Algebraically Defined Graphs and Generalized Quadrangles

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(joint work with Felix Lazebnik)

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Introduction

Definition

A **simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is the set of vertices, and $E \subset \binom{V}{2}$ is the set of edges. We denote the fact that a vertex x is **adjacent** to a vertex y by $x \sim y$. Since Γ is simple, the above definition removes the possibility of multiple edges, directed edges, and loops. In this talk, every graph mentioned will be a simple graph.

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Let Γ be a graph containing at least one cycle. The **girth** of Γ is the length of the shortest cycle in Γ.

Introduction

Definition (Algebraically Defined Graphs)

Let $P = L = \mathbb{F}_q^m$ be two copies of the m-dimensional vector space over $\mathbb{F}_q$ with $q = p^e$. Call the set P points and L lines, with the distinction in notation by $(a) \in P$ and $[a] \in L$. Define $\Gamma_q = \Gamma_q(f_2, f_3, \ldots, f_m)$ to be the bipartite graph with parts P and L and with edge relation defined between them as follows: If $(p) = (p_1, \ldots, p_m) \in P$ and $[\ell] = [\ell_1, \ldots, \ell_m]$, then $(p) \sim [l]$ if and only if

$$\begin{aligned} \ell_2 + p_2 &= f_2(\ell_1, p_1) \\ \ell_3 + p_3 &= f_3(\ell_1, p_1, \ell_2, p_2) \\ &\vdots \\ \ell_m + p_m &= f_m(\ell_1, p_1, \ldots, \ell_{m-1}, p_{m-1}) \end{aligned}$$

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So $[\ell_1, \ell_2, \ldots, \ell_m] \sim (x, p_2, \ldots, p_m)$

Properties of algebraically defined graphs

- Γ_q is q-regular with $|V| = n = 2q^m$ and

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$$|E| = q^{m+1} = \left(\frac{n}{2}\right)^{\frac{m+1}{m}}$$

- For each $b \in \mathbb{F}_q$, there exists an automorphism $t_b \in \mathrm{Aut}(\Gamma_q)$ given by

$$t_b[\ell_1, \ell_2, \ldots, \ell_m] = [\ell_1, \ell_2, \ldots, \ell_m + b]$$
$$t_b(p_1, p_2, \ldots, p_m) = (p_1, p_2, \ldots, p_m - b).$$

Why are these graphs interesting?

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- It has been shown that $\mathrm{ex}(n, C_{2k}) \leq c_k n^{1+1/k}$ for a constant dependent on k. Bondy and Simonovits showed $c_k = 100k$ works, and over time this constant has been improved several times, first by Verstraëte, then Pikhurko, and most recently by Bukh and Jiang who showed $c_k = 80\sqrt{k \log k}$.

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- Lazebnik, Ustimenko, and Woldar used algebraically defined graphs to show that for infinitely many n, $c'_k n^{1+2/(3k+3+\epsilon)} \leq \mathrm{ex}(n, C_{2k})$ where $\epsilon = 1$ when k is odd and $\epsilon = 0$ otherwise.

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Definition

A **generalized** *n*-**gon** of order $q \geq 1$ is a $(q+1)$-regular bipartite graph with diameter $n \geq 2$ and girth $2n$.

- $q = 1$, $n \geq 2$: C_{2n}
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Theorem (Feit and Higman 1964)

There do no exist any generalized n-gons of any order q when $n \notin \{2, 3, 4, 6, 8\}$.

ADG's and Projective Planes

A generalized 3-gon is called a **projective plane**. Let $f_2 = f_2(p_1, \ell_1)$ and consider $\Gamma_q = \Gamma_q(f_2)$. Recall $(p_1, p_2) \sim [\ell_1, \ell_2]$ iff $p_2 + \ell_2 = f_2(p_1, \ell_1)$. If Γ_q has girth 6, there is a unique way to obtain a projective plane from Γ_q.

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- All translation planes can be represented this way by using $f_2(p_1, \ell_1) = p_1 \star \ell_1$ where $\star$ is the multiplication used in a particular quasifield. Translation planes account for many non-isomorphic classes of projective planes.

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Question

Can we construct generalized quadrangles this way too?

ADG's and Generalized Quadrangles

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- No other (non-isomorphic) generalized quadrangles of order q are known.
- Algebraically defined graphs provide one potential method for answering this question. In large part, this question motivated our research.

Results and Methods

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- Many graphs of the form $\Gamma_q(p_1\ell_1, f_3(p_1, \ell_1))$ have been shown to either be isomorphic to $\Gamma_q(p_1\ell_1, p_1\ell_2)$ or have girth less than 8, [1, 3].

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Question

Given $\Gamma_q(f_2, f_3)$ where $f_2 = f_2(p_1, \ell_1)$ and $f_3(p_1, \ell_1, p_2, \ell_2)$, does there exist a function $f_3' = f_3'(p_1, \ell_1)$ so that $\Gamma_q(f_2, f_3) \cong \Gamma_q(f_2, f_3')$?

Results and methods

Observe that for any graph of the form $\Gamma_q(f_2(p_1,\ell_1), f_3(p_1,\ell_1))$.
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then for all $a, b \in \mathbb{F}_q$, the function $t_{a,b}$ where

$$t_{a,b}[\ell_1, \ell_2, \ell_3] = [\ell_1, \ell_2 + a, \ell_3 + b]$$
$$t_{a,b}(p_1, p_2, p_3) = (p_1, p_2 - a, p_3 - b)$$

is an automorphism of Γ_q. Meaning, $q^2 \leq |\mathrm{Aut}(\Gamma_q)|$.

Results and methods

Let $\Gamma = \Gamma_q(p_1\ell_1, p_1\ell_1 p_2(p_1 + p_2 + p_1 p_2))$. This family of graphs appeared in the thesis of Nassau 2021. It was checked via computer that for all odd prime powers $q < 43$, there do not exist functions f_2, f_3 of just p_1 and ℓ_1 such that $\Gamma \cong \Gamma_q(f_2, f_3)$. Though he could not prove it for any infinite sequence of q's.

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Theorem (Lazebnik and T. 2022)

Let p be an odd prime with $p \equiv 1 (mod\ 3)$. If Γ is defined over $\mathbb{F}_p$, then $|Aut(\Gamma)| = p$.

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Corollary (Lazebnik and T. 2022)

Let p be an odd prime with $p \equiv 1 (\ mod\ 3)$. Let $f_2 = f_2(p_1, \ell_1)$ and $f_3 = f_3(p_1, \ell_1)$ be functions of p_1 and ℓ_1. Then $\Gamma_q(f_2, f_3) \not\cong \Gamma$.

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Questions?

Thanks!