2(\mathbb{R}^{>0})$- indeed, it contains $C_c(\mathbb{R})$. Note also that k is unbounded, but a linear operator. It is known as the "formal degree".

Recall: We may think of functions on $\mathbb{R}^{>0}$ as functions on $\mathbb{R}$ for any f defined on $\mathbb{R}^{>0}$ by letting $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{R}^{>0}$.

With the above definitions introduced, we can rewrite the inner product above as

$$\pi^+_{\xi,\eta}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = \int_{\mathbb{R}^{>0}} e^{2\pi ibt}\xi(at)\overline{\eta(t)}dt/t = \int_{\mathbb{R}} k^{-1}(\xi_a\overline{\eta})(t)e^{2\pi ibt}dt = \mathcal{F}(k^{-1}(\xi_a\overline{\eta}))(-b).$$

Here $\mathcal{F}$ denotes the classical Fourier transform.

Remark: Fix $\xi \in \mathrm{Dom}(k^{-1/2})$ (for example, any compactly supported continuous function). Then define the operator

$$V_\xi : L^2(\mathbb{R}^{>0}) \to L^2(G),$$
$$\eta \mapsto \langle \eta, \pi^+(\cdot)\xi\rangle.$$

Note that $\langle \eta, \pi^+(\cdot)\xi\rangle$ is linear. We can also choose ξ so that $\|k^{-1/2}\xi\|_2 = 1$. Under these conditions, V_ξ is an isometry with

$$\langle \eta_1, \eta_2\rangle_{\mathcal{H}} = \langle V_\xi(\eta_1), V_\xi(\eta_2)\rangle = \langle \overline{\pi^+_{\xi,\eta_1}}, \overline{\pi^+_{\xi,\eta_2}}\rangle.$$

Theorem 6.28 (Orthogonality relations)**.** *Let $\xi_1, \xi_2, \eta_1, \eta_2 \in C_c(\mathbb{R}^{>0})$ be arbitrary. Then*

(i) $\langle \pi^+_{\xi_1,\eta_1}, \pi^+_{\xi_2,\eta_2}\rangle_{L^2(G)} = \langle k^{-1/2}\xi_1, k^{-1/2}\xi_2\rangle\langle \eta_1, \eta_2\rangle$.
(ii) $\langle \pi^-_{\xi_1,\eta_1}, \pi^-_{\xi_2,\eta_2}\rangle_{L^2(G)} = \langle k^{-1/2}\xi_1, k^{-1/2}\xi_2\rangle\langle \eta_1, \eta_2\rangle$.
(iii) $\langle \pi^+_{\xi_1,\eta_1}, \pi^-_{\xi_2,\eta_2}\rangle = 0$.

Proof. We begin by showing (i). The left hand side of (i) can be written as

$$\int\limits_{\mathbb{R}^{>0}} \int\limits_{\mathbb{R}} \mathcal{F}(k^{-1}(\xi_{a1}\overline{\eta_1}))(-b)\overline{\mathcal{F}(k^{-1}(\xi_{a2}\overline{\eta_2}))(-b)}\frac{dbda}{a^2}$$
$$= \int\limits_{\mathbb{R}^{>0}} \langle \mathcal{F}(k^{-1}(\xi_{a1}, \overline{\eta_1}))(\cdot), \mathcal{F}(k^{-1}(\xi_{a2}, \overline{\eta_2}))(\cdot)\rangle_{L^2(\mathbb{R})}\frac{da}{a^2}$$
$$= \int\limits_{\mathbb{R}^{>0}} \int\limits_{\mathbb{R}} \frac{\xi_1(at)\overline{\eta_1(t)}}{t}\overline{\frac{\xi_2(at)\overline{\eta_2(t)}}{t}}dt\frac{da}{a^2}$$
$$= \int\limits_{\mathbb{R}^{>0}} \left[\int\limits_{\mathbb{R}^{>0}} \frac{\xi_1(at)}{\sqrt{at}}\frac{\overline{\xi_2(at)}}{\sqrt{at}}\frac{da}{a}\right]\eta_2(t)\overline{\eta_1(t)}dt$$
$$= \langle k^{-1/2}\xi_1, k^{-1/2}\xi_2\rangle\langle \eta_1, \eta_2\rangle.$$

This shows (i). The proof of (ii) is essentially identical, so we omit it here.

To show (iii), recall the map V_ξ^+ for a fixed $\xi \in \mathrm{Dom}(k^{-1/2})$ (here the "+" on V_ξ indicates that we are defining the mapping using irreducible representation π^+- there is an equivalent definition for π^-). We claim that the map V_ξ^+ intertwines π^+ and L (the left-regular representation). To show how, **exercise!**. In a similar way, V_ξ^- intertwines π^- and L. Additionally, $(V_\xi^-)^*V_\xi^+$ intertwines both π^+ and π^-, which is not hard to show. However, as π^+, π^- are irreducible representations that are non-equivalent, this forces $(V_\xi^-)^*V_\xi^+ = 0$. From here, we can easily derive statement (iii) by writing everything out directly by definition. □

Summary:

Fix $\xi \in C_c(\mathbb{R}^{>0}) \subseteq \mathrm{Dom}(k^{-1/2})$. Then

$$V_\xi^+ : \mathcal{H} \to L^2(G),$$
$$\eta \mapsto \langle \eta, \pi^+(\cdot)\xi\rangle$$

is an isometry, and

$$\langle \eta_1, \eta_2\rangle_{\mathcal{H}} = \langle V_\xi^+(\eta_1), V_\xi^+(\eta_2)\rangle_{L^2(G)} = \int \left\langle \eta_1, \pi\left(\begin{bmatrix} a & b \\ 0 & 1\end{bmatrix}\right)\xi\right\rangle\left\langle \pi\left(\begin{bmatrix} a & b \\ 0 & 1\end{bmatrix}\right)\xi, \eta_2\right\rangle\frac{dadb}{a^2}.$$

The above holds for all $\eta_1, \eta_2 \in L^2(\mathbb{R}^{>0})$. "Equivalently",

$$\eta_1 = \int\limits_G \left\langle \eta_1, \pi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right)\xi \right\rangle \pi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right)\xi \frac{dadb}{a^2}$$

in the weak-sense.

Recall: $\pi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right)\xi(t) = e^{2\pi itb}\xi(at) = \xi_{a,b}$.

By the above, this means

$$\eta = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{>0}} \langle \eta, \xi_{a,b} \rangle \xi_{a,b} \frac{dadb}{a^2}.$$

This is called the continuous classical wavelet transform.

We now attempt to generalize the classical wavelet transform to the continuous wavelet transform.

Setting: We consider semi-direct product groups. In what follows, let H, N be two locally compact groups.

Definition 6.29. *A map $\varphi : N \to N$ is an automorphism if it is a group isomorphism and a homeomorphism of topological spaces. We let $\operatorname{Aut}(N)$ denote the group of automorphisms of N, where the group product is composition.*

Suppose there exists $\alpha : H \to \operatorname{Aut}(N)$ which is a group homomorphism such that the map $N \times H \to N$ (where $N \times H$ is equipped with the product topology) defined by

$$(n, h) \mapsto \alpha_h(n)$$

is continuous.

Note: We say H acts on N by α. We sometimes write $\alpha_h(n) := h \cdot n$.

Definition 6.30. *In the above setting, define the semi-direct product of N and H- denoted $N \rtimes H$- to be the set*

$$N \rtimes H = \{(n, h) : \ n \in N, \ h \in H\}$$

equipped with the product topology where the group product is given by

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \alpha_{h_1}(n_2), h_1 \cdot h_2).$$

Facts:

(i) $N \rtimes H$ is a locally compact group.
(ii) (e_N, e_H) is the identity of $N \rtimes H$.
(iii) $(n, h)^{-1} = (\alpha_h(n^{-1}), h^{-1})$.

Notation: If N is abelian, we use "+" to denote the product in N. That is,

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 + h_1 \cdot n_2, h_1 h_2).$$

Examples:

(i) Recall the connected $ax + b$ group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a > 0, \ b \in \mathbb{R} \right\}.$$

It is clear by matrix multiplication mechanisms that if we pick $H = \mathbb{R}^{>0}$ and $N = \mathbb{R}$, then

$$G \cong N \rtimes H = \mathbb{R} \rtimes \mathbb{R}^{>0}.$$

Here $h : H \to \mathrm{Aut}(N)$ where $h \mapsto$ multiplication by h. So the action of H on N is just multiplication.

(ii) Let $N = \mathbb{R}^N$, and $H \leq \mathrm{GL}_n(\mathbb{R})$ (here H is a closed subgroup- hence it is automatically locally compact), and let the action of $h \in H$ on $X \in N$ be matrix multiplication hX. Our set is

$$\{(X, h) : \ x \in \mathbb{R}^n, \ h \in H\}.$$

This class of examples contains a wide variety of groups, including the $ax+b$ group, shearlet groups, and Euclidean motion groups (i.e. SO(3), actions by rotations, etc.).

(iii) Let $N = \mathbb{R}^2 = \left\{ \begin{bmatrix} y \\ z \end{bmatrix} : \ y, z \in \mathbb{R} \right\}$ and $H = \mathbb{R}$. We let $x \in H$ act on $\begin{bmatrix} y \\ z \end{bmatrix} \in N$ by

$$x \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z + xy \end{bmatrix}.$$

Exercise: Check that this is an action! With this choice of N and H, we have

$$N \rtimes H = \left\{ \left(\begin{bmatrix} y \\ z \end{bmatrix}, x \right) : \ y, z, x \in \mathbb{R} \right\}.$$

Group multiplication is as follows:

$$\left(\begin{bmatrix} y_1 \\ z_1 \end{bmatrix}, x_1 \right) \left(\begin{bmatrix} y_2 \\ z_2 \end{bmatrix}, x_2 \right) = \left(\begin{bmatrix} y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \end{bmatrix}, x_1 + x_2 \right).$$

Note that $N \rtimes H \cong \mathbb{H}$, where $\mathbb{H}$ is the Heisenberg group

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \ x, y, z \in \mathbb{R} \right\}.$$

Left Haar measure for semi-direct products

Proposition 6.31. *There exists a continuous homomorphism $\delta : H \to \mathbb{R}^{>0}$ such that for all $f \in C_c(N)$,*

$$\int_N f(n) dn = \delta(h) \int_N f(\alpha_h(n)) dn$$

where dn is the left Haar measure on N.

Proof. Fix $h \in H$. Let λ_N denote the left Haar measure on N. Define a new measure on N:

$$\lambda_N^h(E) = \lambda_N(\alpha_h(E)).$$

Then λ_N^h is left-translation invariant. Indeed, we see

$$\lambda_N^h(nE) = \lambda_N(\alpha_h(nE)) = \lambda_N(\alpha_n(n) \cdot \alpha_h(E)) = \lambda_N(\alpha_h(E)) = \lambda_N^h(E).$$

By uniqueness of the left Haar measure, there exists a positive constant (depending on h)- say $\delta(h)$- so that

$$\lambda_N(E) = \delta(h)^{-1} \lambda_N(\alpha_h(E)).$$

Then

$$\int_N f(x) dx = \int_N f(\alpha_h(x)) d(\alpha_h(x)) = \delta(h) \int_N f(\alpha_h(x)) dx.$$

To finish the proof, we need to show that δ is a homomorphism. **Exercise!** $\square$

Examples:

(i) Consider the connected $ax+b$ group- i.e., $G = \mathbb{R} \rtimes \mathbb{R}^{>0}$. We want $\delta : \mathbb{R}^{>0} \to (0,\infty)$ so that

$$\int_{\mathbb{R}} f(x)dx = \delta(a) \int_{\mathbb{R}} f(ax)dx.$$

Clearly, $\delta(a) = a$.

(ii) Let $G = \mathbb{H} = \mathbb{R}^2 \rtimes \mathbb{R}$. We want $\delta : \mathbb{R}^{>0} \to (0,\infty)$ so that

$$\int_{\mathbb{R}^2} f\left(\begin{bmatrix} y \\ z \end{bmatrix}\right) dydz = \delta(x) \int_{\mathbb{R}^2} f\left(\begin{bmatrix} y \\ xy+z \end{bmatrix}\right) dydz$$

$$= \delta(x) \int_{\mathbb{R}^2} f\left(\begin{bmatrix} y \\ xy+z \end{bmatrix}\right) dzdy = \delta(x) \int_{\mathbb{R}^2} f\left(\begin{bmatrix} y \\ z \end{bmatrix}\right) dzdy.$$

This shows $\delta(x) = 1$.

(iii) Let $G = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$. For $\delta : \mathrm{GL}_n(\mathbb{R}) \to (0,\infty)$ to satisfy the properties above, we let $\delta(A) = |\det(A)|$.

Theorem 6.32. *For the semi-direct product $N \rtimes H$ and $\delta : H \to (0,\infty)$ the homomorphism above (i.e., the scaling of the action of H on N), we have*

$$d\lambda_G(n,h) = \frac{dndh}{\delta(h)},$$

where dn is the left Haar measure on N and dh is the left Haar measure on H.

Examples:

(i) If $G = \mathbb{R} \rtimes \mathbb{R}^{>0}$, then

$$d\lambda_G(b,a) = \frac{db(da/a)}{a} = \frac{dbda}{a^2}.$$

(ii) If $G = \mathbb{H}$, then

$$d\lambda_G((y,z),x) = dydzdx.$$

(iii) If $G = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, then

$$d\lambda_G\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, A\right) = \frac{dx_1 \cdots dx_n \prod_{i,j=1}^n da_{ij}}{|\det(A)|^{n+1}}.$$

(iv) If G is the group of two dimensional Euclidean motion, $G = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. Here $\delta : \mathrm{SO}(2) \to (0,\infty)$ is $\delta = 1$. Thus,

$$d\lambda_G\left(\begin{bmatrix} y \\ z \end{bmatrix}, A\right) = dxdydA.$$

Remark: We can define the modular function for semi-direct products as well: if $G = N \rtimes H$, then

$$\Delta_G(n,h) = \frac{\Delta_N(n)\Delta_H(h)}{\delta(h)}.$$

To prove this (we do not do so here), fix $(m,k) \in N \rtimes H$ and $f \in C_c(G)$, and consider

$$\int_N \int_H f((n,h)(m,k)) \frac{dndh}{\delta(h)}.$$

Quasi-regular representation
Fix $H \subseteq \mathrm{GL}_d(\mathbb{R})$ as a closed subgroup, and let $N = \mathbb{R}^d$. We let $G = N \rtimes H = \mathbb{R}^d \rtimes H$. The specific representation we wish to consider is $\pi : G \to \mathcal{U}(L^2(\mathbb{R}^d))$ defined by

$$(\pi(x,h)f)(y) = \frac{1}{\sqrt{|\det h|}} f(h^{-1} \cdot (y-x)).$$

We can check this is indeed a continuous unitary representation- **exercise!**

What are closed invariant subspaces of π?

Example: If $G = \mathbb{R} \rtimes \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we'll use

$$(\pi(b,a)f)(y) = \frac{1}{\sqrt{|a|}} f(a^{-1}(y-b)).$$

We saw this representation in Assignment 3, and showed it was irreducible.

Considering the most general case, let $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\hat{\mathbb{R}^d})$ be the Fourier transform.

Recall: We have

$$\hat{\mathbb{R}^d} = \{\chi_\omega : \mathbb{R}^d \to \mathbb{T},\ \omega \in \mathbb{R}^d\} \cong \mathbb{R}^d.$$

That is, we can index all characters in $\hat{\mathbb{R}^d}$ by elements of $\mathbb{R}^d$. We can explicitly write

$$\chi_\omega(\mathbf{x}) = e^{2\pi i \langle \omega, \mathbf{x} \rangle}.$$

We can find an equivalent form for the quasi-regular representation π, which we denote as σ. We have $\sigma : \mathbb{R}^d \rtimes H \to \mathcal{U}(L^2(\hat{\mathbb{R}^d})) \cong \mathcal{U}(L^2(\mathbb{R}^d))$, where

$$\sigma(\cdot) = \mathcal{F}\pi(\cdot)\mathcal{F}^{-1}.$$

Specifically, we will have

$$(\sigma(x,h)\hat{f})(\omega) = \mathcal{F}(\pi(x,h)f)(\omega)$$

$$= \int_{\mathbb{R}^d} \frac{1}{\sqrt{|\det h|}} f(h^{-1} \cdot (y-x)) \overline{\chi_\omega}(y) dy = \frac{|\det h|}{\chi_\omega(x)\sqrt{|\det h|}} \int_{\mathbb{R}^d} f(y) e^{-2\pi i \langle \omega, h \cdot y \rangle} dy$$

$$= \frac{\sqrt{|\det h|}}{\chi_\omega(x)} \int_{\mathbb{R}^d} f(y) e^{-2\pi i \langle \omega, h \cdot y \rangle} dy.$$

The equivalent form, therefore, is for $\sigma : G \to \mathcal{U}(L^2(\hat{\mathbb{R}^d}))$ where

$$(\sigma(x,h)\hat{f})(\omega) = \sqrt{|\det h|} e^{-2\pi i \langle x, \omega \rangle} \hat{f}(h^t \omega).$$

Fix $U \subseteq \mathbb{R}^d$ such that U is H-invariant: for every $\omega \in U$, $(h^t)^{-1}\omega \in U$. Define $\mathcal{H}_U = \{\phi \in L^2(\hat{\mathbb{R}^d}) : \operatorname{supp}(\phi) \subseteq U\}$.

Fact: All σ-invariant subspaces of $L^2(\hat{\mathbb{R}^d})$ are of this form.

Example: Let $G = \mathbb{R} \rtimes \mathbb{R}^*$. Our goal is to find all $U \subseteq \mathbb{R}$ such that for all $\omega \in U$, $a^{-1} \cdot \omega \in U$ for $a \in \mathbb{R}^*$. It should be clear that we have only the following:

(i) $U = \{0\}$;
(ii) $U = \mathbb{R} \setminus \{0\}$;
(iii) $U = \mathbb{R}$.

Then we have no non-trivial closed σ-invariant subspaces.

Example: If $G = \mathbb{R} \rtimes \mathbb{R}^+$, then σ has 2 closed invariant subspaces: $\mathcal{H}_{(0,\infty)}$ and $\mathcal{H}_{(-\infty,0)}$.

Theorem 6.33. *Let $G = \mathbb{R}^d \rtimes H$, with π and σ as above. Suppose there exists a unique open orbit of the above action of H on $\mathbb{R}^d$- say O- such that*

(i) O has full measure.

(ii) O is free (meaning for all $\xi \in O$, if $h\xi = \xi$ then $h = e$).

Then π is irreducible. Also, π is square integrable: that is, there exists $\xi, \eta \neq 0$ such that $\pi_{\xi,\eta} \in L^2(G)$.

Fix one such $\xi \in L^2(\mathbb{R}^d)$. Define

$$V_\xi : L^2(\mathbb{R}^d) \to L^2(G),$$
$$\eta \mapsto \langle \eta, \pi(\cdot)\xi \rangle.$$

Note: V_ξ is an isometry, with $L_x V_\xi = V_\xi \pi(x)$ for all $x \in G$. Here L_x is the left-regular representation.

If we pick $\eta_1, \eta_2 \in L^2(\mathbb{R}^d)$, as our action in the above case is free this implies that

$$\eta_1 = \int_G \langle \eta_1, \pi(x)\xi \rangle \pi(x)\xi dx$$

giving us a sort of reconstruction formula.