NOTES ON ORBIT EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS AND CK ALGEBRAS

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ABSTRACT. These notes were used as the basis for a series of lectures I gave as part of the Fall 2022 "Noncommutative Analysis Seminar" at University of Delaware. The main focus is on explaining semi-recent results of Matsumoto on connections between what we call a topological one-sided shift space, and certain Cuntz-Krieger algebras. The majority of the material was taken from the two Matsumoto papers referenced in the last section; I do not claim any of the proof ideas as new or novel. Indeed, the purpose of these notes is only meant to be expository, and as an educational resource; it should not be referenced.

CONTENTS

1. NOTATION AND PRELIMINARIES

We will use this section to lay out the basics in notation, and discuss some preliminary results that will be necessary for what follows.

We write $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. Characteristic functions of a set S are denoted as χ_S.

Definition 1.1. *A topological space is said to be totally disconnected if there are no non-trivial connected subsets.*

Remark. In what follows, it is often enough to assume the topological spaces we deal with are generated by a basis of clopen sets in order to guarantee total disconnectivity.

Definition 1.2. *A Cantor set is any compact, metrizable, totally disconnected space with no isolated points.*

Remark: Any two spaces which are compact Hausdorff, metrizable, totally disconnected and have no isolated points are homeomorphic via Brower's Theorem; hence, any two Cantor spaces are homeomorphic.

Let $A = (A(i,j))_{i,j}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$, where $1 < N \in \mathbb{N}$. In what follows, we will assume A has no rows or columns identical to zero. Define

$$X_A := \{(x_n)_{n\in\mathbb{N}} | \; x_n \in [N], \; A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}.$$

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We note that X_A is a compact Hausdorff space inside the compact product space $[N]^{\mathbb{N}}$ (via Tychonoff's Theorem). The shift transformation σ_A on X_A is defined via

$$\sigma_A((x_n)_n) = (x_{n+1})_{n\in\mathbb{N}}.$$

Definition 1.3. *The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A.*

Remark: We assume A satisfies what is known as Condition (I) as defined in [1]. This is equivalent to requiring that X_A is a Cantor set.

A word $\mu = \mu_1 \cdots \mu_k$ with $\mu_i \in [N]$ is said to be admissible for X_A if μ appears at any point in an element $x \in X_A$.

Definition 1.4. *If (X_A, σ_A) is the one-sided topological Markov shift for A, the Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by N partial isometries $S_1, \dots, S_N$ subject to the relations*

$$\sum_{j=1}^{N} S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*.$$

The subalgebra $\mathcal{D}_A$ of $\mathcal{O}_A$ is generated by elements of the form $S_\mu S_\mu^*$, where μ is an admissible word of X_A. Here, S_μ is the notation used for the product of partial isometries

$$S_\mu = S_{i_1} \cdots S_{i_k},$$

where $\mu = (i_1, \dots, i_k)$.

Remarks:

(i) When we say $\mathcal{O}_A$ is a "universal" C^*-algebra, (without going into too much detail) this is an algebra which can be described in terms of generators and relations- however, these generators and relations must be chosen in such a way so that we can realize them as subalgebras of $\mathcal{B}(\mathcal{H})$ for some legitimate choice of a Hilbert space $\mathcal{H}$ (so there are norm conditions that we must take into consideration when defining a universal C^*-algebra, as compared to something like a free algebra). We don't have to worry too much though, as (we can take for granted) that $\mathcal{O}_A$ can be concretely represented on any separable Hilbert space $\mathcal{H}$.

(ii) For more explicit detail on a concrete representation of $\mathcal{O}_A$, let $\mathcal{H}_A$ be a Hilbert space with complete orthonormal system $\{e_x\}$, for $x \in X_A$. Consider the partial isometries T_i for $1 \le i \le N$ defined by

$$T_i e_x = \begin{cases} e_{ix}, & \text{if } ix \in X_A, \\ 0, & \text{otherwise.} \end{cases}$$

These are indeed partial isometries, as $T_i T_i^* T_i = T_i$ (which can easily be seen by determining T_i^* via its action on the orthonormal system given). It is easy to check (**exercise!**) that the partial isometries T_i satisfy the relations

$$\sum_{j=1}^{N} T_j T_j^* = 1, \quad T_i^* T_i = \sum_{j=1}^{N} A(i,j) T_j T_j^* \text{ for } i = 1, \dots, N.$$

As A satisfies (I), X_A has no isolated points- therefore, there are no points $x \in X_A$ such that we can find a neighborhood $U \subseteq X_A$ surrounding x which contains no other points of X_A. Thus, $T_i \neq 0$ for each $i \in [N]$. This yields a faithful representation of $\mathcal{O}_A$ on $\mathcal{H}_A$ via

$$S_i \mapsto T_i.$$

(iii) The subalgebra $\mathcal{D}_A$ is isomorphic to $C(X_A)$ via the following correspondence:

$$S_\mu S_\mu^* \in \mathcal{D}_A \iff \chi_\mu \in C(X_A),$$

where χ_μ is the characteristic function for the set

$$U_\mu := \{(x_n)_{n\in\mathbb{N}} \in X_A |\, x_1 = \mu_1, \ldots, x_k = \mu_k\}.$$

Indeed, $C(X_A)$ is generated by shifts of characteristic functions on cylinder sets in X_A; that is, sets of the form

$$Z(i) = \{x \in X_A |\, x_1 = i\}$$

(see the comments before [1, Proposition 2.5]). That the mapping above is a well-defined homomorphism is simple to check, using the product formula(s) written in [1, Lemma 2.1].

We will take the following proposition for granted, without proving it directly. For a complete proof, see [1, Remark 2.18].

Proposition 1.5. *The algebra $\mathcal{D}_A$ is maximal abelian in $\mathcal{O}_A$.*

We will close this section by including a few words on short, exact, and split sequences (specifically for abelian groups).

Definition 1.6. *If G_n are abelian groups, and $\varphi_n : G_n \to G_{n+1}$ are group homomorphisms, then the sequence*

$$\cdots \xrightarrow{\varphi_{n-2}} G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} G_{n+1} \xrightarrow{\varphi_{n+1}} \cdots$$

is called exact if $\mathrm{rng}(\varphi_n) = \ker(\varphi_{n+1})$ *for all $n \in \mathbb{N}$.*

Definition 1.7. *An exact sequence of abelian groups (modules over $\mathbb{Z}$) of the form*

$$0 \to A \to B \to C \to 0$$

is called short exact.

Example: If N is a normal subgroup of abelian group G, then

$$0 \to N \xrightarrow{\iota} G \xrightarrow{q} G/N \to 0$$

is a short exact sequence, where ι is the inclusion mapping and q is the canonical quotient.

Definition 1.8. *Suppose we have a short exact sequence*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0.$$

If there exists a homomorphism $\lambda : C \to B$ so that $\psi \circ \lambda = \mathrm{id}_C$, then λ is called a lift of φ and the short exact sequence is called split exact.

Remark: There is an equivalent characterization of split exact sequences: we say

$$0 \to A \to B \to C \to 0$$

is split exact if there exists an isomorphism $f : B \to A \oplus C$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \xhookrightarrow[\iota]{} & A \oplus C & \xrightarrow[q]{} & C & \longrightarrow & 0.
\end{array}$$

Note that this *only holds* in the case that A, B, C are abelian groups.

2. Continuous orbit equivalence

Definition 2.1. *For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the orbit $\mathrm{orb}_{\sigma_A}(x)$ of x under σ_A is defined by*

$$\mathrm{orb}_{\sigma_A}(x) = \bigcup_{k=0}^{\infty}\bigcup_{\ell=0}^{\infty} \sigma_A^{-k}(\sigma_A^{\ell}(x)) \subseteq X_A.$$

It is easy to see that $y \in \mathrm{orb}_{\sigma_A}(x)$ if and only if there exists an admissible word $\mu = \mu_1 \cdots \mu_k$ for X_A such that

$$y = (\mu_1, \ldots, \mu_k, x_{\ell+1}, x_{\ell+2}, \ldots)$$

for some $k, \ell \in \mathbb{N}_+$. For what follows, let $\mathrm{Homeo}(X_A)$ be the group of all homeomorphisms on X_A.

Definition 2.2. *Let $[\sigma_A]$ be the set of all homeomorphisms $\tau \in \mathrm{Homeo}(X_A)$ such that $\tau(x) \in \mathrm{orb}_{\sigma_A}(x)$ for all $x \in X_A$, and let Γ_A be the set of $\tau \in [\sigma_A]$ such that there exist continuous functions $k, \ell : X_A \to \mathbb{N}_+$ satisfying*

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{\ell(x)}(x)$$

for all $x \in X_A$. We call $[\sigma_A]$ the full group of (X_A, σ_A) and Γ_A the topological full group of (X_A, σ_A).

Remark: Both $[\sigma_A]$ and Γ_A are groups; to show this, **exercise!**

Example: (Section 3, [2]) Let $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Define $\tau \in \mathrm{Homeo}(X_F)$ by setting

$$\tau(x_1, x_2, \ldots) = \begin{cases} (2, 1, 1, x_4, x_5, \ldots), & \text{if } (x_1, x_2, x_3) = (1, 1, 1), \\ (1, 1, 1, x_4, x_5, \ldots), & \text{if } (x_1, x_2, x_3) = (2, 1, 1), \\ (x_1, x_2, \ldots), & \text{otherwise.} \end{cases}$$

As $\sigma_F(\tau(x)) = \sigma_F(x)$ for all $x \in X_F$ (as is easy to see), letting $k(x) = \ell(x) = 1$ for all $x \in X_F$ we have $\tau \in \Gamma_F$.

Definition 2.3. *Let (X_A, σ_A) and (X_B, σ_B) be two one-sided topological Markov shifts. If there exists a homeomorphism $h : X_A \to X_B$ such that*

$$h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x)), \quad x \in X_A$$

then (X_A, σ_A) and (X_B, σ_B) are said to be topologically orbit equivalent. In this case, we can find $k_1, \ell_1 : X_A \to \mathbb{N}_+$ and $k_2, \ell_2 : X_B \to \mathbb{N}_+$ so that

$$\begin{aligned} \sigma_B^{k_1(x)}(h(\sigma_A(x))) &= \sigma_B^{\ell_1(x)}(h(x)), \quad x \in X_A, \\ \sigma_A^{k_2(x)}(h^{-1}(\sigma_B(x))) &= \sigma_A^{\ell_2(x)}(h^{-1}(x)), \quad x \in X_B. \end{aligned} \tag{1}$$

If we may choose k_1, ℓ_1, k_2, ℓ_2 to be continuous, then (X_A, σ_A) and (X_B, σ_B) are said to be continuously orbit equivalent.

Example: If two one-sided topological Markov shifts are topologically conjugate (so there exists a homeomorphism $h : X_A \to X_B$ so that $\sigma_A = h^{-1} \circ \sigma_B \circ h$), then they are also continuously orbit equivalent.

3. Étale groupoids

We briefly review some terminology involving groupoids provided during previous lectures.

Recall:

(i) For $x \in G^{(0)}$, $r(Gx)$ is called the G-orbit of x. If each G-orbit is dense in $G^{(0)}$ (so the smallest closed set in $G^{(0)}$ containing $r(Gx)$ is $G^{(0)}$ itself), we say G is *minimal*.

(ii) We say G is *principal* if $\mathrm{Iso}(G) = G^{(0)}$ (where $\mathrm{Iso}(G) = \{\gamma \in G : \ r(\gamma) = s(\gamma)\}$).

(iii) When $\mathrm{Int}(\mathrm{Iso}(G)) = G^{(0)}$, we say G is *essentially principal.*
(iv) A subset $U \subset G$ is called a G-set if $r|_U, s|_U$ are injective.
(v) For an open G-set U, we let π_U be the homeomorphism $r \circ (s|_U)^{-1}$ from $s(U)$ to $r(U)$.

Definition 3.1. *Let G be an essentially principal étale groupoid whose unit space $G^{(0)}$ is compact.*

(i) The set of all $\alpha \in \mathrm{Homeo}(G^{(0)})$ such that for each $x \in G^{(0)}$ there exists a $g \in G$ so that $r(g) = x$ and $s(g) = \alpha(x)$ is called the full group of G, and is denoted by $[G]$.
(ii) The set of all $\alpha \in \mathrm{Homeo}(G^{(0)})$ for which there exists a compact open G-set U satisfying $\alpha = \pi_U$ is called the topological full group of G, and is denoted by $[[G]]$.

Remarks:

(i) $[G]$ is a subgroup of $\mathrm{Homeo}(G^{(0)})$, and $[[G]]$ is a subgroup of $[G]$. For the proof of these facts: **exercise!**
(ii) If $\alpha \in [[G]]$, then the compact open G-set U is unique as G is essentially principal.
(iii) As G is second-countable (i.e., it has a topological basis consisting of countably many sets), it has countably many compact open subsets $\Rightarrow$ the group $[[G]]$ is at most countable.

Definition 3.2. *The étale groupoid G_A for a topological Markov shift (X_A, σ_A) is given by*

$$\{(x, n, y) \in X_A \times \mathbb{Z} \times X_A |\ \text{there exists } k, \ell \in \mathbb{N}_+, n = k - \ell,\ \sigma_A^k(x) = \sigma_A^\ell(y)\}.$$

The topology of G_A is generated by the sets

$$\{(x, k - \ell, y) \in G_A |\ x \in V,\ y \in W,\ \sigma_A^k(x) = \sigma_A^\ell(y)\}$$

where $V, W \subseteq X_A$ are open and $k, \ell \in \mathbb{N}_+$. Two elements (x, n, y) and (x', n', y') are composable if and only if $y = x'$; in this case, the multiplication and inverse are

$$(x, n, y) \cdot (y, n', y') = (x, n + n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

The range and source maps are given by

$$r(x, n, y) = (x, 0, x), \quad s(x, n, y) = (y, 0, y).$$

Finally, we can identify X_A with the unit space $G_A^{(0)}$ via the mapping

$$x \mapsto (x, 0, x).$$

Remark: The topological full group $[[G_A]]$ is isomorphic to the continuous full group Γ_A; **exercise!**

4. Proving the main theorem

Our main focus for the rest of the lectures is proving the following.

Theorem 4.1. *Let $(X_A, \sigma_A), (X_B, \sigma_B)$ be two irreducible one-sided topological Markov shifts. The following are equivalent:*

(i) (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.
(ii) The étale groupoids G_A and G_B are isomorphic.
(iii) There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ so that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.

To prove Theorem 2.3 in [3], we need to spend some time studying Γ_A and its connection to what is called the normalizer $N(\mathcal{O}_A, \mathcal{D}_A)$.

Definition 4.2. *Let $\mathcal{U}(\mathcal{O}_A), \mathcal{U}(\mathcal{D}_A)$ denote the groups of unitaries of $\mathcal{O}_A$ and $\mathcal{D}_A$ (respectively). The normalizer of $\mathcal{D}_A$ in $\mathcal{O}_A$ is defined by*

$$N(\mathcal{O}_A, \mathcal{D}_A) = \{v \in \mathcal{U}(\mathcal{O}_A) |\ v\mathcal{D}_A v^* = \mathcal{D}_A\}.$$

Notation: For $v \in \mathcal{U}(\mathcal{O}_A)$, put $\mathrm{Ad}(v)(a) = vav^*$ for $a \in \mathcal{O}_A$.

Given no objection from the audience, we will take the following four propositions/lemmas for granted (i.e., without explicit proof) as showing them is quite technical (and not very illuminating in the opinion of the author).

Proposition 4.3. *For $\tau \in \Gamma_A$, there exists a unitary $u_\tau \in N(\mathcal{O}_A, \mathcal{D}_A)$ such that*

$$\mathrm{Ad}(u_\tau)(f) = f \circ \tau^{-1}, \quad f \in \mathcal{D}_A$$

and $\tau \mapsto u_\tau$ is a group homomorphism. (Note that we identify $\mathcal{D}_A$ with $C(X_A)$ here).

Lemma 4.4. *There exists a family v_m for $m \in \mathbb{Z}$ of partial isometries in $\mathcal{O}_A$ such that all but finitely many are zero, and with these properties:*

(i) $v = \sum_{m \in \mathbb{Z}} v_m$, *where the nonzero v_m are finite.*
(ii) $v_m \mathcal{D}_A v_m^* \subset \mathcal{D}_A$ *and* $v_m^* \mathcal{D}_A v_m \subset \mathcal{D}_A$ *for* $m \in \mathbb{Z}$.
(iii) $v_m^* v_m$ *and* $v_m v_m^*$ *are projections in* $\mathcal{D}_A$ *for* $m \in \mathbb{Z}$.
(iv) $v_m^* v_{m'} = v_m v_{m'}^* = 0$ *for* $m \neq m'$.
(v) $v_0 \in \mathcal{F}_A$.

Remark: By the definition of $\mathcal{O}_A$, we get an action $\rho : \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_A)$ given by the correspondence

$$S_i \to e^{it} S_i$$

for $1 \leq i \leq N$ and $e^{it} \in \mathbb{T}$. It is "well-known" that the fixed point algebra of $\mathcal{O}_A$ under ρ is the AF-algebra $\mathcal{F}_A$ generated by elements of the form $S_\mu S_\nu^*$ where μ, ν are admissible words with $|\mu| = |\nu|$.

Approximately-finite dimensional C^*-algebra; formed as an inductive limit of increasing chain of finite-dimensional C^*-algebras

To see why, start by letting $\mathcal{F}_A^n$ be the C*-subalgebra generated by elements of the form $S_\mu S_\nu^*$, where μ, ν are admissible words for X_A of length n; if $\mathcal{F}_A^{\mathrm{alg}} = \cup_{n=1}^\infty \mathcal{F}_A^n$, then $\mathcal{F}_A^{\mathrm{alg}}$ is a dense $*$-subalgebra of $\mathcal{F}_A$. We will see how the gauge action behaves on $\mathcal{F}_A^{\mathrm{alg}}$: if $x \in \mathcal{F}_A^{\mathrm{alg}}$, then (without loss of generality) $x = S_\mu S_\nu^*$ for admissible words with $|\mu| = |\nu|$. We see

$$\begin{aligned} \rho_t(S_\mu S_\nu^*) &= \rho_t(S_{\mu_1}) \cdots \rho_t(S_{\mu_{|\mu|}}) \rho_t(S_{\nu_1}^*) \cdots \rho_t(S_{\nu_{|\mu|}}^*) \\ &= (e^{it} S_{\mu_1}) \cdots (e^{it} S_{\mu_{|\mu|}})(e^{-it} S_{\nu_1}^*) \cdots (e^{-it} S_{\nu_{|\mu|}}^*) \\ &= S_\mu S_\nu^*, \end{aligned}$$

for any $e^{it} \in \mathbb{T}$. As this shows ρ fixes $\mathcal{F}_A^n$ for each $n \in \mathbb{N}$, ρ fixes $\mathcal{F}_A^{\mathrm{alg}}$ (and thus $\mathcal{F}_A$ as well, by density).

To show that this is the *only* subalgebra of $\mathcal{O}_A$ which is fixed under the gauge action, let $\mathcal{O}_A^\rho$ denote the fixed point algebra associated to the C.K. algebra, with mapping $P : \mathcal{O}_A \to \mathcal{O}_A^\rho$ given by

$$x \mapsto \int_{\mathbb{T}} \rho_t(x) dt,$$

where dt is the normalized Haar measure on $\mathbb{T}$. If we start with some $x \in \mathcal{O}_A^\rho$ where $x = \lim_k x_k$, where the latter sequence lies inside the $*$-algebra generated by partial isometries $\{S_1, \ldots, S_N\}$ we clearly have $x = P(x) = \lim_k P(x_k)$. By construction, $P(x_k)$ must belong to $\mathcal{F}_A^n$ for n large enough, and hence $x \in \mathcal{F}_A$. A similar argument can be used to show that for $x \in \mathcal{F}_A^{\mathrm{alg}}$ we also have $P(x) = x$, and hence true for all $x \in \mathcal{F}_A$. Therefore, $\mathcal{F}_A = \mathcal{O}_A^\rho$.

Lemma 4.5. *For a fixed $n \in \mathbb{N}$, there exists partial isometries $v_\mu, v_{-\mu} \in \mathcal{F}_A$ for each admissible word μ of length n satisfying the following conditions:*

(i) $v_n = \sum_{\mu \text{ admissible, length } n} S_\mu v_\mu$ *and* $v_{-n} = \sum_{\mu \text{ admissible, length } n} v_{-\mu} S_\mu^*$.

(ii) $v_\mu^* v_\mu, S_\mu v_\mu v_\mu^* S_\mu^*, S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*$ *and* $v_{-\mu} v_{-\mu}^*$ *are projections in* $\mathcal{D}_A$ *such that*

$$v_n^* v_n = \sum v_\mu^* v_\mu, \quad v_n v_n^* = \sum S_\mu v_\mu v_\mu^* S_\mu^*,$$
$$v_{-n}^* v_{-n} = \sum S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*, \quad v_{-n} v_{-n}^* = \sum v_{-\mu} v_{-\mu}^*,$$

where the sums are over all admissible words μ of length n.

(iii) $v_\mu v_\nu^* = v_{-\mu}^* v_{-\nu} = 0$ *for μ, ν admissible distinct words of length n.*

(iv) $v_\mu \mathcal{D}_A v_\mu^*, v_\mu^* \mathcal{D}_A v_\mu, v_{-\mu} \mathcal{D}_A v_{-\mu}^*$ *and* $v_{-\mu}^* \mathcal{D}_A v_{-\mu}$ *are contained in* $\mathcal{D}_A$.

Lemma 4.6. *For a partial isometry $u \in \mathcal{F}_A$ satisfying*

$$u \mathcal{D}_A u^* \subset \mathcal{D}_A, \quad u^* \mathcal{D}_A u \subset \mathcal{D}_A,$$

there exists $k_u \in \mathbb{N}$ such that the homeomorphism $h_u : \mathrm{supp}(u^*u) \to \mathrm{supp}(uu^*)$ defined by $\mathrm{Ad}(u)(g) = g \circ h_u^{-1}$ for $g \in \mathcal{D}_A u^* u$ satisfies the condition

$$\sigma_A^{k_u}(h_u(x)) = \sigma_A^{k_u}(x)$$

for $x \in \mathrm{supp}(u^*u)$.

Proposition 4.7. *For any $v \in N(\mathcal{O}_A, \mathcal{D}_A)$ the homeomorphism τ_v on X_A induced by the automorphism of $\mathcal{D}_A$ defined by the restriction of $\mathrm{Ad}(v)$ to $\mathcal{D}_A$ gives rise to an element of Γ_A.*

Proof. For $v \in N(\mathcal{O}_A, \mathcal{D}_A)$, let $v_m, m \in \mathbb{Z}$ be the partial isometries in $\mathcal{O}_A$ as in Lemma 4.4. Let $K \in \mathbb{N}$ so that $v_m = 0$ for all $m \in \mathbb{Z}$ with $|m| > K$, meaning $v = \sum_{m=-K}^{K} v_m$. We have

$$\mathrm{Ad}(v)(f) = \sum_{n=1}^{K} v_n f v_n^* + v_0 f v_0^* + \sum_{n=1}^{K} v_{-n} f v_{-n}^*$$

for $f \in \mathcal{D}_A$. As $v_m^* v_m, v_m v_m^*$ are projections in $\mathcal{D}_A$, we can put clopen sets

$$X_A^{(m)} = \mathrm{supp}(v_m^* v_m), \quad Y_A^{(m)} = \mathrm{supp}(v_m v_m^*)$$

for $m \in \mathbb{Z}$ with $|m| \le K$ in X_A so that

$$X_A = \bigcup_{|m| \le K} X_A^{(m)} = \bigcup_{|m| \le K} Y_A^{(m)}.$$

By Lemma 4.6, as $v_0 \in \mathcal{F}_A$ there exists $k_0 \in \mathbb{N}$ such that

$$\sigma_A^{k_0}(\tau_0(x)) = \sigma_A^{k_0}(x), \quad \text{for } x \in X_A^{(0)},$$

where $\tau_0 : X_A^{(0)} \to Y_A^{(0)}$ is the homeomorphism satisfying $\mathrm{Ad}(v_0)(f) = f \circ \tau_0^{-1}$ with $f \in \mathcal{D}_A v_0^* v_0$. For v_n, v_{-n} and $1 \le n \le K$, by Lemma 4.5 we have (for $f \in \mathcal{D}_A$)

$$v_n f v_n^* = \sum S_\mu v_\mu f v_\mu^* S_\mu^* \text{ and } v_{-n} f v_{-n}^* = \sum v_{-\mu} S_\mu^* f S_\mu v_{-\mu}^*$$

where the sums are taken over all admissible words μ of length n. Put

$$X_A^{(n,\mu)} = \mathrm{supp}(v_\mu^* v_\mu), \quad X_A^{(-n,\mu)} = \mathrm{supp}(S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*),$$
$$Y_A^{(n,\mu)} = \mathrm{supp}(S_\mu v_\mu v_\mu^* S_\mu^*), \quad Y_A^{(-n,\mu)} = \mathrm{supp}(v_{-\mu} v_{-\mu}^*).$$

Again, by Lemma 4.5

$$X_A^{(m)} = \bigcup_\mu X_A^{(m,\mu)}, \quad Y_A^{(m)} = \bigcup_\mu Y_A^{(m,\mu)}$$

where μ is an admissible word of length $|m|$, and $|m| \leq K$. There exists a homeomorphism

$$\tau_{(m,\mu)} : X_A^{(m,\mu)} \to Y_A^{(m,\mu)}$$

for $m \in \mathbb{Z}$ with $|m| \leq K$ such that

$$\begin{aligned} \mathrm{Ad}(S_\mu v_\mu)(f) &= f \circ \tau_{(n,\mu)}^{-1}, \quad f \in \mathcal{D}_A v_\mu^* v_\mu, \\ \mathrm{Ad}(v_{-\mu} S_\mu^*)(g) &= g \circ \tau_{(-n,\mu)}^{-1}, \quad g \in \mathcal{D}_A S_\mu v_{-\mu}^* v_{-\mu} S_\mu^* \end{aligned}$$

where $n \in \mathbb{N}$ with $1 \leq n \leq K$. As $v_\mu, v_{-\mu} \in \mathcal{F}_A$, by Lemma 4.6 there exist $k_{(n,\mu)}, k_{(-n,\mu)} \in \mathbb{N}$ so that

$$\begin{aligned} \sigma_A^{k(n,\mu)}(\tau_{(n,\mu)}(x)) &= \sigma_A^{k(n,\mu)+n}(x), \quad x \in X_A^{(n,\mu)}, \\ \sigma_A^{k(n,\mu)+n}(\tau_{(-n,\mu)}(x)) &= \sigma_A^{k(n,\mu)}(x), \quad x \in X_A^{(-n,\mu)}. \end{aligned}$$

As we have

$$\tau_v(x) = \begin{cases} \tau_{(n,\mu)}(x), & x \in X_A^{(n,\mu)}, \\ \tau_0(x), & x \in X_A^{(0)}, \\ \tau_{(-n,\mu)}(x), & x \in X_A^{(-n,\mu)} \end{cases}$$

and X_A is made of disjoint unions of clopen sets $X_A^{(0)}, X_A^{(m,\mu)}, Y_A^{(0)}, Y_A^{(m,\mu)}$ we conclude that $\tau_v \in \Gamma_A$. □

Remark: We can naturally embed $\mathcal{U}(\mathcal{D}_A)$ into $N(\mathcal{O}_A, \mathcal{D}_A)$; indeed, using $\mathrm{id} : \mathcal{U}(\mathcal{D}_A) \to \mathcal{U}(\mathcal{O}_A)$ we have that for $u \in \mathcal{U}(\mathcal{D}_A)$, $u \in \mathcal{U}(\mathcal{O}_A)$ as well with $u\mathcal{D}_A u^* = \mathcal{D}_A$. Hence, $\mathrm{id}(u) \in N(\mathcal{O}_A, \mathcal{D}_A)$. In what follows, we use this embedding.

Theorem 4.8. *The sequence* $0 \to \mathcal{U}(\mathcal{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\tau} \Gamma_A \to 0$ *is exact and splits.*

Proof. By Proposition 4.7, the map $\lambda : N(\mathcal{O}_A, \mathcal{D}_A) \to \Gamma_A$ given by

$$v \mapsto \tau_v$$

is a homomorphism. It is surjective, by Proposition 4.3. If we suppose that $\tau_v = \mathrm{id}$ on X_A for some $v \in N(\mathcal{O}_A, \mathcal{D}_A)$ (i.e., $v \in \ker(\tau)$) this means $\mathrm{Ad}(v) = \mathrm{id}$ on $\mathcal{D}_A$. Therefore, v commutes with every element in $\mathcal{D}_A$; as $\mathcal{D}_A$ is maximal abelian via Proposition 1.5, we have that $v \in \mathcal{D}_A$ (otherwise we could generate a larger abelian subalgebra using $\mathcal{D}_A$, and v). Therefore, $v \in \mathcal{U}(\mathcal{D}_A)$ with $\ker(\tau) = \mathcal{U}(\mathcal{D}_A)$. It is clear that $\mathrm{rng}(\mathrm{id}) = \mathcal{U}(\mathcal{D}_A)$, and so the sequence is exact.

Again by Proposition 4.3, for $\tau \in \Gamma_A$ the unitary u_τ defined by

$$u_\tau e_x = e_{\tau(x)}$$

for $x \in X_A$ gives rise to a section of the exact sequence; indeed, u_τ is well-defined (as τ is a homeomorphism, it will send elements in the complete orthonormal system to other elements in the system) and clearly unitary. If $\pi : \Gamma_A \to N(\mathcal{O}_A, \mathcal{D}_A)$ is the map described in Proposition 4.3 sending τ to u_τ, we see

$$(\lambda \circ \pi)(\tau) = \lambda(u_\tau) = \tau_{u_\tau} = \tau$$

for each $\tau \in \Gamma_A$. Thus, $\lambda \circ \pi = \mathrm{id}_{\Gamma_A}$, showing the sequence splits. □

Proposition 4.9. *If there is an isomorphism* $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ *such that* $\Psi(\mathcal{D}_A) = \mathcal{D}_B$, *then there is a homeomorphism* $h : X_A \to X_B$ *such that* $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$.

Proof. By Theorem 4.8, there exists a group isomorphism $\tilde{\Psi} : \Gamma_A \to \Gamma_B$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{U}(\mathcal{D}_A) & \xrightarrow{\text{id}} & N(\mathcal{O}_A, \mathcal{D}_A) & \xrightarrow{\tau} & \Gamma_A & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \Psi|_{\mathcal{U}(\mathcal{D}_A)}} & & \Big\downarrow {\scriptstyle \Psi} & & \Big\downarrow {\scriptstyle \tilde{\Psi}} & & \\
0 & \longrightarrow & \mathcal{U}(\mathcal{D}_B) & \xrightarrow{\text{id}} & N(\mathcal{O}_B, \mathcal{D}_B) & \xrightarrow{\tau} & \Gamma_B & \longrightarrow & 0.
\end{array}
$$

Indeed: as the sequence is split exact, we know that

$$N(\mathcal{O}_A, \mathcal{D}_A) \cong \mathcal{U}(\mathcal{D}_A) \oplus \Gamma_A,$$
$$\Psi(N(\mathcal{O}_A, \mathcal{D}_A)) = N(\mathcal{O}_B, \mathcal{D}_B) \cong \mathcal{U}(\mathcal{D}_B) \oplus \Gamma_B = \Psi(\mathcal{U}(\mathcal{D}_A)) \oplus \Gamma_B$$

where the identity $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ guarantees the existence of such a $\tilde{\Psi}$. For $v \in N(\mathcal{O}_A, \mathcal{D}_A)$, put $\mathrm{Ad}(v)(f) = vfv^*$ for $f \in \mathcal{D}_A$; let $\tau_v \in \mathrm{Homeo}(X_A)$ be the homeomorphism on X_A which satisfies $\mathrm{Ad}(v)(f) = f \circ \tau_v^{-1}$ for $f \in \mathcal{D}_A$. Let $h : X_A \to X_B$ be the homeomorphism satisfying $\Psi(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_A$. As $\Psi : N(\mathcal{O}_A, \mathcal{D}_A) \to N(\mathcal{O}_B, \mathcal{D}_B)$ is an isomorphism and

$$\{\tau_v |\ v \in N(\mathcal{O}_A, \mathcal{D}_A)\} = \Gamma_A,$$

the identity $\Psi \circ \mathrm{Ad}(v) \circ \Psi^{-1} = \mathrm{Ad}(\Psi(v))$ implies $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$. $\square$

Proposition 4.10. *If $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$ for some homeomorphism $h : X_A \to X_B$, then (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

Proof. Assume that $h \circ \Gamma_A \circ h^{-1} = \Gamma_B$. For any $y \in X_B$, let $x = h^{-1}(y)$; so $h(\Gamma_A(x)) = \Gamma_B(h(x))$. Through a simple check (see "Addendum") we have $\Gamma_A(x) = \mathrm{orb}_{\sigma_A}(x)$ and $\Gamma_B(h(x)) = \mathrm{orb}_{\sigma_B}(h(x))$, so $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$.

We next will show there exist continuous cocycle functions for h. For any admissible word μ with $|\mu| = 2$, there exist $\tau_\mu \in \Gamma_A$ and $k_{\tau_\mu}, \ell_{\tau_\mu} : X_A \to \mathbb{N}_+$ satisfying the conditions

$$\begin{cases} \sigma_A^{k_{\tau_\mu}(x)}(\tau_\mu(x)) = \sigma_A^{\ell_{\tau_\mu}(x)}(x), \quad x \in X_A, \\ \tau_\mu(y) = \sigma_A(y), \quad y \in U_\mu, \\ k_{\tau_\mu}(y) = 0, \ell_{\tau_\mu}(y) = 1, \quad y \in U_\mu. \end{cases}$$

Put $\tau_h = h \circ \tau_\mu \circ h^{-1} \in \Gamma_B$. For $x \in U_\mu$, as $\tau_\mu(y) = \sigma_A(y)$ for $y \in U_\mu$ we have $h(\sigma_A(x)) = \tau_h(h(x))$. Since $\tau_h \in \Gamma_B$, we can find $k^\mu_{\tau_h}, \ell^\mu_{\tau_h} : X_B \to \mathbb{N}_+$ so that

$$\sigma_B^{k^\mu_{\tau_h}(y)}(\tau_h(y)) = \sigma_B^{\ell^\mu_{\tau_h}(y)}(y).$$

For $y \in h(U_\mu)$, put $x = h^{-1}(y)$ so that

$$\sigma_B^{k^\mu_{\tau_h}(y)}(h \circ \sigma_A(x)) = \sigma_B^{\ell^\mu_{\tau_h}(h(x))}(h(x)).$$

Let $\{\mu^{(1)}, \dots, \mu^{(M)}\}$ be the set of all admissible words in X_A of length 2. Define $k_1^h, \ell_1^h : X_A \to \mathbb{N}_+$ by setting

$$k_1^h(x) = k^{\mu^{(i)}}_{\tau_h}(h(x)), \quad \ell_1^h(x) = \ell^{\mu^{(i)}}_{\tau_h}(h(x)), \quad x \in U_{\mu^{(i)}}.$$

They are continuous, and satisfy

$$\sigma_B^{k_1^h(x)}(h \circ \sigma_A(x)) = \sigma_B^{\ell_1^h(x)}(h(x)), \quad x \in X_A.$$

Using a similar argument, we can find continuous functions $k_2^h, \ell_2^h : X_B \to \mathbb{N}_+$ satisfying the corresponding relations for $y \in X_B$, showing (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. $\square$

Proposition 4.11. *If (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then there exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.*

Proof. Let $h : X_A \to X_B$ be a homeomorphism which gives rise to continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B). Take continuous functions $k_1, \ell_1 : X_A \to \mathbb{N}_+$ and $k_2, \ell_2 : X_B \to \mathbb{N}_+$ which satisfy the relations (1); representing $\mathcal{O}_A, \mathcal{O}_B$ on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively (as shown before), we will show that there exists a unitary $u_h : \mathcal{H}_A \to \mathcal{H}_B$ such that

$$\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B, \quad \mathrm{Ad}(u_h)(f) = f \circ h^{-1}, \text{ for } f \in \mathcal{D}_A.$$

Let $\{e_x^A\}$ for $x \in X_A$ (resp. $\{e_y^B\}$ for $y \in X_B$) denote the complete orthonormal system on $\mathcal{H}_A$ (resp. $\mathcal{H}_B$). Define the unitary $u_h : \mathcal{H}_A \to \mathcal{H}_B$ by setting $u_h e_x^A = e_{h(x)}^B$ for $x \in X_A$. We first want to show that $\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B$. Denote by S_i^A, S_i^B the canonical generating partial isometries for $\mathcal{O}_A, \mathcal{O}_B$. For $y \in X_B$, we have

$$u_h S_i^A u_h^* e_y^B = \begin{cases} e^B_{h(ih^{-1}(y))}, & \text{if } ih^{-1}(y) \in X_A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_B^{(i)} = \{y \in X_B |\ ih^{-1}(y) \in X_A\}$. Put $z = ih^{-1}(y) \in X_A$. By relations (1) and the equality $h(\sigma_A(z)) = y$, we have $h(z) \in \sigma_B^{-\ell_1(z)}(\sigma_B^{k_1(z)}(y))$. Thus,

$$h(z) = (\mu_1(z), \ldots, \mu_{\ell_1(z)}(z), y_{k_1(z)+1}, y_{k_1(z)+2}, \ldots)$$

for some admissible word $\mu_1(z) \ldots \mu_{\ell_1(z)}(z)$ for X_B. As the maps k_1, ℓ_1 and $y \mapsto z = ih^{-1}(y)$ are continuous, there exists finite numbers

$$\tilde{k}_1 = \max\{k_1(z) |\ y \in X_B^{(i)}\}, \quad \tilde{\ell}_1 = \max\{\ell_1(z) |\ y \in X_B^{(i)}\}.$$

The set $\{(\mu_1(z), \ldots, \mu_{\ell_1(z)}(z))$ admissible $|\ y \in X_B^{(i)}\}$ of words is a finite subset of $W_{\tilde{\ell}_1}(X_B)$, which consists of the union of all admissible words of length j for X_B, where j ranges from 0 to $\tilde{\ell}_1$. The map

Note that we are introducing new notation here!

$$y \in X_B^{(i)} \to (\mu_1(z), \ldots, \mu_{\ell_1(z)}(z)) \in W_{\tilde{\ell}_1}(X_B)$$

is continuous (when the latter set is endowed with the discrete topology). For a word $\nu = \nu_1 \ldots \nu_j \in W_{\tilde{\ell}_1}(X_B)$ and $0 \le n \le \tilde{k}_1$, the sets

$$E_\nu^{(i)} = \{y \in X_B^{(i)} |\ \mu_1(z) = \nu_1, \ldots, \mu_{\ell_1(z)}(z) = \nu_j\},$$
$$F_n^{(i)} = \{y \in X_B^{(i)} |\ k_1(z) = n\}$$

are clopen in $X_B^{(i)}$, where $z = ih^{-1}(y)$. We define projections in $\mathcal{D}_B$:

$$Q_\nu^{(i)} = \chi_{E_\nu^{(i)}}, \quad P_n^{(i)} = \chi_{F_n^{(i)}}, \quad P^{(i)} = \chi_{X_B^{(i)}}.$$

As we have disjoint unions

$$X_B^{(i)} = \bigcup_{\nu \in W_{\tilde{\ell}_1}(X_B)} E_\nu^{(i)} = \bigcup_{0 \le n \le \tilde{k}_1} F_n^{(i)},$$

we then have

$$P^{(i)} = \sum_{\nu \in W_{\tilde{\ell}_1}(X_B)} Q_\nu^{(i)} = \sum_{0 \le n \le \tilde{k}_1} P_n^{(i)}.$$

For $y \in X_B^{(i)}$ and $\nu \in W_{\tilde{\ell}_1}(X_B)$ with $0 \le n \le \tilde{k}_1$, we have $y \in E_\nu^{(i)} \cap F_n^{(i)}$ if and only if $h(ih^{-1}(y)) = \nu\sigma_B^n(y)$, where the latter condition is equivalent to the condition that

$$e^B_{h(ih^{-1}(y))} = S_\nu^B e^B_{\sigma_B^n(y)}.$$

As $y \in E_\nu^{(i)} \cap F_n^{(i)}$ if and only if $P_n^{(i)} Q_\nu^{(i)} e_y^B = e_y^B$, and $e^B_{\sigma_B^n(y)} = \sum (S_\xi^B)^* e_y^B$ (where the sum is taken over all admissible words ξ of length n for X_B), we have

$$e^B_{h(ih^{-1}(y))} = \sum_{0 \le n \le \tilde{k}_1} \sum_{\nu \in W_{\tilde{\ell}_1}(X_B)} \left(S_\nu^B \sum_\xi (S_\xi^B)^* \right) P_n^{(i)} Q_\nu^{(i)} e_y^B, \quad y \in X_B^{(i)}.$$

Hence,

$$u_h S_i^A u_h^* e_y^B = \sum_{0 \le n \le \tilde{k}_1} \sum_{\nu \in W_{\tilde{\ell}_1}(X_B)} \left(S_\nu^B \sum_\xi (S_\xi^B)^* \right) P_n^{(i)} Q_\nu^{(i)} e_y^B, \quad y \in X_B^{(i)}.$$

Therefore, we have

$$u_h S_i^A u_h^* = \sum_{0 \le n \le \tilde{k}_1} \sum_{\nu \in W_{\tilde{\ell}_1}(X_B)} \left(S_\nu^B \sum_\xi (S_\xi^B)^* \right) P_n^{(i)} Q_\nu^{(i)} P^{(i)}.$$

As $P_n^{(i)}, Q_\nu^{(i)}$ and $P^{(i)}$ are projections in $\mathcal{D}_B$, we have $\mathrm{Ad}(u_h)(S_i^A) \in \mathcal{O}_B$, so that $\mathrm{Ad}(u_h)(\mathcal{O}_A) \subset \mathcal{O}_B$. As $u_h^* = u_{h^{-1}}$, by symmetry we have $\mathrm{Ad}(u_h^*)(\mathcal{O}_B) \subset \mathcal{O}_A$- so $\mathrm{Ad}(u_h)(\mathcal{O}_A) = \mathcal{O}_B$.

Directly from the definition of u_h, where $u_h e_x^A = e^B_{h(x)}$ for $x \in X_A$ we have $\mathrm{Ad}(u_h)(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_A$, and thus $\mathrm{Ad}(u_h)(\mathcal{D}_A) = \mathcal{D}_B$. □

Remark: Stitching together Propositions 4.9, 4.10, and 4.11 we have been able to prove the equivalence of $(i) \iff (iii)$ for Theorem 4.1.

It can be shown (with some work) if G_A is the groupoid associated to (X_A, σ_A) there is a natural isomorphism $\Psi_1 : \mathcal{O}_A \to \mathrm{C}^*_\mathrm{r}(G_A)$ with $\Psi_1(\mathcal{D}_A) = C_0(G_A^{(0)})$ (and $\Psi_2 : \mathcal{O}_B \to \mathrm{C}^*_\mathrm{r}(G_B)$ with the corresponding properties; see [6, Section 1.3]). Using [5, Proposition 4.11], we have the following result.

Theorem 4.12. *Let G_1, G_2 be essentially principal étale groupoids. The following are equivalent:*

(i) G_1 and G_2 are isomorphic.

(ii) There exists an isomorphism $\pi : \mathrm{C}^*_r(G_1) \to \mathrm{C}^*_r(G_2)$ so that $\pi(\mathrm{C}_0(G_1^{(0)})) = \mathrm{C}_0(G_2^{(0)})$.

For G_A and G_B, if we assume they are isomorphic by Theorem 4.12 we have the existence of an isomorphism $\pi : \mathrm{C}^*_\mathrm{r}(G_A) \to \mathrm{C}^*_\mathrm{r}(G_B)$ so that $\pi(\mathrm{C}_0(G_A^{(0)})) = \mathrm{C}_0(G_B^{(0)})$. Using the identification of $G_A^{(0)} \cong X_A$, $G_B^{(0)} \cong X_B$ as both of the latter spaces are compact, we have

$$\mathrm{C}_0(G_A^{(0)}) \cong C(X_A), \quad \mathrm{C}_0(G_B^{(0)}) \cong C(X_B).$$

Stitching everything together, we get an isomorphism

$$\Psi_2^{-1} \circ \pi \circ \Psi_1 : \mathcal{O}_A \to \mathcal{O}_B$$

which sends $\mathcal{D}_A$ to $\mathcal{D}_B$. This proves the equivalence of $(ii) \iff (iii)$ in Theorem 4.1, finishing the proof.

5. Addendum

There is a small comment made in the proof of Proposition 4.10; if asked, the sketch of the proof is as follows:

Proof. If we first start with $\tau \in \Gamma_A$, we know there exist continuous $k, \ell : X_A \to \mathbb{N}_+$ so that

$$\tau(x) = (\mu_1(x), \ldots, \mu_{k(x)}(x), x_{\ell(x)+1}, \ldots)$$

for $(\mu_1(x), \ldots, \mu_{k(x)}(x))$ an admissible word for X_A. Thus, $\tau(x) \in \mathrm{orb}_{\sigma_A}(x)$, and as our choice of x was arbitrary we have $\Gamma_A(x) \subseteq \mathrm{orb}_{\sigma_A}(x)$.

For the other direction, we can show that for $x = (x_n)_{n\in\mathbb{N}} \in X_A$ and $j \in \{1, \ldots, N\}$ with $jx \in X_A$ for each j, there exist $\tau_1, \tau_2 \in \Gamma_A$ so that

$$\tau_1(x) = (j, x_1, x_2, \ldots), \quad \tau_2(x) = (x_2, x_3, \ldots).$$

As Γ_A is a group, we can then find $(\mu_1, \ldots, \mu_k)$ admissible with

$$(\mu_1, \ldots, \mu_k, x_{\ell+1}, \ldots) \in \Gamma_A(x)$$

for all $k, \ell \in \mathbb{N}_+$. This implies $\mathrm{orb}_{\sigma_A}(x) \subseteq \Gamma_A(x)$, and thus the proof is complete. □

More detail can be found in the proof of [2, Lemma 3.4].

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