Existence of Pappus Configurations in Projective Planes containing Perspectivities

Vladislav Taranchuk

Notation: If A and B are points of an affine or projective plane, we write $\overline{AB}$ for the line containing them. If ℓ and m are lines of an affine or projective plane, then we denote their point of intersection by $\ell \cap m$. If ℓ and m are lines of an affine plane, we write $\ell \parallel m$ to denote that ℓ is parallel to m.

The following definitions and lemmas are variations of definitions and lemmas found in Artin's Geometric Algebra [1] and Casse's Projective Geometry.

Definition (Casse [2], p.56)**.** A one-to-one map of a projective plane Π onto itself is said to be a **collineation** if and only if the images of collinear points are collinear.

Definition (Casse [2], p.111)**.** Let σ be a collineation of a projective plane Π. Then

1. A point P is **fixed** by σ is $\sigma(P) = P$.
2. A line ℓ is **fixed** by σ if $\sigma(\ell) = \sigma$.
3. A line ℓ is **fixed pointwise** if $\sigma(P) = P$ for all $P \in \ell$.
4. A point P is **fixed linewise** if σ fixes every line passing through P.

Definition (Casse [2], p.112)**.** Let σ be a collineation of a projective plane Π that fixes a line ℓ pointwise and a point P linewise. Then:

1. σ is called (P, ℓ)**-perspectivity** or a **central collineation**.
2. If $P \in \ell$ then σ is called an **elation**.
3. If $P \notin \ell$ then σ is called a **homology**.

Definition (Artin [1], Ch. 2)**.** Let $\mathcal{A}$ be an affine plane and $\mathcal{P}$ be its set of points. A map $\sigma : \mathcal{P} \to \mathcal{P}$ is called a **dilation** of $\mathcal{A}$, if for any points P and Q, the line ℓ' parallel to $\overline{PQ}$ passing through $\sigma(P)$ contains $\sigma(Q)$. We will not call the map δ which maps all points of $\mathcal{A}$ into a single point a dilation.

Lemma (Artin [1], Ch. 2)**.** *A dilation σ of an affine plane $\mathcal{A}$ is uniquely determined by the images of two distinct points P and Q. Furthermore, σ is a bijection on the set of points.*

Definition (Artin [1], Ch. 2)**.** Let σ be a dilation, if P is any point, then the line ℓ containing P and $\sigma(P)$ is called a **trace** of σ.

Lemma (Artin [1], Ch. 2)**.** *If σ is a dilation of an affine plane $\mathcal{A}$ that is not the identity, then the set of traces of σ are either:*

1. *A parallel class of lines if σ has no fixed points.*
2. *The pencil of lines through a point P, if P is a fixed point. (By the first lemma above, a dilation that is not the identity can have at most one fixed point.)*

Definition (Artin [1], Ch. 2)**.** A dilation τ, shall be called a **translation** if τ is the identity map, or if τ has no fixed points. If τ has no fixed point, the parallel class of traces for τ will be called the **direction** of τ.

Proposition. *Let Π be a projective plane, σ a (P, ℓ)-perspectivity of Π, and $\mathcal{A}$ the affine plane obtained from Π by removing the line ℓ: Then:*

1. *If σ is an elation of Π, it's induced action on $\mathcal{A}$ is a translation.*

2. *If σ is a homology of Π, it's induced action on $\mathcal{A}$ is a dilation which is not a translation with P as the fixed point.*

To the author's knowledge, the existence of Pappus configurations in an arbitrary projective plane is an open question. The following theorem gives a sufficient condition for the existence of a Pappus configuration in a projective plane.

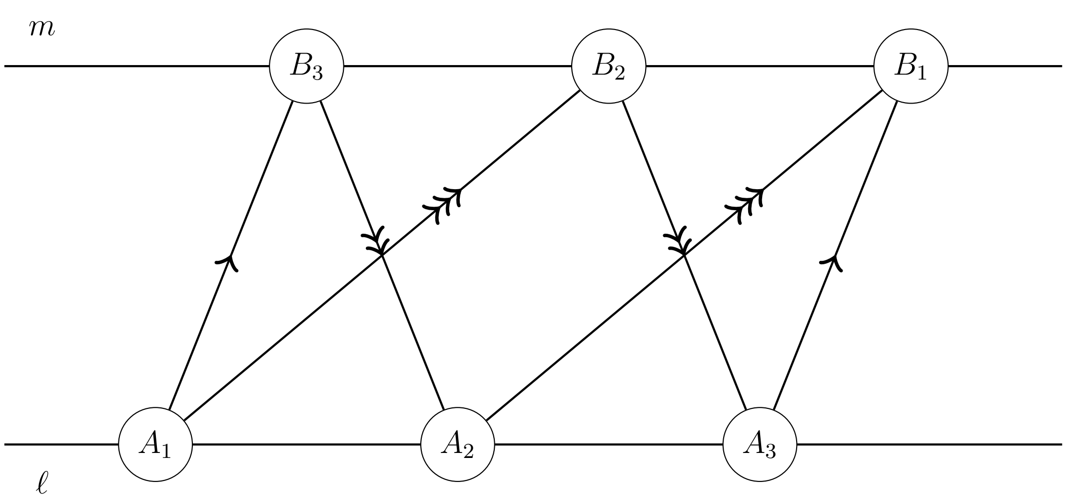
Theorem 1. *Let Π be an projective plane such that for some point P and line ℓ there exists a (P, ℓ)-perspectivity σ of Π. Then Π contains a Pappus Configuration(many in fact).*

Proof. Let $\mathcal{A}$ be the affine plane obtained from Π by removing ℓ. Then the induced action of σ on $\mathcal{A}$ is a dilation. We present the proof in two distinct cases: First we assume that $\sigma^2 \neq 1$, where 1 is the identity map, then we assume that $\sigma^2 = 1$.

Suppose that $\sigma^2 \neq 1$. Let $\mathcal{T}$ denote the set of traces of σ. Let $\ell \in \mathcal{T}$ and suppose that $A_1 \in \ell$ is a point. Then denote $\sigma(A_1) = A_2$ and $\sigma(A_2) = A_3$. Note that since $\sigma^2 \neq 1$, then A_1, A_2, A_3 are necessarily distinct and lie on ℓ, since ℓ is trace of σ. Now, choose any point not on ℓ and call it B_3. Then $B_3 \in m$ for some distinct line $m \in \mathcal{T}$. Denote $\sigma(B_3) = B_2$ and $\sigma(B_2) = B_1$, and note that B_1, B_2, B_3 are all distinct and all lie on m.

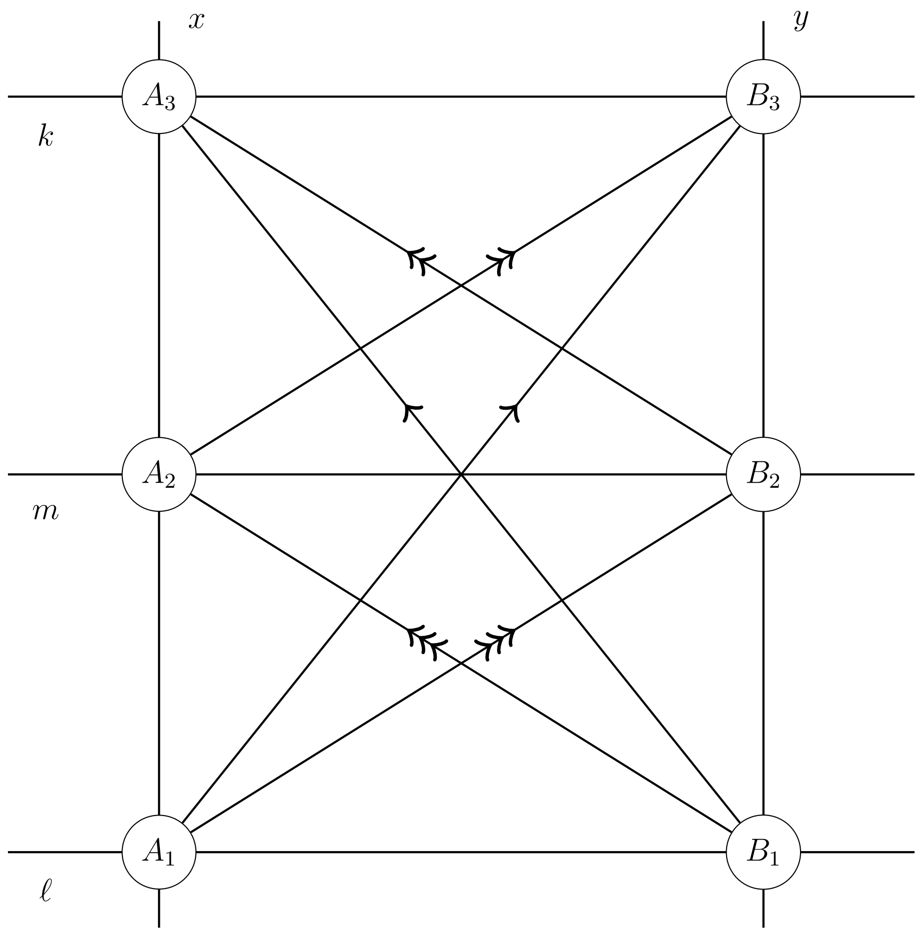
A dilation must preserve parallelism and incidence. Thus σ must map the line $\overline{A_1B_3}$ to $\overline{A_2B_2}$ and it must be that $\overline{A_2B_2}$ is then mapped to $\overline{A_3B_1}$, which further implies that $\overline{A_1B_3} \parallel \overline{A_2B_2} \parallel \overline{A_3B_1}$, the key here being that $\overline{A_1B_3} \parallel \overline{A_3B_1}$. Likewise, $\overline{B_3A_2} \parallel \overline{B_2A_3}$.

As a further consequence, since $\sigma(A_1) = A_2$ and $\sigma(B_2) = B_1$, then $\overline{A_1B_2} \parallel \overline{A_2B_1}$, which gives us a special type of Pappus Configuration.



Now suppose instead that $\sigma^2 = 1$ the identity map on $\mathcal{A}$. Let $\mathcal{T}$ again represent the traces of σ and let $\ell, m, k \in \mathcal{T}$. Let x be a line of $\mathcal{A}$, $x \notin \mathcal{T}$ and such that x is not parallel to any of the lines ℓ, m, k. Now define the following points: $A_1 = x \cap \ell, A_2 = x \cap m, A_3 = x \cap m$. Let y be the line parallel to x containing the point $B_1 = \sigma(A_1)$. Since $\ell \in \mathcal{T}$, then B_1 lies on ℓ. Denote then $B_2 = y \cap k$ and $B_3 = y \cap m$ and note that since σ is a dilation, then $\sigma(x) = y$, and as $m, k \in \mathcal{T}$ then we have $\sigma(A_2) = B_2$ and $\sigma(A_3) = B_3$.

Since $\sigma^2 = 1$, observe that $\sigma(B_1) = A_1, \sigma(B_2) = A_2$ and $\sigma(B_3) = A_3$. Consider now the following lines, $\overline{A_1B_2}, \overline{A_1B_3}, \overline{A_2B_3}$. Note that by definition of σ we must have that $\sigma(\overline{A_1B_2}) \parallel A_1B_2$. We also know where σ sends all of the defined points, and so we obtain that $\sigma(\overline{A_1B_2}) = \overline{B_1A_2} \parallel \overline{A_1B_2}$. Following in suit, it is clear then that $\sigma(\overline{A_1B_3}) = \overline{B_1A_3} \parallel \overline{A_1B_3}$ and $\sigma(\overline{A_2B_3}) = \overline{B_2A_3} \parallel \overline{A_2B_3}$. Thus we obtain the desired result with the following illustration as a guide.



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The following corollaries follow quickly from the theorem above.

Corollary. Suppose that A is a finite affine plane of odd order n, if τ is a translation of A and $P \in A$ a point, then P is contained in at least n^2 distinct Pappus configurations of the type constructed int he theorem above.

Corollary. Suppose that A is a finite affine plane of even order n, let τ be a dilation of A such that the traces of τ is the pencil of lines through some point O. Then if $P \in A$ a point $P \neq O$, then P is contained in at least $(n-1)^2$ distinct Pappus configurations of the type constructed in the theorem above.

References

[1] E. Artin, *Geometric Algebra*, New York, NY, Interscience Publishers, (1957).

[2] R. Casse, *Projective Geometry: An Introduction*, Oxford University Press, (2006)