Algebraically Defined Graphs and their Applications

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Introduction

Definition

A **simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is the set of vertices, and $E \subset \binom{V}{2}$ is the set of edges. We denote the fact that a vertex x is **adjacent** to a vertex y by $x \sim y$. Since Γ is simple, the above definition removes the possibility of multiple edges, directed edges, and loops. In this talk, every graph mentioned will be a simple graph.

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Definition (Algebraically Defined Graph - ADG)

Let $P = L = \mathbb{F}_q^m$ be two copies of the m-dimensional vector space over $\mathbb{F}_q$ with $q = p^e$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $\Gamma_q = \Gamma_q(f_2, f_3, \ldots, f_m)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_m) \in P$ and $[\ell] = [\ell_1, \ldots, \ell_m]$, then $(p) \sim [l]$ if and only if

$$\begin{aligned}
\ell_2 + p_2 &= f_2(\ell_1, p_1) \\
\ell_3 + p_3 &= f_3(\ell_1, p_1, \ell_2, p_2) \\
&\vdots \\
\ell_m + p_m &= f_m(\ell_1, p_1, \ldots, \ell_{m-1}, p_{m-1})
\end{aligned}$$

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- Spectral Graph Theory: Build families of graphs satisfying particular spectral properties, like families of expanders.
- And more!

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A **generalized** n**-gon** of order $q \geq 1$ is a $(q+1)$-regular bipartite graph with diameter $n \geq 2$ and girth $2n$.

- $q = 1$, $n \geq 2$: C_{2n}
- $n = 2$, $q \geq 1$: $K_{(q+1),(q+1)}$

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Theorem (Feit and Higman 1964)

There do no exist any generalized n-gons of any order q when $n \notin \{2, 3, 4, 6, 8\}$.

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A generalized 3-gon is called a **projective plane**. Let $f_2 = f_2(p_1, \ell_1)$ and consider $\Gamma_q = \Gamma_q(f_2)$. Recall $(p_1, p_2) \sim [\ell_1, \ell_2]$ iff $p_2 + \ell_2 = f_2(p_1, \ell_1)$. If Γ_q has girth 6, there is a unique way to obtain a projective plane from Γ_q.

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- The classical projective plane can be obtained using $f_2(p_1, \ell_1) = p_1\ell_1$.
- All translation planes can be constructed this way by using $f_2(p_1, \ell_1) = p_1 \star \ell_1$ where $\star$ is the multiplication used in a particular quasifield. Translation planes account for many non-isomorphic classes of projective planes.

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Question

What other families of projective planes could be constructed via $\Gamma_q(f_2)$?

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- Finally, one canstruct the biaffine part of the generalized hexagon. In particular, this is given by $\Gamma_q(p_1\ell_1, p_1\ell_2, p_1\ell_3, p_2\ell_3 - p_3\ell_2)$.

Extremal Problems

Let $\mathrm{ex}(n, \mathcal{F})$ denote the largest number of edges in n-vertex graph that does not contain a copy of any graph $F \in \mathcal{F}$ as a subgraph. Denote $\mathcal{C}_k = \{C_k, C_{k-1}, \ldots, C_4, C_3\}$, the family of all cycles up to length k.

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- It has been shown that $\mathrm{ex}(n, \mathcal{C}_{2k+1}) \leq c_k n^{1+1/k}$ for a constant dependent on k. Bondy and Simonovits showed $c_k = 100k$ works, and over time this constant has been improved several times, first by Verstraëte, then Pikhurko, and most recently by Bukh and Jiang who showed $c_k = 80\sqrt{k \log k}$.

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- Lazebnik, Ustimenko, and Woldar used algebraically defined graphs to show that for infinitely many n, $c'_k n^{1+2/(3k+3+\epsilon)} \leq \text{ex}(n, \mathcal{C}_{2k+1})$ where $\epsilon = 1$ when k is odd and $\epsilon = 0$ otherwise.

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What do mathematicians do when they cannot answer a particular question? **Generalize and try again!**

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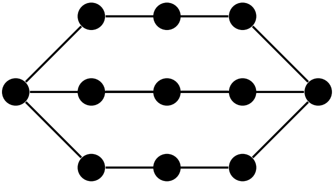

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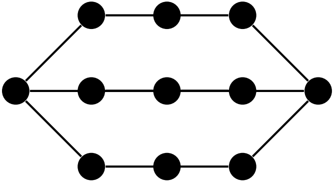

- A $\theta_{4,3}$
- Any cycle of length $2k$ can be thought of as a $\theta_{k,2}$ graph.

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- $c_1 n^{1+1/15} \leq \mathrm{ex}(n, n^{2/3}, \mathcal{C}_6) \leq c_2 n^{1+1/9}$ (bipartite graphs with differing number of vertices in each part) - Lower bound using ADG's due to: Lazebnik, Ustimenko, Woldar (1994)

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- Your favorite extremal problem?

Spectral Problems

The *Cheeger constant* $h(\Gamma)$ is an important measure of the connectivity of graph Γ. It is defined as

$$h(\Gamma) = \min\left\{ \frac{|\partial S|}{|S|} : S \subset V(\Gamma), 0 < |S| \leq \frac{|V(\Gamma)|}{2} \right\}$$

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- Expander families have many applications in computer science, electrical engineering and more!
- Building a family of expander graphs is generally a hard problem.

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Theorem (Alon-Milman (1986), Dodziuk (1984), and Mohar (1989))

Let Γ be a q-regular connected graph with second largest eigenvalue λ_2. Then

$$\frac{(q-\lambda_2)}{2} \leq h(\Gamma) \leq \sqrt{q^2-\lambda_2^2}.$$

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Theorem (Alon-Boppana (1986, 1991))

Let $\{\Gamma_n\}$ be an infinite family of q-regular connected graphs with $|V(\Gamma_n)| \to \infty$ as $n \to \infty$. Then

$$\liminf_{n\to\infty} \lambda_2(\Gamma_n) \geq 2\sqrt{q-1}.$$

Spectral Problems

Conjecture (Ustimenko)

The family of graphs $CD(k, q)$, the same family built by Lazebnik, Ustimenko and Woldar to improve lower bound on the girth problem, have second largest eigenvalue at most $2\sqrt{q}$.

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The graphs $D(2, q), D(3, q), D(4, q)$ have second largest eigenvalue $2\sqrt{q}$ for any prime power q.

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Theorem (Gupta and T., 2022)

For odd prime powers q, $D(5, q)$ has second largest eigenvalue bounded by $2\sqrt{q}$.

Questions?

Thanks!