The Anti-Ramsey Problem for the Sidon Equation

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Introduction

The **Ramsey number** $R_k(H)$ is the smallest integer n such that every k-edge coloring of K_n contains a monochromatic copy of H.

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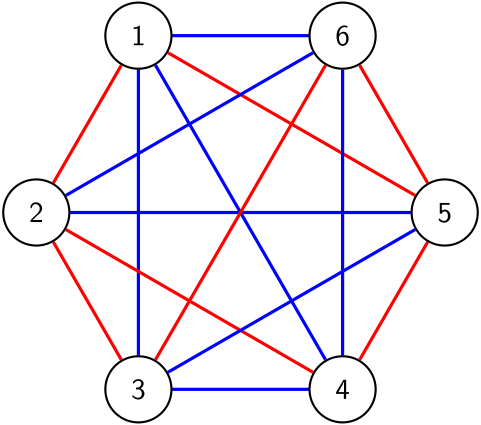

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Define the **Ramsey Multiplicity** factor $RM_k(n; H)$ as the smallest integer such that every k-edge coloring of K_n contains at least $RM_k(n; H)$ monochromatic copies of H.
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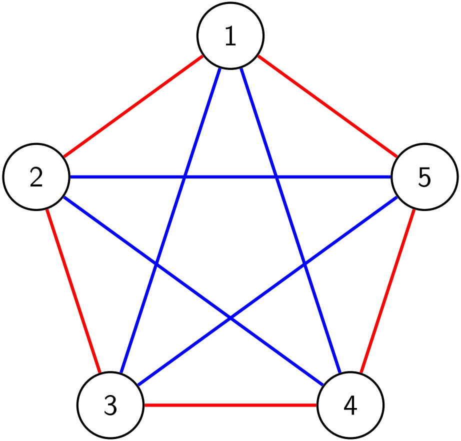

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For n large

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Theorem (Conlon 2007)

For n large

$$\frac{n^t}{C^{(1+O(1))t^2}} \leq RM_2(n; K_t)$$

where $C \approx 2.18$.

Introduction

In 2017, Saad and Wolf introduced Ramsey multiplicity type problems in abelian groups. Given a k-coloring of the elements of a finite abelian group, how many monochromatic solutions are there to:

- $X + Y = T$(Schur Triples)
- $X + Y = 2Z$(3-Term AP's)
- $X + Y = Z + T$(Sidon Equation)

Introduction

In 2017, De Silva, Si, Tait, Tunçbilek, Yang, and Young introduced **anti-Ramsey Multiplicity**.

Question

Given a graph H, what is the maximum number of rainbow copies of H over all k-edge colorings of K_n?

They studied this question in regards to matchings, stars, and other graphs.

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Theorem (Fox, Mahdian,and Radoičić 2008)

For every $n \geq 4$, every partition of [n] into four color classes R, B, G, and Y, with $min\{|R|, |B|, |G|, |Y|\} > \frac{n+1}{6}$, contains a rainbow solution to $x + y = z + t$.

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Question

What is the maximum amount of rainbow solutions to the Sidon equation($X + Y = Z + T$) over all colorings?

New Results

Denote $AR^k_{X+Y=Z+T}(n)$ to be the maximum number of rainbow Sidon 4-sets over all k-colorings of $[n]$. We obtained the following theorems that answer the above question up to the correct order of growth.

For arbitrary k, we answer this question up to the correct order of growth for both n and k.

Theorem (T and Timmons 2018)

For $k \geq 4$,

$$\left(\frac{1}{12} - \frac{1}{3k} + \frac{\theta}{k^2}\right) n^3 - O_k(n^2) \leq AR^k_{X+Y=Z+T}(n)$$

where $\theta = \frac{1}{3}$ if k is even, and $\theta = \frac{1}{4}$ if k is odd.

New Results

Theorem (T and Timmons 2018)

For $k \geq 4$

$$AR^k_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{24k}\right) n^3 + O_k(n^2).$$

In the case that $k = 4$, we obtain bounds that are quite close, where the lower bound is a result of our general lower bound and the upper bound uses a different argument.

Theorem (T and Timmons 2018)

When $k = 4$,

$$\frac{2n^3}{96} - O(n^2) \leq AR^4_{X+Y=Z+T}(n) \leq \frac{3n^3}{96} + O(n^2).$$

The Lower Bound

Using an explicit coloring we obtain a stronger lower bound. This coloring is defined by assigning each color to a unique modulo class mod k. For example, if $k = 4$, then our coloring would look as follows.

1 2 3 4 5 6 7 8 9 10 11 12

This method of coloring give us our lower bound.

$$\left(\frac{1}{12} - \frac{1}{3k} + \frac{\theta}{k^2}\right) n^3 - O_k(n^2) \leq AR^k_{X+Y=Z+T}(n)$$

Upper Bound for $k > 4$

- Count the exact number Sidon 4 sets in $[n]$, which can be shown to be

$$\frac{n^3}{12} - \frac{3n^2}{8} + \frac{5n}{12} - \theta$$

where $\theta = 0$ if n is even, and $\theta = \frac{1}{8}$ if n is odd.

- Obtain lower bound for the number of non-rainbow solutions.
    - Given two elements, a and b, there are at least $\frac{n}{2} - 4$ Sidon 4-sets that contain a and b.

$$a + b = x + y \mid a + x = b + y$$

    - Use worst case analysis and averaging arguments to obtain lower bound on the amount of non-rainbow Sidon 4-sets.
    - Combine to obtain upper bound on $AR^k_{X+Y=Z+T}(n)$.

Upper Bound for $k > 4$

Using this method we obtain that there are at least $\frac{n^3}{24k} - O_k(n^2)$ non-rainbow solutions, which in turn gives us the upper bound

$$AR^k_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{24k}\right) n^3 + O_k(n^2).$$

Upper Bound for $k = 4$

In the case that $k = 4$, we use a different argument to strengthen the upper bound. This argument relies on a key lemma from a paper by Lev.

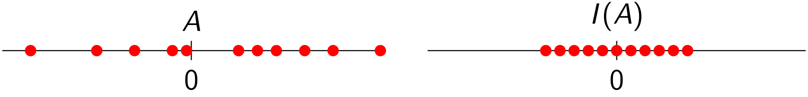
Let the function $E_t(A_1, A_2, \ldots, A_t)$ be defined by $|\{(a_1, a_2, \ldots, a_t) \in A_1 \times A_2 \times \cdots \times A_t : a_1 + a_2 + \cdots + a_t = 0\}|$.

Lemma (Lev 1998)

Let $t \geq 2$ be an integer. For any finite sets $A_1, A_2, \ldots, A_t \subset \mathbb{Z}$,

$$E_t(A_1, A_2, \ldots, A_t) \leq E_t(I(A_1), I(A_2), \ldots, I(A_t))$$

where $I(A)$ is the set of integers within the interval $(\frac{-|A|}{2}, \frac{|A|}{2})$.



Upper Bound for $k = 4$

Let $j = \frac{|A_1|+|A_2|+|A_3|+|A_4|}{4}$, and define J to be the set of integers contained within the interval $(\frac{-j}{2}, \frac{j}{2})$.

Lemma (Taranchuk and Timmons 2018)

Let $A_1, A_2, A_3, A_4 \subset [n]$ *, then*

$$E_4(I(A_1), I(A_2), I(A_3), I(A_4)) \leq E_4(J, J, J, J)$$

In our case, the sets we are working with a partition of $[n]$ then $E_4(J, J, J, J)$ stays the same regardless of the coloring. Thus computing $E_4(J, J, J, J)$ and applying it gives us our upper bound. Together with the lower bound we obtain that

$$\frac{2n^3}{96} - O(n^2) \leq AR^4_{X+Y=Z+T}(n) \leq \frac{3n^3}{96} + O(n^2).$$

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Thanks!