On the existence of Pappus configurations in projective planes

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Projective Planes

Definition

A projective plane $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an incidence structure whose elements are the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$ together with an incidence relation $\mathcal{I}$ satisfying the following three axioms:

1. Given any two points there exists a unique line containing them.
2. Any two lines intersect at exactly one point.
3. There exist four points, no three of which are collinear.

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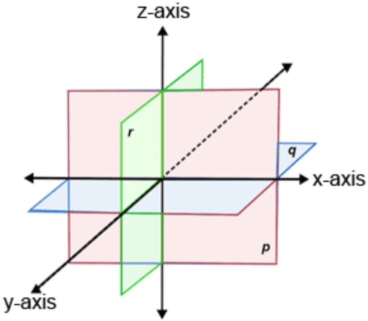
Facts:

- If Π is finite, then every lines has $n + 1$ points (for some n) and we call n the order of Π.
- If Π is finite, then every point lies on $n + 1$ lines.
- If Π is finite, then Π has $n^2 + n + 1$ points and $n^2 + n + 1$ lines.

Classical Projective Planes

Let $\mathbb{F}$ be a field and $\mathbb{F}^3$ be the three dimensional vector space over $\mathbb{F}$. Define the projective plane Π to have $\mathcal{P}$ be the set of all 1-dimensional subspaces of $\mathbb{F}^3$ and the $\mathcal{L}$ be the set of all 2-dimensional subspaces of $\mathbb{F}^3$, where $\mathcal{I}$ is defined naturally via subset containment. Over $\mathbb{R}$, a geometric interpretation of the above gives $\mathcal{P}$ to be the set of all lines through the origin, and $\mathcal{L}$ as all planes through the origin.



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Theorem (Bruck-Ryser)

Let Π be a finite projective plane of order n. If $n \equiv 1 (mod\ 4)$ or $n \equiv 2 (mod\ 4)$, then there cannot exist a projective plane of order n unless n can be expressed as the sum of two square integers.

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Any finite projective plane of prime order is classical.

Configurations

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In a projective plane Π, a *configuration* is a finite collection of points and lines.

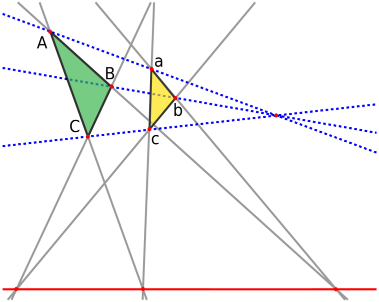

Figure: Desargues Configuration

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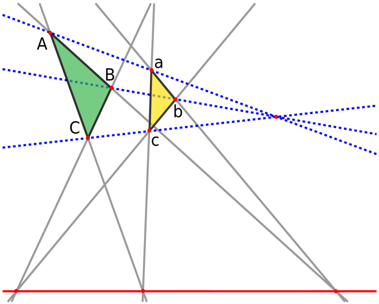

Figure: Desargues Configuration

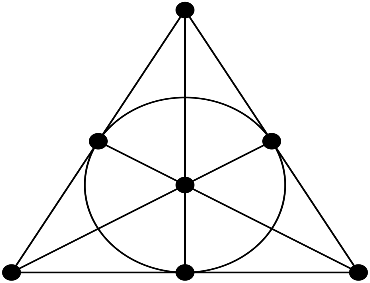

Figure: Fano Configuration

Results on Configurations

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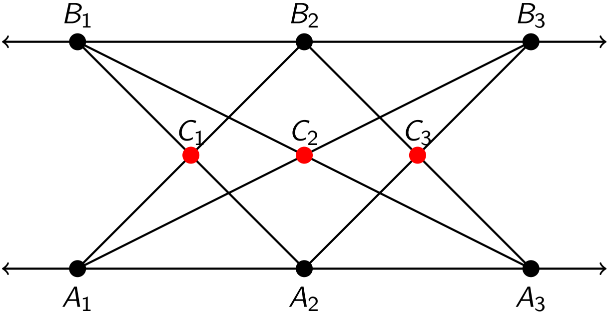
Does every finite projective plane contain a Pappus Configuration?

Pappus Configuration



In a projective plane Π, let ℓ and m be lines and A_1, A_2, A_3 lie on ℓ and B_1, B_2, B_3 lie on m. Obtain the points C_1, C_2, C_3 as in the picture above. If C_1, C_2, C_3 are collinear, then we call this configuration of points and lines a Pappus configuration.

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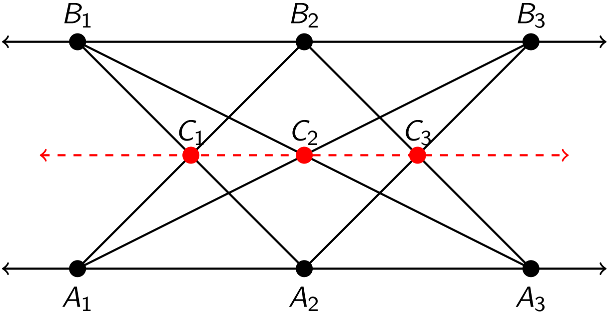

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Collineations and Perspectivities

Definition

A *collineation* of a projective(or affine) plane Π is a bijection $\sigma : \mathcal{P} \to \mathcal{P}$ that maps lines to lines. That is, if $\ell \in \mathcal{L}$ where $\ell = \{P_1, P_2, \dots\}$, then $\sigma(\ell) := \{\sigma(P_1), \sigma(P_2), \dots\} \in \mathcal{L}$.

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Definition

Let σ be a collineation of a projective plane Π. We say σ *fixes* a point P if $\sigma(P) = P$, likewise, we say σ *fixes* a line ℓ if $\sigma(\ell) = \ell$. We say σ fixes a line ℓ *pointwise* if for every $P \in \ell$, $\sigma(P) = P$. We say σ fixes a point P *linewise* if for every line ℓ through P, σ fixes ℓ.

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Definition

Let σ be a collineation of a projective plane Π. If there exists a line ℓ and a point V (not necessarily on ℓ) such that σ fixes ℓ pointwise, and fixes V linewise, then σ is called a (V, ℓ)-*perspectivity*.

My Research

Theorem (T. 2020)

Let Π be a projective plane whose collineation group admits at least one (V, ℓ)-perspectivity, then Π has a Pappus configuration. (Many, in fact)

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Corollary (T. 2020)

Let Π *be a finite projective plane of non-square order whose collineation group admits a collineation of order 2. Then* Π *has a Pappus configuration. All known finite planes have collineation groups of even order, implying the existence of a collineation of order 2.*

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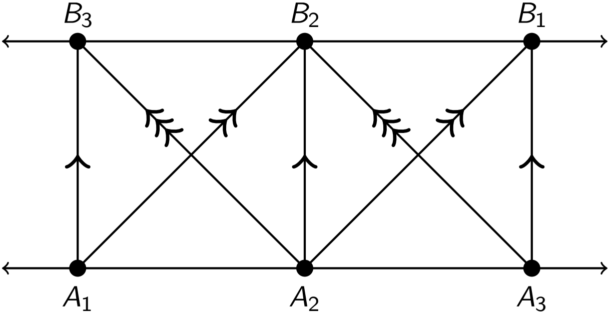
Theorem (T. 2020)

Let Π be a projective plane whose collineation group admits a quasiperspectivity of order 3, then Π has a Pappus configuration.

Proof Outline



Suppose Π is a projective plane containing σ a (V, ℓ)-perspectivity whose order is not 2. Consider the affine plane $\mathcal{A}$ obtained by removing ℓ and note that σ induces a collineation on $\mathcal{A}$. If $\sigma(A_1) = A_2$ and $\sigma(A_2) = A_3$, and likewise $\sigma(B_3) = B_2$ and $\sigma(B_2) = B_1$, then we may build a the above Pappus configuration. Since for any $\ell \in \mathcal{A}$, $\sigma(\ell)$ is parallel to ℓ, we have our result.

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Lemma (Hughes and Piper)

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Every projective plane Π *whose collineation group admits a quasiperspectivity contains a Pappus configuration.*

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Conjecture

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Question

Do there exist two non-isomorphic projective planes with the same number of Pappus configuraitons?

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Questions?

Thanks!