On a new family of algebraically defined graphs with small automorphism group

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Introduction

Definition

A **simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is the set of vertices, and $E \subset \binom{V}{2}$ is the set of edges. We denote the fact that a vertex x is **adjacent** to a vertex y by $x \sim y$. Since Γ is simple, the above definition removes the possibility of multiple edges, directed edges, and loops. In this talk, every graph mentioned will be a simple graph.

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For a graph Γ, denote the group of all automorphisms of Γ by $\mathrm{Aut}(\Gamma)$.

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Definition (Algebraically Defined Graphs)

Let $P = L = \mathbb{F}_q^m$ be two copies of the m-dimensional vector space over $\mathbb{F}_q$ with $q = p^e$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $\Gamma_q = \Gamma_q(f_2, f_3, \ldots, f_m)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_m) \in P$ and $[\ell] = [\ell_1, \ldots, \ell_m]$, then $(p) \sim [l]$ if and only if

$$\begin{aligned} \ell_2 + p_2 &= f_2(\ell_1, p_1) \\ \ell_3 + p_3 &= f_3(\ell_1, p_1, \ell_2, p_2) \\ &\vdots \\ \ell_m + p_m &= f_m(\ell_1, p_1, \ldots, \ell_{m-1}, p_{m-1}) \end{aligned}$$

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So $[\ell_1, \ell_2, \ldots, \ell_m] \sim (x, p_2, \ldots, p_m)$

Properties of algebraically defined graphs

- Γ_q is q-regular with $|V| = n = 2q^m$ and

$$|E| = q^{m+1} = \left(\frac{n}{2}\right)^{\frac{m+1}{m}}$$

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- Γ_q is q-regular with $|V| = n = 2q^m$ and

$$|E| = q^{m+1} = \left(\frac{n}{2}\right)^{\frac{m+1}{m}}$$

- For each $b \in \mathbb{F}_q$, there exists an automorphism $t_b \in \mathrm{Aut}(\Gamma_q)$ given by

$$t_b[\ell_1, \ell_2, \ldots, \ell_m] = [\ell_1, \ell_2, \ldots, \ell_m + b]$$
$$t_b(p_1, p_2, \ldots, p_m) = (p_1, p_2, \ldots, p_m - b).$$

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Definition

A **generalized** n**-gon** of order $q \geq 1$ is a $(q+1)$-regular bipartite graph with diameter $n \geq 2$ and girth $2n$.

- $q = 1$, $n \geq 2$: C_{2n}
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For $n = 3, 4, 6$, there exists a generalized n-gon of order q for every prime power q.

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Theorem (Tits 1959)

For $n = 3, 4, 6$, there exists a generalized n-gon of order q for every prime power q.

Theorem (Feit and Higman 1964)

There do no exist any generalized n-gons of any order q when $n \notin \{2, 3, 4, 6, 8\}$.

Why are these graphs interesting?

A generalized 3-gon is called a **projective plane**. Let $f_2 = f_2(p_1, \ell_1)$ and consider $\Gamma_q = \Gamma_q(f_2)$. Recall $(p_1, p_2) \sim [\ell_1, \ell_2]$ iff $p_2 + \ell_2 = f_2(p_1, \ell_1)$. If Γ_q has girth 6, there is a unique way to obtain a projective plane from Γ_q.

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- All André Planes can be represented this way by using $f_2(p_1, \ell_1) = p_1 \star \ell_1$ where $\star$ is the multiplication used in a particular quasifield. André planes account for many non-isomorphic classes of projective planes.

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Question

What other families of projective planes could be constructed via $\Gamma_q(f_2)$?

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Do there exist generalized quadrangles of odd order q that are not isomorphic to the classical generalized quadrangle?

Algebraically defined graphs provide one potential method for answering this question. In large part, this question motivated our research.

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- It has been shown that $\text{ex}(n, C_{2k}) \leq c_k n^{1+1/k}$ for a constant dependent on k. Bondy and Simonovits showed $c_k = 100k$ works, and over time this constant has been improved several times, first by Verstraëte, then Pikhurko, and most recently by Bukh and Jiang who showed $c_k = 80\sqrt{k \log k}$.

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- Lazebnik, Ustimenko, and Woldar used algebraically defined graphs to show that for infinitely many n, $c'_k n^{1+2/(3k+3+\epsilon)} \leq \text{ex}(n, C_{2k})$ where $\epsilon = 1$ when k is odd and $\epsilon = 0$ otherwise.

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Question

Given $\Gamma_q(f_2, f_3)$ *where* $f_2 = f_2(p_1, \ell_1)$ *and* $f_3(p_1, \ell_1, p_2, \ell_2)$*, does there exist a function* $f_3' = f_3'(p_1, \ell_1)$ *so that* $\Gamma_q(f_2, f_3) \cong \Gamma_q(f_2, f_3')$*?*

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Let $\Gamma = \Gamma_q(p_1\ell_1, p_1\ell_1 p_2(p_1 + p_2 + p_1 p_2))$. This family of graphs appeared in the thesis of Nassau 2021. It was checked via computer that for all odd prime powers $q < 43$, there do not exist functions f_2, f_3 of just p_1 and ℓ_1 such that $\Gamma \cong \Gamma_q(f_2, f_3)$. Though he could not prove it for any infinite sequence of q's.

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Theorem (Lazebnik and T. 2021+)

Let p be an odd prime with $p \equiv 1 (\mod 3)$. Let $f_2 = f_2(p_1, \ell_1)$ and $f_3 = f_3(p_1, \ell_1)$ be functions of p_1 and ℓ_1. Then $\Gamma_q(f_2, f_3) \not\cong \Gamma$.

Results and methods

Consider any graph of the form $\Gamma_q(f_2(p_1, \ell_1), f_3(p_1, \ell_1))$. Observe that since

$$p_2 + \ell_2 = f_2(p_1, \ell_1)$$
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then for all $a, b \in \mathbb{F}_q$, the function $t_{a,b}$ where

$$t_{a,b}[\ell_1, \ell_2, \ell_3] = [\ell_1, \ell_2 + a, \ell_3 + b]$$
$$t_{a,b}(p_1, p_2, p_3) = (p_1, p_2 - a, p_3 - b)$$

is an automorphism of Γ_q. Meaning, $q^2 \leq |\mathrm{Aut}(\Gamma_q)|$.

Results and methods

For the rest of this talk, let $\Gamma = \Gamma_q(p_1\ell_1, p_1\ell_1 p_2(p_1 + p_2 + p_1 p_2))$.

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The proof is broken into two main parts:

- The first part works for prime powers $q \equiv 1 (\mathrm{mod}\ 3)$.
- The second part works when q is prime, under the assumption of the first part.

Proof Outline

- Nassau showed that
  - $r_3([0,1,0]) = q^3 - 4q^2 + 9q - 8$
  - $r_3([0,0,0]) = q^3 - 4q^2 + 8q - 6$

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- We show that that for all vertices x in Γ that are not of the form $[0,1,r]$ or $[0,0,r]$

$$r_3(x) < r_3([0,0,0])$$

- Therefore, for all $\phi \in \mathrm{Aut}(\Gamma)$:
  - $\phi[0,1,r] = [0,1,s]$
  - $\phi[0,0,r] = [0,0,t]$

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Vertices at distance three of $[A, B, 0]$ can be described as follows:

$$[A, B, 0] \sim (a, \star, \star) \sim [x, \star, \star] \sim (b, c, P_{A,B}(b, c; a)/(b - a))$$

where $P_{A,B}(b, c; a)$ is a 4th degree polynomial in a.

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where $P_{A,B}(b, c; a)$ is a 4th degree polynomial in a. In particular:

$$\begin{aligned} P_{A,B}(b, c; a) &= A^2(Ab - B - c)a^4 \\ &+ A(A - 2B + 1)(Ab - B - c)a^3 \\ &- B(2A - B + 1)(Ab - B - c)a^2 \\ &- (Ac^2b^2 - AB^2b + Ac^2b + ACb^2 + B^3 + B^2c)a \\ &+ cb(cb + c + b)(c + B). \end{aligned}$$

Proof Outline

The set of all distance three neighbors of lines of the form $[A, B, 0]$ is given by

$$\left\{\left(b, c, \frac{P_{A,B}(b,c;a)}{b-a}\right) : \quad a, b, c \in \mathbb{F}_q, c \neq Ab - B, a \neq b\right\}.$$

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To bound $r_3([A, B, 0])$ we need to bound the range of $P_{A,B}(b, c; a)/(b-a)$.

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The set of all distance three neighbors of lines of the form $[A, B, 0]$ is given by

$$\left\{\left(b, c, \frac{P_{A,B}(b, c; a)}{b-a}\right) : \quad a, b, c \in \mathbb{F}_q, c \neq Ab - B, a \neq b\right\}.$$

To bound $r_3([A, B, 0])$ we need to bound the range of $P_{A,B}(b, c; a)/(b-a)$.

Theorem (Lazebnik and T. 2021+)

Let $q \equiv 1 (mod\ 3)$ be an odd prime power, then the rational function

$$x^3 + c_2 x^2 + c_1 x + \frac{c_{-1}}{x}$$

with $c_2, c_1, c_{-1} \in \mathbb{F}_q$ has range with size at most $q - 3$.

Proof Outline

Note that for all $x \in \mathbb{F}_q$, $x \neq 0$:

$$x^3 + c_2x^2 + c_1x + \frac{c_{-1}}{x} = x^3 + c_2x^2 + c_1x + c_{-1}x^{q-2}.$$

Proof Outline

Note that for all $x \in \mathbb{F}_q$, $x \neq 0$:

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Theorem (Hermite's Criterion)

Let $q = p^e$ be a prime power. If $p(x)^t (\text{mod } x^q - x)$ has degree $q - 1$ for some $t \not\equiv 0 (\text{mod } p)$, then $p(x)$ is not a permutation polynomial of $\mathbb{F}_q$.

Proof Outline

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Theorem (Wan 1993)

If a polynomial $p(x)$ of degree n is not a permutation polynomial of $\mathbb{F}_q$, then

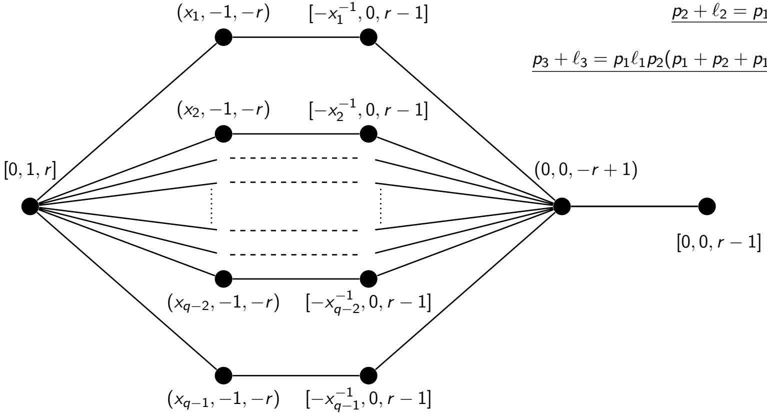
$$|Rng(p(x))| \leq q - \left\lceil \frac{q-1}{n} \right\rceil.$$

Proof Outline

Observe the following structure in Γ.

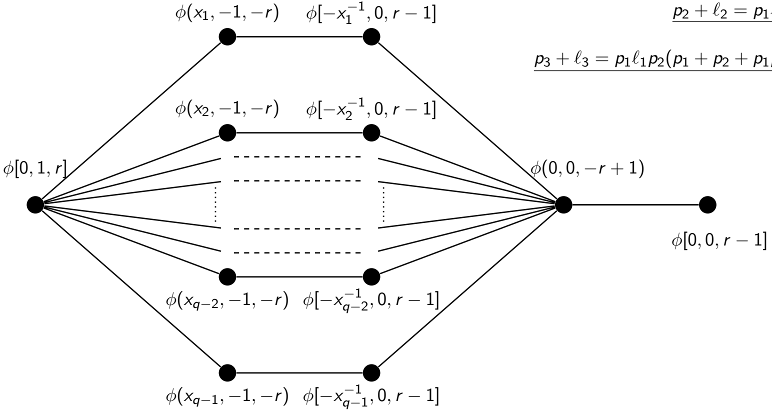
$\underline{p_2 + \ell_2 = p_1\ell_1}$

$\underline{p_3 + \ell_3 = p_1\ell_1 p_2(p_1 + p_2 + p_1p_2)}$



Proof Outline

Let $\phi \in \mathrm{Aut}(\Gamma)$ and apply ϕ. Suppose $\phi[0, 1, r] = [0, 1, s]$ and $\phi[0, 0, r - 1] = [0, 0, w]$.



$p_2 + \ell_2 = p_1 \ell_1$

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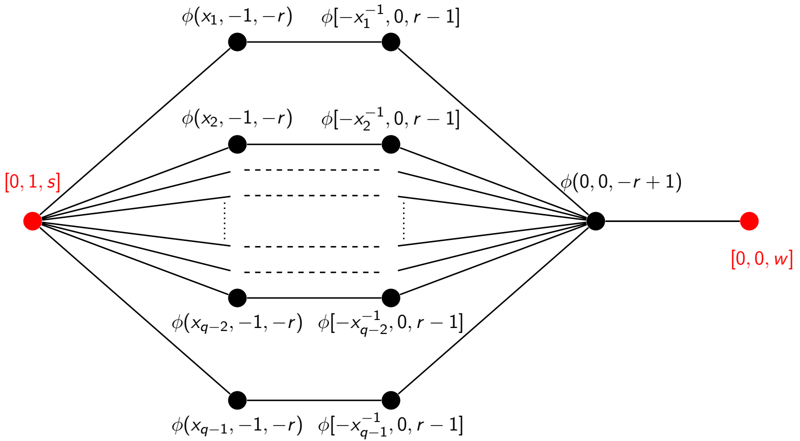
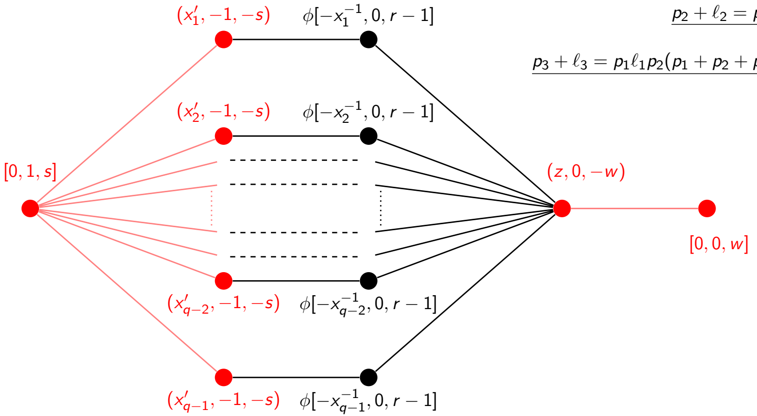
Proof Outline



Proof Outline



$$\underline{p_2 + \ell_2 = p_1 \ell_1}$$

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Proof Outline

$$\underline{p_2 + \ell_2 = p_1\ell_1}$$

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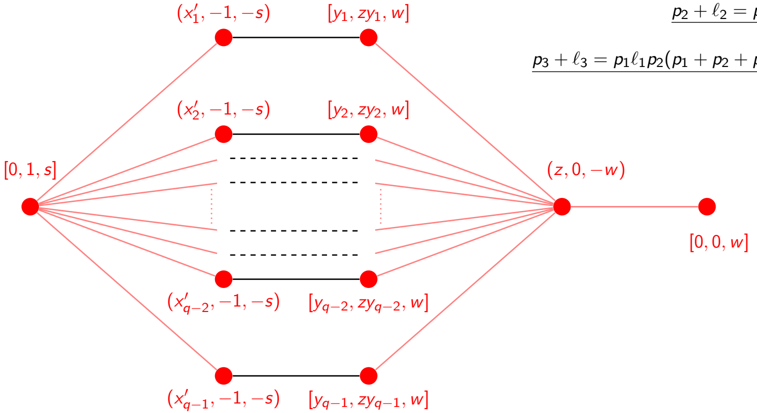
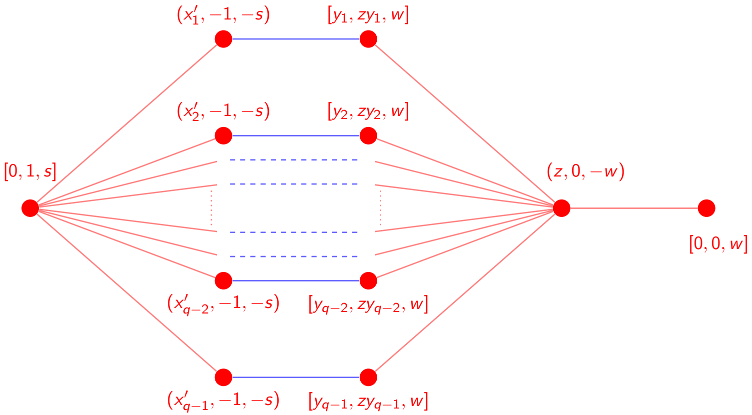


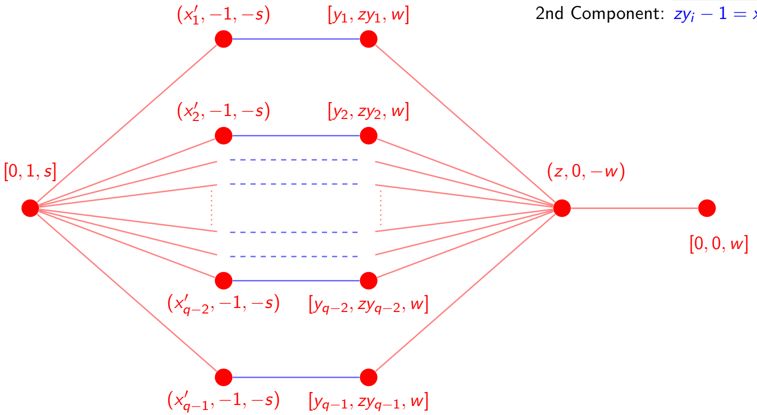
Proof Outline



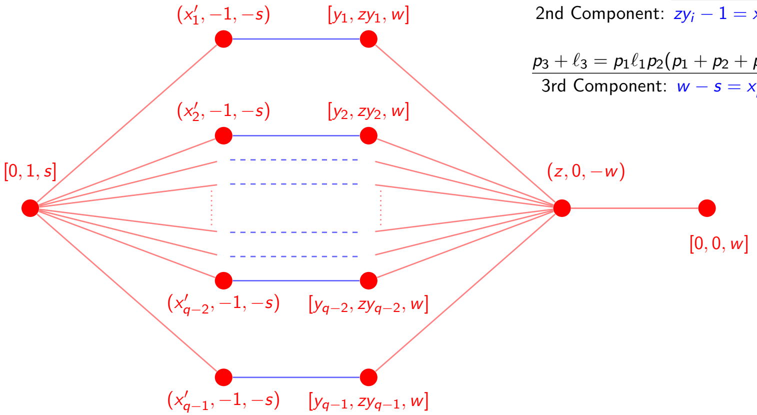
Proof Outline

$$\underline{p_2 + \ell_2 = p_1 \ell_1}$$

2nd Component: $zy_i - 1 = x_i' y_i$



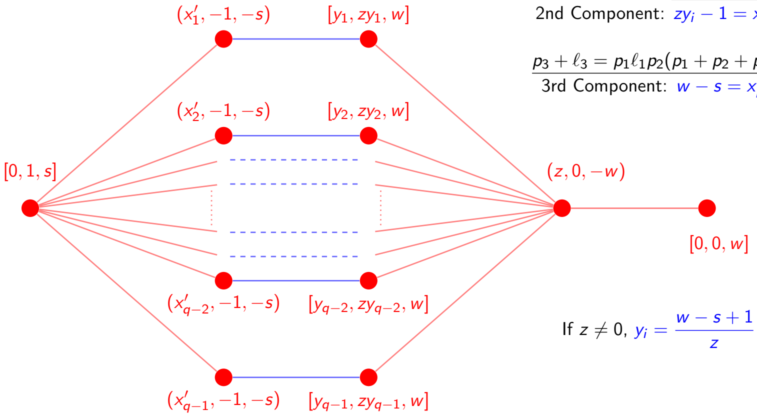
Proof Outline



$$\frac{p_2 + \ell_2 = p_1 \ell_1}{\text{2nd Component: } zy_i - 1 = x_i' y_i}$$

$$\frac{p_3 + \ell_3 = p_1 \ell_1 p_2 (p_1 + p_2 + p_1 p_2)}{\text{3rd Component: } w - s = x_i' y_i}$$

Proof Outline



$\underline{p_2 + \ell_2 = p_1 \ell_1}$

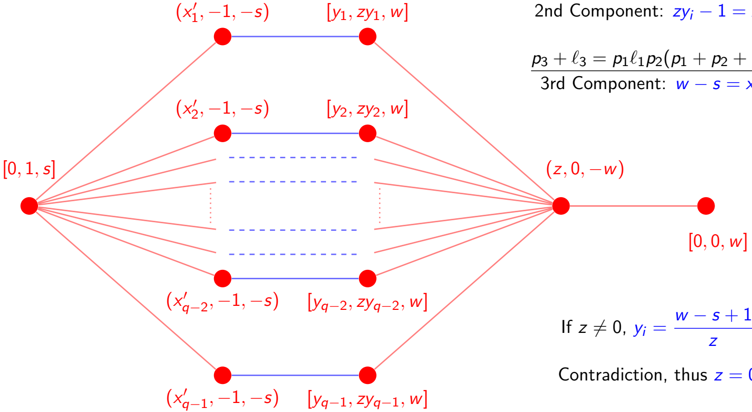
2nd Component: $zy_i - 1 = x'_i y_i$

$\underline{p_3 + \ell_3 = p_1 \ell_1 p_2 (p_1 + p_2 + p_1 p_2)}$

3rd Component: $w - s = x'_i y_i$

If $z \neq 0$, $y_i = \dfrac{w - s + 1}{z}$

Proof Outline



$\underline{p_2 + \ell_2 = p_1\ell_1}$

2nd Component: $zy_i - 1 = x'_i y_i$

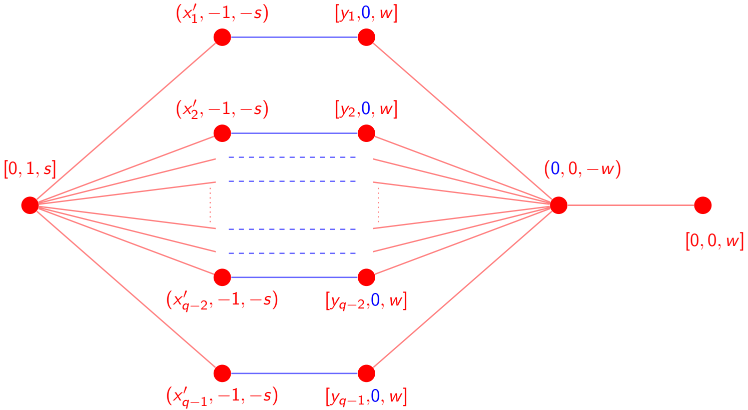
$\underline{p_3 + \ell_3 = p_1\ell_1 p_2(p_1 + p_2 + p_1 p_2)}$

3rd Component: $w - s = x'_i y_i$

If $z \neq 0$, $y_i = \dfrac{w - s + 1}{z}$

Contradiction, thus $z = 0$

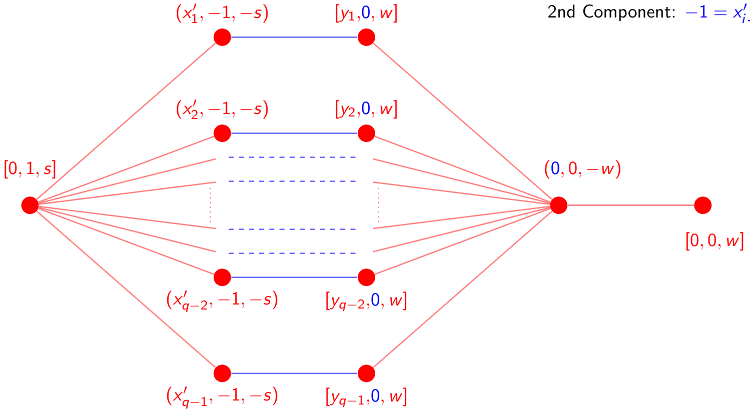
Proof Outline



Proof Outline

$\underline{p_2 + \ell_2 = p_1 \ell_1}$

2nd Component: $-1 = x_i' y_i$



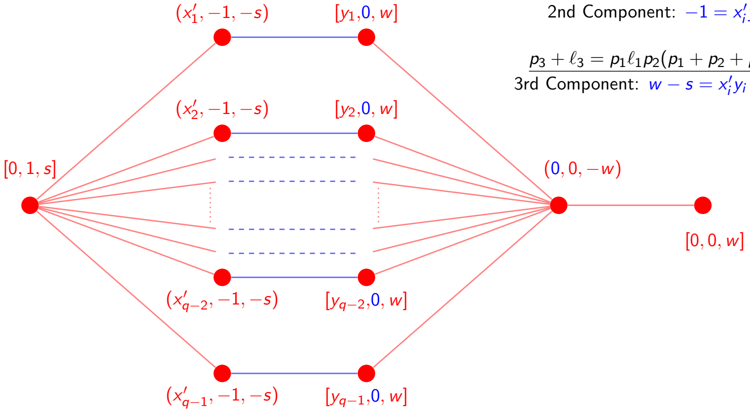
Proof Outline

$\underline{p_2 + \ell_2 = p_1 \ell_1}$

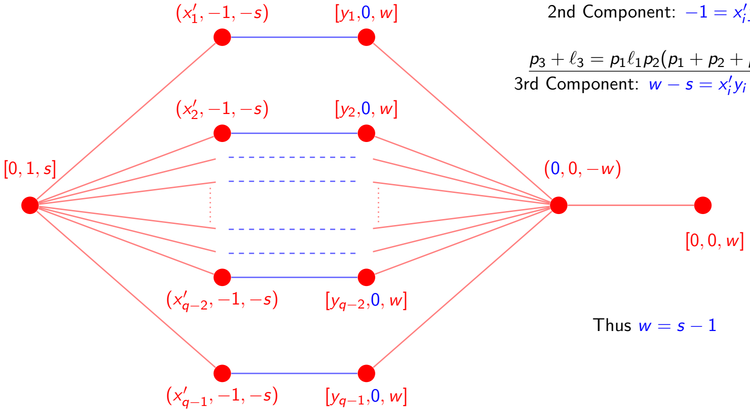
2nd Component: $-1 = x_i' y_i$

$\underline{p_3 + \ell_3 = p_1 \ell_1 p_2 (p_1 + p_2 + p_1 p_2)}$

3rd Component: $w - s = x_i' y_i = -1$



Proof Outline



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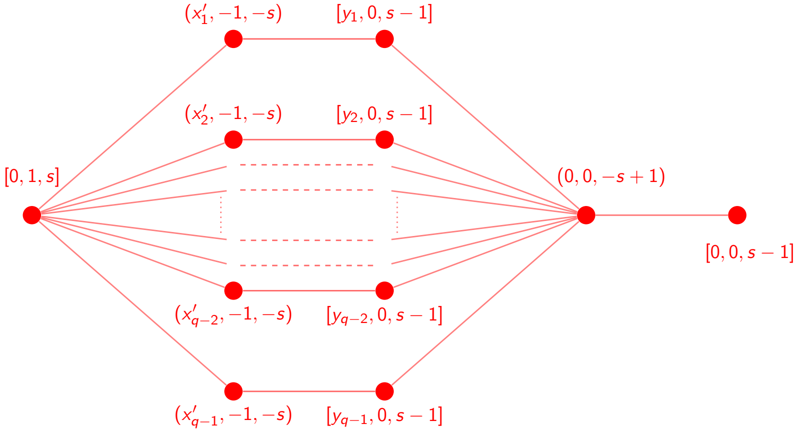
$\underline{p_3 + \ell_3 = p_1 \ell_1 p_2 (p_1 + p_2 + p_1 p_2)}$

3rd Component: $w - s = x_i' y_i = -1$

Thus $w = s - 1$

Proof Outline



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Questions?

Thanks!