Algebraically Defined Graphs and their Applications

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Introduction

Definition

A **simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where $V = V(\Gamma)$ is the set of vertices, and $E = E(\Gamma) \subset \binom{V}{2}$ is the set of edges. We denote the fact that a vertex x is **adjacent** to a vertex y by $x \sim y$. Since Γ is simple, the above definition removes the possibility of multiple edges, directed edges, and loops. In this talk, every graph mentioned will be a simple graph.

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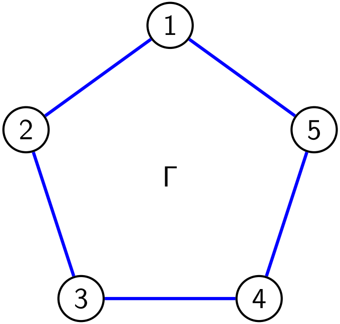
For a graph Γ, denote the group of all automorphisms of Γ by $\mathrm{Aut}(\Gamma)$.

Example



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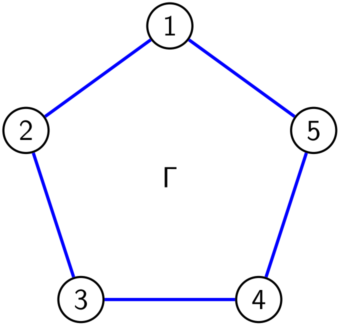

Facts about Γ:

- $V(\Gamma) = \{1, 2, 3, 4, 5\}$.

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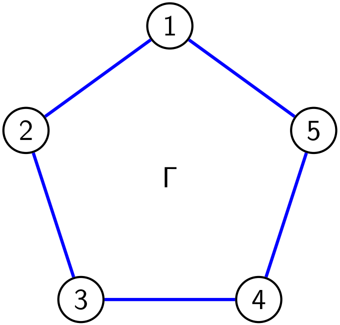

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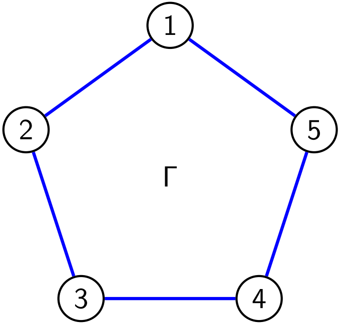

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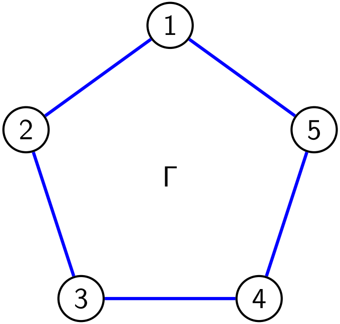

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- Γ is a 5-cycle.
- The permutation $\sigma = (12345)$ is an automorphism of Γ.

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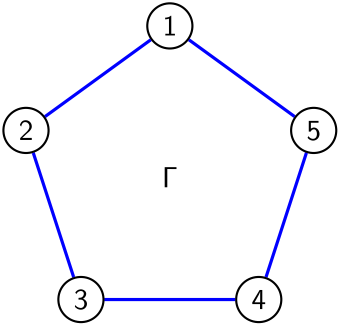
Definition

Let Γ be a graph and $x \in V$. The **degree** of x is the number of vertices y such that $y \sim x$. We say that Γ is **regular** if every vertex has the same degree.

Example - Your Turn!



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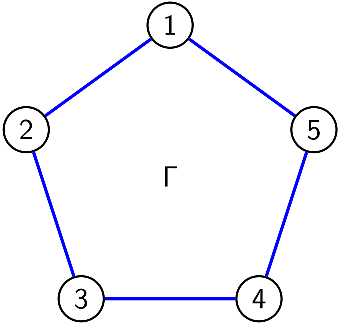

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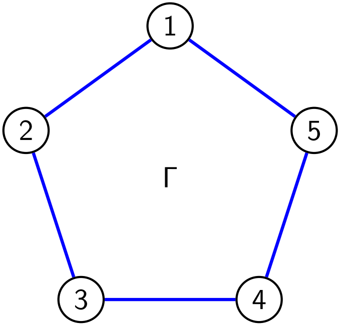

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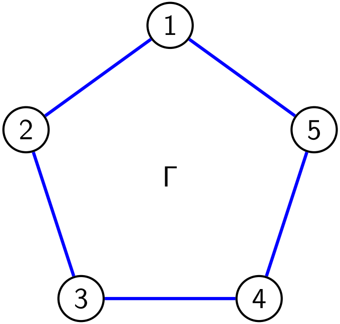

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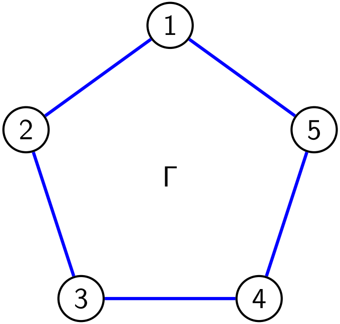

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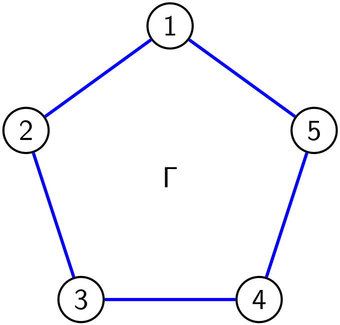

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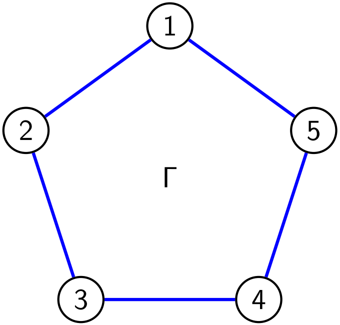

Questions about Γ:

- What is the maximum distance between any two vertices of Γ? 2 ✓
- What is the girth of Γ? 5 ✓
- Is Γ regular? Yes, every vertex has degree 2 ✓

Introduction

Definition (Algebraically Defined Graphs)

Let $P = L = \mathbb{F}_q^m$ be two copies of the m-dimensional vector space over $\mathbb{F}_q$ with $q = p^e$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $\Gamma_q = \Gamma_q(f_2, f_3, \ldots, f_m)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_m) \in P$ and $[\ell] = [\ell_1, \ldots, \ell_m]$, then $(p) \sim [l]$ if and only if

$$\begin{aligned} \ell_2 + p_2 &= f_2(\ell_1, p_1) \\ \ell_3 + p_3 &= f_3(\ell_1, p_1, \ell_2, p_2) \\ &\vdots \\ \ell_m + p_m &= f_m(\ell_1, p_1, \ldots, \ell_{m-1}, p_{m-1}) \end{aligned}$$

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So $[\ell_1, \ell_2, \ldots, \ell_m] \sim (x, p_2, \ldots, p_m)$

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- For each $b \in \mathbb{F}_q$, there exists an automorphism $t_b \in \mathrm{Aut}(\Gamma_q)$ given by

$$t_b[\ell_1, \ell_2, \ldots, \ell_m] = [\ell_1, \ell_2, \ldots, \ell_m + b]$$
$$t_b(p_1, p_2, \ldots, p_m) = (p_1, p_2, \ldots, p_m - b).$$

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- It has been shown that $\text{ex}(n, C_{2k}) \leq c_k n^{1+1/k}$ for a constant dependent on k. Bondy and Simonovits showed $c_k = 100k$ works, and over time this constant has been improved several times, first by Verstraëte, then Pikhurko, and most recently by Bukh and Jiang who showed $c_k = 80\sqrt{k \log k}$.

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- Lazebnik, Ustimenko, and Woldar used algebraically defined graphs to show that for infinitely many n, $c'_k n^{1+2/(3k+3+\epsilon)} \leq \text{ex}(n, C_{2k})$ where $\epsilon = 1$ when k is odd and $\epsilon = 0$ otherwise.

Example: C_4-free graph

Let $\Gamma_q = \Gamma_q(p_1\ell_1)$, be an algebraically defined graph. This graphs has $P = L = \mathbb{F}_q^2$ for any prime power q. The edge relation is $[\ell_1, \ell_2] \sim (p_1, p_2)$ iff

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Suppose we have a cycle, meaning there are points $(p) = (p_1, p_2)$, $(q) = (q_1, q_2)$ and lines $[\ell] = [\ell_1, \ell_2], [k] = [k_1, k_2]$ such that

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Suppose we have a 4-cycle, meaning there are points $(p) = (p_1, p_2)$, $(q) = (q_1, q_2)$ and lines $[\ell] = [\ell_1, \ell_2], [k] = [k_1, k_2]$ such that

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$$p_2 - q_2 = (p_1 - q_1)\ell_1 = (p_1 - q_1)k_1 \implies \ell_1 = k_1 \text{ X}$$

Example: C_4-free graph

So $\Gamma_q(p_1\ell_1)$ is C_4-free. The number of edges in this graph provides a suitable lower bound for $\mathrm{ex}(n, C_4)$ where $n = 2q^2$. So in particular, this graph gives

$$\frac{\sqrt{2}}{4} n^{\frac{3}{2}} \leq \mathrm{ex}(n, C_4)$$

when $n = 2q^2$. Here the magnitude of the lower bound matches that of the upper bound. As it turns out, $\Gamma_q(p_1, \ell_1)$ has girth 6. There are many known non-isomorphic graphs $\Gamma_q(f_2(p_1, \ell_1))$ that have girth 6.

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Theorem

Let $k = 2, 3, 5$, *then*

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Let q be an odd prime power. The graph $\Gamma_q(p_1\ell_1, p_1\ell_2)$ is the only known example of a q-regular graph with girth 8, in particular, girth 8 implies it does not have C_6 as a subgraph. So this particular graph contributes to the theorem above for $k = 3$.

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Do there exist other graphs that are q-regular and with girth 8?

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Do there exist other graphs that are q-regular and with girth 8?

Algebraically defined graphs provide one potential method for answering this question. In large part, this question motivated the research conducted by Dr. Lazebnik and I this summer.

Results

Question

Given $\Gamma_q(f_2, f_3)$ *where* $f_2 = f_2(p_1, \ell_1)$ *and* $f_3(p_1, \ell_1, p_2, \ell_2)$*, does there exist a function* $f_3' = f_3'(p_1, \ell_1)$ *so that* $\Gamma_q(f_2, f_3) \cong \Gamma_q(f_2, f_3')$*?*

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Let $\Gamma = \Gamma_q(p_1\ell_1, p_1\ell_1 p_2(p_1 + p_2 + p_1 p_2))$. This family of graphs appeared in the thesis of Nassau 2020. It was checked via computer that for all odd prime powers $q < 43$, there do not exist functions f_2, f_3 of just p_1 and ℓ_1 such that $\Gamma \cong \Gamma_q(f_2, f_3)$. Though he could not prove it for any infinite sequence of q's.

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Theorem (Lazebnik and T. 2021+)

Let p be an odd prime with $p \equiv 1 (\mod 3)$. Let $f_2 = f_2(p_1, \ell_1)$ and $f_3 = f_3(p_1, \ell_1)$ be functions of p_1 and ℓ_1. Then $\Gamma_q(f_2, f_3) \ncong \Gamma$.

Questions?

Thanks!