On the Spectrum of Wenger Graphs

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Introduction

Definition

A **simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is the set of vertices, and $E \subset \binom{V}{2}$ is the set of edges. Since Γ is simple, the above definition removes the possibility of multiple edges, directed edges, and loops.

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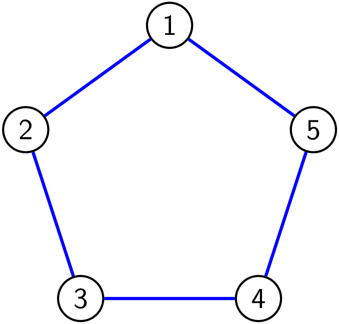
Definition

Let $V = \{1, 2, \ldots, n\}$. Define the adjacency matrix $A = (a_{ij})$ of a simple graph $\Gamma = \Gamma(V, E)$ as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

This matrix records the adjacency relations between vertices in Γ.

Example



$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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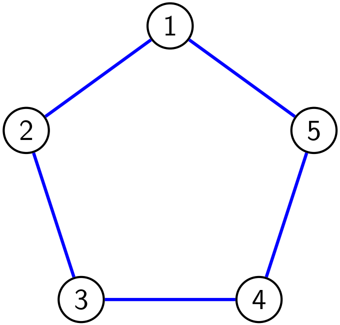
Let G be a finite abelian group with $S \subset G \setminus \{0\}$. Let Γ be a graph with the vertex set G and if $x, y \in G$, then $x \sim y$ if and only if $y - x \in S$. Then Γ is called a **Cayley Graph** of G with **generating set** S. If $S = -S$, then Γ is undirected.

Example



The 5 cycle is a Cayley Graph of $(\mathbb{Z}_5, +)$ with generating set $S = \{1, -1\}$.

Definition (Wenger Graph)

Let $P = L = \mathbb{F}_q^{m+1}$ be two copies of the $(m+1)$-dimensional vector space over $\mathbb{F}_q$ with $q = p^r$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $W_m(q)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_{m+1}) \in P$ and $[l] = [l_1, \ldots, l_{m+1}]$, then $(p) \sim [l]$ if and only if

$$
\begin{aligned}
l_2 + p_2 &= l_1 p_1, \\
l_3 + p_3 &= l_1 p_1^2, \\
&\vdots \\
l_{m+1} + p_{m+1} &= l_1 p_1^m
\end{aligned}
$$

Main Result

Theorem (Cioabă, Lazebnik, Li 2014)

For all prime powers q and $1 \leq m \leq q-1$, the distinct eigenvalues of $W_m(q)$ are

$$\pm q, \pm\sqrt{mq}, \pm\sqrt{(m-1)q}, \ldots, \pm\sqrt{2q}, \pm\sqrt{q}, 0.$$

The multiplicity of the eigenvalue $\pm\sqrt{iq}$ of $W_m(q)$, $0 \leq i \leq m$, is

$$(q-1)\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}.$$

Why is this interesting?

- Wenger-type graphs have been used in producing simple and asymptotically best known constructions for $\mathrm{ex}(n, C_{2k}) = \max\{|E(G)| : |V(G)| = n, C_{2k} \not\subset G\}$. In fact for $k = 2, 3, 5$, these graphs have a matching order of growth as the general known upper bound, giving us $\mathrm{ex}(n, C_{2k}) \sim \theta(n^{1+1/k})$ for these k.
- In general, finding the spectrum of increasingly large graphs is a difficult problem. The main result of this paper completely resolves the question by not only finding the eigenvalues of this family of graphs but their multiplicities as well.

Proof Outline

We need some key facts and observations:

- Since $W_m(q)$ is bipartite, we may order the vertices in a manner so that the adjacency matrix of $W_m(q)$ is

$$A = \begin{array}{c c} & \begin{array}{cc} P & L \end{array} \\ \begin{array}{c} P \\ L \end{array} & \begin{pmatrix} 0 & N^T \\ N & 0 \end{pmatrix} \end{array} \implies A^2 = \begin{array}{c c} & \begin{array}{cc} P & L \end{array} \\ \begin{array}{c} P \\ L \end{array} & \begin{pmatrix} N^T N & 0 \\ 0 & NN^T \end{pmatrix} \end{array}$$

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- We have then that $N^T N$ and NN^T have the same spectrum so that the spectrum of A^2 is completely determined by the spectrum of NN^T.
- Since $W_m(q)$ is bipartite, the spectrum is symmetric around 0, including multiplicity. Then λ an eigenvalue of A^2 implies $\pm\sqrt{\lambda}$ are eigenvalues of A.

Proof Outline

- Recall $(p) \sim [l]$ in $W_m(q)$ if and only if $p_i + l_i = p_1^i l_1$ for $2 \leq i \leq m+1$.

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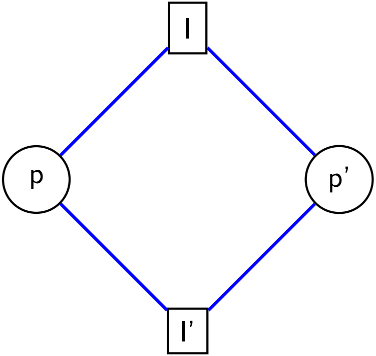
- Recall $(p) \sim [l]$ in $W_m(q)$ if and only if $p_i + l_i = p_1^i l_1$ for $2 \leq i \leq m+1$.
- $W_m(q)$ is q regular, which implies diagonal entries of A^2 are q.
- $W_m(q)$ contains no 4-cycles, which implies off-diagonal entries of A^2 are either 0 or 1.



Proof Outline

- If $NN^T = H + qI$, then H is the adjacency matrix of the point graph of $W_m(q)$ on L, call it Γ, in which $[l] \sim [l']$ if there exists $(p) \in P$ so that $[l] \sim (p) \sim [l']$ in $W_m(q)$.

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- Now $[l] \sim [l']$ in Γ implies there exists a $(p) \in P$ so that $l_{i+1} + p_{i+1} = l_1 p_1^i$ and $l'_{i+1} + p_{i+1} = l'_1 p_1^i$ which implies

$$l_{i+1} - l'_{i+1} = p_1^i (l_1 - l'_1)$$

So as vectors in $l, l' \in \mathbb{F}_q^{m+1}$, the two vertices in Γ are adjacent if and only if $l - l' = (t, tu, \ldots, tu^m)$, where $t = l_1 - l'_1$ and $u = p_1$, so that the edge relation is entirely determined by t and u. So we see that in fact Γ is the Cayley graph of the group $(\mathbb{F}_q^{m+1}, +)$ with generating set $S = \{(t, tu, \ldots, tu^m) : t \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}$.

Proof Outline

Lemma

Let Γ be the Cayley graph of a group G, with $|G| = n$, and generating set S. If $\chi : G \to \mathbb{C}^*$ is a character, and $x \in \mathbb{C}^G = \mathbb{C}^n$, with $x_a = \chi(a)$, then x an eigenvector of Γ with eigenvalue

$$\sum_{s \in S} \chi(s).$$

In fact every eigenvector and eigenvalue is of this form.

- Then with Γ the point graph on L, $G = (\mathbb{F}_q^{m+1}, +)$ and S as given the eigenvalues are

$$\lambda_{(w_1,\ldots,w_{m+1})} = \sum_{(t,tu,\ldots,tu^m) \in S} \omega^{\mathrm{tr}(tw_1)} \cdot \omega^{\mathrm{tr}(tuw_2)} \cdots \omega^{\mathrm{tr}(tu^m w_{m+1})}$$

where ω is a pth root of unity and $\mathrm{tr}(x) = \sum_{i=1}^{r} x^{p^i}$.

Proof Outline

- For any $\alpha \in \mathbb{F}_q$, $\mathrm{tr}(\alpha) \in \mathbb{F}_p$ and acts as a linear functional from $\mathbb{F}_q$ onto $\mathbb{F}_p$. So we have that

$$\sum_{(t,tu,\ldots,tu^m)\in S} \omega^{\mathrm{tr}(tw_1)} \cdot \omega^{\mathrm{tr}(tuw_2)} \ldots \omega^{\mathrm{tr}(tu^m w_{m+1})}$$
$$= \sum_{(t,tu,\ldots,tu^m)\in S} \omega^{\mathrm{tr}(t(w_1+w_2u+\cdots+w_{m+1}u^m))} = \sum_{(t,tu,\ldots,tu^m)\in S} \omega^{\mathrm{tr}(tf(u))}$$

where $f(u) = w_1 + w_2u + \cdots + w_{m+1}u^m$. Where finally we write

$$\sum_{(t,tu,\ldots,tu^m)\in S} \omega^{\mathrm{tr}(tf(u))} = \sum_{t\in\mathbb{F}_q^*, f(u)=0} \omega^{\mathrm{tr}(tf(u))} + \sum_{t\in\mathbb{F}_q^*, f(u)\neq 0} \omega^{\mathrm{tr}(tf(u))}$$

Proof Outline

- We have

$$\sum_{t \in \mathbb{F}_q^*} \omega^{\mathrm{tr}(tx)} = \begin{cases} q-1 & \text{if } x = 0 \\ -1 & \text{otherwise} \end{cases}$$

which leads us to

$$\lambda_{(w_1,\ldots,w_{m+1})} = |\{u \in \mathbb{F}_q : f(u) = 0\}|(q-1) - |\{u \in \mathbb{F}_q : f(u) \neq 0\}|.$$

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- If $f = 0$ given then the corresponding eigenvalue is $\lambda_{(0,0,\ldots,0)} = q(q-1)$
- If $f \neq 0$, then the size of each set is given by the number of distinct roots f has in $\mathbb{F}_q$. If f has i distinct many roots in $\mathbb{F}_q$, where $0 \leq i \leq m$ in $\mathbb{F}_q$, then the corresponding eigenvalue is $i(q-1) - (q-i) = iq - q$.

Proof Outline

- So the eigenvalues of H the adjacency matrix of Γ are either $q(q-1)$ or $iq - q$ for $0 \le i \le m$. Now $NN^T = H + qI$ implies the eigenvalues of NN^T are also the eigenvalues of A^2 which we find to be q^2 and iq for $0 \le i \le m$. So the eigenvalues of A are $\pm q, \pm\sqrt{mq}, \pm\sqrt{(m-1)q}, \ldots, \pm\sqrt{2q}, \pm\sqrt{q}, 0$ as claimed.

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- $\pm q$ necessarily have multiplicity one. To count multiplicity of $\sqrt{iq}$, we must count the number of polynomials of degree at most m having exactly i disinct roots in $\mathbb{F}_q$. By a lemma from "Counting Polynomials with a given number of zeros in a finite field" by A. Knopfmacher and J. Knopfmacher, we obtain the desired result,the multiplcity of $\pm\sqrt{iq}$ is exactly

$$(q-1)\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}.$$

Consequences

- As a consequence of the sum of multiplicities summing to the order of the graph, the following identity is obtained.

Corollary (Cioabă, Lazebnik, Li 2014)

For every prime power q and $1 \leq m \leq q - 1$,

$$\sum_{i=0}^{m} \binom{q}{i} \sum_{d=i}^{m} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k} = \frac{q^{m+1} - 1}{q - 1}.$$

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- It is also the case that whenever $m \geq q$, we have that $W_m(q)$ is a disconnected graph with each component isomorphic to $W_{q-1}(q)$ and having q^{m-q+1} many such components, giving us the full spectrum with multiplcity for $W_m(q)$ for $m \geq q$ by application of the main result.

Questions?

Thanks!