On the number of k-gons in finite projective planes

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August 21, 2022

Abstract

Let Π be a projective plane of order n and Γ_Π be its Levi graph (the point-line incidence graph). For fixed $k \geq 3$, let $c_{2k}(\Gamma_\Pi)$ denote the number of $2k$-cycles in Γ_Π. In this paper we show that

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k}n^{2k} + O(n^{2k-2}), \qquad n \to \infty.$$

We also state a conjecture regarding the third and fourth largest terms in the asymptotic of the number of $2k$-cycles in Γ_Π. This result was also obtained independently by Voropaev [26] in 2012.

Let $\mathrm{ex}(v, C_{2k}, \mathcal{C}_{\mathrm{odd}} \cup \{C_4\})$ denote the greatest number of $2k$-cycles amongst all bipartite graphs of order v and girth at least 6. As a corollary of the result above, we obtain

$$\mathrm{ex}(v, C_{2k}, \mathcal{C}_{\mathrm{odd}} \cup \{C_4\}) = \left(\frac{1}{2^{k+1}k} - o(1)\right) v^k, \qquad v \to \infty.$$

1 Introduction

Over the years, many questions have surfaced regarding counting the number of certain substructures within a projective plane. In this paper we contribute to an open question in the area. We omit the standard definitions related to finite geometries and graph theory. For all undefined notions in finite geometries we refer the reader to Casse [7], and to Bollobas [5] for all graph theoretic notions. We will also need the following definitions and notation.

Let Π denote a projective plane of order n. Then $N = n^2+n+1$ represents the number of points and the number of lines in Π. If A and B are points of Π, we write AB for the line containing them. We write S_k for the group of all permutations of $\{1, 2, \ldots, k\}$, the symmetric group.

The *point-line incidence graph* Γ_Π of Π, also known as the *Levi graph* of Π, is the bipartite graph with the set of points of Π to be one vertex part and the set of lines of Π to be the other vertex part. A point P is adjacent to a line ℓ in Γ_Π if P lies on ℓ in Π. We write $P \sim \ell$ to denote adjacency of a point and line in Γ_Π.

Let H be a graph and $\mathcal{F}$ be a family of (forbidden) graphs. Let $\mathrm{ex}(n, H, \mathcal{F})$ denote the maximum number of copies of H in an n-vertex graph containing no graphs in $\mathcal{F}$ as a subgraph. When $H = K_2$ (just an edge), then a simplified notation is used for

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$\mathrm{ex}(n, K_2, \mathcal{F})$, namely $\mathrm{ex}(n, \mathcal{F})$, and it is often called the *Turán number* of $\mathcal{F}$. Clearly, $\mathrm{ex}(n, \mathcal{F})$ denotes the largest number of edges an n-vertex graph can have without containing any subgraphs from $\mathcal{F}$. Any n-vertex graph with $\mathrm{ex}(n, \mathcal{F})$ edges is called an extremal graph for $\mathcal{F}$. The problem of determining $\mathrm{ex}(n, \mathcal{F})$ is usually referred to as a *Turán type* problem. For the extensive literature related to Turán type problems, see Bollobas [6], Füredi [11], Füredi and Simonovits [12], Verstraëte [19], Mubayi and Verstraëte [24], Lazebnik, Sun, and Wang [23].

Let C_k denote a cycle of length k, $\mathcal{C}_k = \{C_3, C_4, \ldots, C_k\}$, and $\mathcal{C}_{\mathrm{odd}} = \{C_3, C_5, C_7, \ldots\}$ the set of all odd cycles. Some early attention that $\mathrm{ex}(n, H, \mathcal{F})$ received was from Erdös [9] who stated a conjecture regarding the extremal graph of $\mathrm{ex}(n, C_5, \{C_3\})$. This conjecture was resolved by Hatami, Hladkýi, Král, Norine, and Razborov [18] and independently by Grzesik [16], building on the work of Györi [17]. The more recent wave of interest in $\mathrm{ex}(n, H, \mathcal{F})$ was initiated by Alon and Shikelman [2]. There have been several new results regarding the growth rate of $\mathrm{ex}(n, H, \mathcal{F})$ where $\mathcal{F} = \mathcal{C}_{2m}$ or $\mathcal{F} = \{C_{2m}\}$, with resolution up to the leading term in certain cases. We refer the reader to the papers [13], [14], and [25] for the most up to date reading regarding $\mathrm{ex}(n, H, \mathcal{C}_{2m})$ and $\mathrm{ex}(n, H, \{C_{2m}\})$.

Note that $(\mathcal{C}_{\mathrm{odd}} \cup \{C_4\})$-free graphs are bipartite graphs of girth at least 6. Given that the Levi graph of a projective plane is bipartite and has girth 6, it serves as a good candidate for obtaining lower bounds on $\mathrm{ex}(n, H, \mathcal{F})$. In this paper, we will be counting the number of cycles of length $2k$ in the Levi graph, and consequently we obtain lower bounds on $\mathrm{ex}(v, C_{2k}, \mathcal{C}_{\mathrm{odd}} \cup \{C_4\})$. We denote the number of cycles of length $2k$ in a graph G by $c_{2k}(G)$.

As a simpler problem, one can count the number of closed walks of length $2k$ in the Levi graph of a projective plane Π of order n. One can easily show that this number depends only on k and n, and not on the actual plane. Let Π be a projective plane of order n and Γ_Π its Levi graph. Let A be the adjacency matrix of Γ_Π. As A is a real symmetric matrix, then all of its eigenvalues are real. Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{2N}$ be the eigenvalues of A. By considering eigenvalues of A^2, it can be deduced that $\lambda_1 = n + 1$ and $\lambda_{2N} = -(n + 1)$, each with multiplicity one, and all other eigenvalues are equal to $\pm\sqrt{n}$ each with multiplicity $N - 1$. It follows, see [4], that the number of closed walks of length $2k$ in Γ_Π is given by

$$\mathrm{Trace}(A^{2k}) = \sum_{i=1}^{2N} \lambda_i^{2k} = 2(n+1)^{2k} + 2(N-1)n^k$$

and so the number of closed walks of length $2k$ in Γ_Π depends only on n and k, and so it is the same for all projective planes of order n.

This may lead one to ask, what other structures appear in a finite projective plane Π and does the number of these structures depend only the order of the plane? In general the answer is no, as can be observed in the case of Desargues and Pappus configurations. Another interesting example concerns the number of k-arcs. Define a *k-arc* in a projective plane Π to be a set of k points of Π, no three of which are collinear. For $k \leq 6$, Glynn [15] showed that the number of k-arcs in a plane of order n does not depend on the plane. Furthermore, in [15], Glynn computes an expression for the number of 7-arcs in any finite projective plane, and using this expression deduces that there do not exist projective planes of order 6, as evaluating the formula at 6 yields a negative value. Glynn's work counting k-arcs was recently extended by Kaplan, Kimport, Lawrence, Peilen and Weinreich [20] who determined an expression for the number of 9-arcs in an arbitrary projective plane. It is worth mentioning that for $k = 7, 8, 9$, the formula for the number of k-arcs depends on more than just k and the order of the plane.

In [21] Lazebnik, Mellinger, and Vega demonstrate that it is possible to embed a $2k$-cycle of every possible size into the Levi graph of any finite affine or projective plane. This was further extended by Aceves, Heywood, Klahr, and Vega [1] who showed that one can embed a $2k$-cycle of every possible size into the Levi graph of the projective space $PG(d, q)$. Moreover, in a different paper Lazebnik, Mellinger, and Vega [22] motivated the study of counting $2k$-cycles in the Levi graph with the following two questions. For fixed $k \geq 3$:

1. Which C_4-free bipartite graphs with partitions of size N contain the greatest number of $2k$-cycles?

2. Do the Levi graphs of all affine or projective planes of order n contain the same number of $2k$-cycles?

Fiorini and Lazebnik [10] show that the Levi graph of a projective plane has the largest number of C_6's among all C_4-free bipartite graphs with size parts. This work was extended by De Winter, Lazebnik, and Verstraëte [8] who show that the same holds when $k = 4$ and $n \geq 157$. In [22], progress towards question 2 is made as the exact value of $c_{2k}(\Gamma_\Pi)$ is determined for $k = 3, 4, 5, 6$, showing that in these cases, $c_{2k}(\Gamma_\Pi)$ depends only on the order of Π. This work was further extended by Voropaev [26], again demonstrating that $c_{2k}(\Gamma_\Pi)$ depends only on the order of Π up to $k = 10$. Determining explicit formulas for larger k may very well have interesting consequences just like in the example of formula for the number of 7-arcs in a projective plane.

In this paper, we make some progress towards resolving question 2 as we determine the first and second leading terms in the asymptotic of the number of k-gons in an arbitrary projective plane. The magnitude of the leading term for the number of k-gons is shown to be the same as that of the number of closed walks of length $2k$ in the Levi graph of a finite projective plane, but with a different leading coefficient.

Here we list our main results.

Theorem 1. *Let Π be a projective plane of order n and Γ_Π be its Levi graph. Then for fixed $k \geq 4$,*

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k} n^{2k} + O(n^{2k-2}), \qquad n \to \infty$$

Theorem 2. *Let $k \geq 4$, then*

$$ex(v, C_{2k}, \mathcal{C}_{odd} \cup \{C_4\}) = \left(\frac{1}{2^{k+1}k} - o(1) \right) v^k, \qquad v \to \infty.$$

The structure of our paper is as follows. In section 2 we state some definitions and prove some important lemmas. In section 3 we begin to place bounds on the number of certain types of subgraphs in Γ_Π. In section 4 we prove Theorems 1 and 2 mentioned above. Finally, in our concluding remarks, we give a table of coefficient data on the number of $2k$-cycles in Γ_Π for small k and leave the reader with a conjecture.

2 Symmetries of QG_k

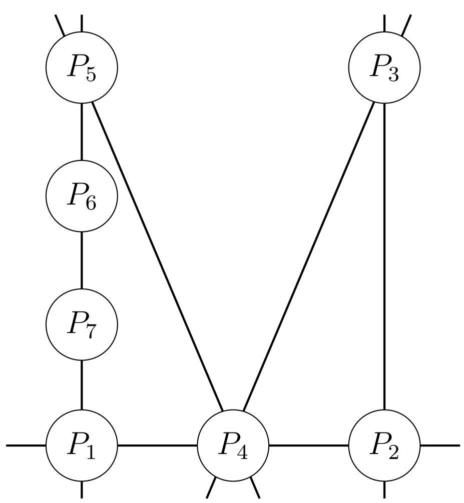
Let Π be a projective plane and Γ_Π be its Levi graph. For $k \geq 3$, we define a *quasi k-gon* to be a sequence $QG_k = (P_1, P_2, \ldots, P_k)$ of k distinct points of Π. Here, P_k and P_1 are thought of as consecutive elements in QG_k. All arithmetic done in the indices is considered to be modulo k, where we will use $\{1, 2, \ldots k\}$ as the representatives of each equivalence class mod k. We call $\mathcal{P}_{QG_k} = \{P_1, \ldots, P_k\}$ the set of points of QG_k and $\mathcal{L}_{QG_k}$ is the set of all distinct lines amongst $P_1P_2, P_2P_3, \ldots, P_kP_1$. For convenience, we will write $\mathcal{L}_{QG_k} = \{P_iP_{i+1} : 1 \leq i \leq k\}$. If $|\mathcal{L}_{QG_k}| = k$, meaning all lines of the form P_iP_{i+1} for $1 \leq i \leq k$ are distinct, then we call QG_k a *k-gon* and instead denote it by G_k. We will denote the number of k-gons in a projective plane Π by $c_k(\Pi)$. It will be shown later that in fact

$$\frac{1}{2k} c_k(\Pi) = c_{2k}(\Gamma_\Pi).$$

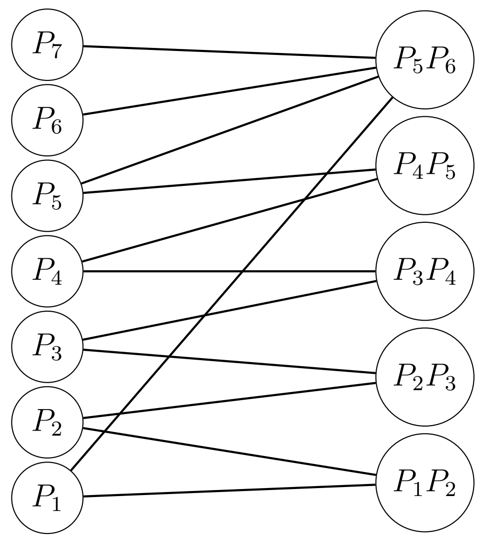
Define the subgraph Γ_{QG_k} of Γ_Π corresponding to QG_k as follows: The set of vertices $V(\Gamma_{QG_k})$ is given by $\mathcal{P}_{QG_k} \cup \mathcal{L}_{QG_k}$. The edges $E(\Gamma_{QG_k})$ are obtained by joining a vertex P_i to all vertices in the set $\{P_{i-1}P_i, P_iP_{i+1}\}$ for $1 \leq i \leq k$. If $P_{i-1}P_i = P_iP_{i+1}$ then P_i has only one neighbor. It is clear that in the case where QG_k is actually a k-gon, the corresponding graph Γ_{QG_k} is a cycle of length $2k$.

Here we provide an example. Let $QG_7 = (P_1, P_2, \ldots, P_7)$ be a quasi 7-gon given by the following figure. We use QG_7 to demonstrate the corresponding graph Γ_{QG_7}.

QG_7



Γ_{QG_7}



Let us take a moment to comment on the above figure. Here $QG_7 = (P_1, P_2, \ldots, P_7)$, with the corresponding set of lines $\{P_iP_{i+1} : 1 \leq i \leq k\}$. We assume that $P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6$ are all distinct lines. Observe that P_4 lies on the line P_1P_2, however, $P_4 \not\sim P_1P_2$ in Γ_{QG_7}. By definition of Γ_{QG_7}, we have only that $P_4 \sim P_3P_4$ and $P_4 \sim P_4P_5$. Furthermore, note that $P_5P_6 = P_6P_7 = P_7P_1$ and therefore P_6 and P_7 each only have one neighbor, namely P_5P_6.

The symmetric group S_k acts on quasi k-gons in Π in the following way: If $QG_k = (P_1, \ldots, P_k)$ and $\sigma \in S_k$, then $\sigma(QG_k) := (P_{\sigma(1)}, \ldots, P_{\sigma(k)})$. Hence,

$$\mathcal{P}_{\sigma(QG_k)} = \{P_{\sigma(1)}, \ldots, P_{\sigma(k)}\} = \{P_1, \ldots, P_k\} = \mathcal{P}_{QG_k}$$

and the set of lines of $\sigma(QG_k)$ is

$$\mathcal{L}_{\sigma(QG_k)} = \{P_{\sigma(i)}P_{\sigma(i+1)} : 1 \leq i \leq k\}.$$

Note that in general, $\mathcal{L}_{QG_k}$ is not necessarily equal to $\mathcal{L}_{\sigma(QG_k)}$.

We call two quasi k-gons $QG_k = (P_1, \ldots, P_k)$ and $QG'_k = (P'_1, \ldots, P'_k)$ *equivalent*, and write $QG_k \equiv QG'_k$, if $\Gamma_{QG_k} = \Gamma_{QG'_k}$, that is, they have the same vertex set and the same edge set. It is obvious that equivalence of quasi k-gons is an equivalence relation and if $QG_k \equiv QG'_k$, then there exists $\sigma \in S_k$ such that $\sigma(QG_k) = QG'_k$. Therefore $S(QG_k) := \{\sigma \in S_k : QG_k \equiv \sigma(QG_k)\}$ is a subgroup of S_k.

Remark: Given a quasi k-gon QG_k and permutation $\sigma \in S_k$ we stress the following point: We do not view QG_k as a partial plane in Π defined by the points and lines of QG_k. Therefore, if $QG_k \equiv \sigma(QG_k)$, then σ should not be thought of as a collineation. As an example, we refer to the figure above of QG_7 and consider $\sigma = (1234567)$. In the lemma that follows, we demonstrate that $QG_7 \equiv \sigma(QG_7)$, however, observe that while P_5, P_6, P_7, P_1 lie on one line in Π, $P_{\sigma(5)} = P_6$, $P_{\sigma(6)} = P_7$, $P_{\sigma(7)} = P_1$ and $P_{\sigma(1)} = P_2$ are not collinear in Π.

Let D_k denote the dihedral group of order $2k$, which is defined as the group of automorphisms of the graph C_k. It is well known that $D_k = \langle a, b\rangle$ where $a^n = b^2 = (ab)^2 = 1$.

Lemma 1. *Let Π be a projective plane and $QG_k = (P_1, \ldots, P_k)$ in Π. Then $S(QG_k)$ contains D_k as a subgroup.*

Proof. Let $\sigma = (12\ldots k)$, so that $\sigma(QG_k) = (P_{\sigma(1)}, \ldots, P_{\sigma(k)}) = (P_2, \ldots, P_k, P_1)$. We wish to show that $\sigma(QG_k) \equiv QG_k$. Note that the vertex set $V(\Gamma_{\sigma(QG_k)}) = \mathcal{P}_{\sigma(QG_k)} \cup \mathcal{L}_{\sigma(QG_k)} = \mathcal{P}_{QG_k} \cup \mathcal{L}_{\sigma(QG_k)}$. Here

$$\begin{aligned}
\mathcal{L}_{\sigma(QG_k)} &= \{P_{\sigma(i)}P_{\sigma(i+1)} : 1 \leq i \leq k\} = \{P_{i+1}P_{i+2} : 1 \leq i \leq k\} \\
&= \{P_iP_{i+1} : 1 \leq i \leq k\} = \mathcal{L}_{QG_k}.
\end{aligned}$$

Thus we have $V(\Gamma_{\sigma(QG_k)}) = V(\Gamma_{QG_k})$. The edge set $E(\Gamma_{\sigma(QG_k)})$ is given by joining $P_{\sigma(i)} = P_{i+1}$ to all distinct lines in $\{P_{\sigma(i-1)}P_{\sigma(i)}, P_{\sigma(i)}P_{\sigma(i+1)}\} = \{P_iP_{i+1}, P_{i+1}P_{i+2}\}$ where $1 \leq i \leq k$. These are exactly the same edges that appear in Γ_{QG_k}. Thus, $\sigma \in S(QG_k)$ and has order k.

Now consider the permutation ρ of $\{1, 2, \ldots, k\}$, such that $\rho(i) = k + 1 - i$. That is, $\rho(QG_k) = (P_k, P_{k-1}, \ldots, P_1)$. Set $j = k + 1 - i$, then $\rho(i) = j$ and $\rho(i + 1) = j - 1$. Observe that $V(\Gamma_{\rho(QG_k)}) = \mathcal{P}_{QG_k} \cup \mathcal{L}_{\rho(QG_k)}$ where

$$\begin{aligned}
\mathcal{L}_{\rho(QG_k)} &= \{P_{\rho(i)}P_{\rho(i+1)} : 1 \leq i \leq k\} = \{P_jP_{j-1} : 1 \leq j \leq k\} \\
&= \{P_{j-1}P_j : 1 \leq j \leq k\} = \mathcal{L}_{QG_k}.
\end{aligned}$$

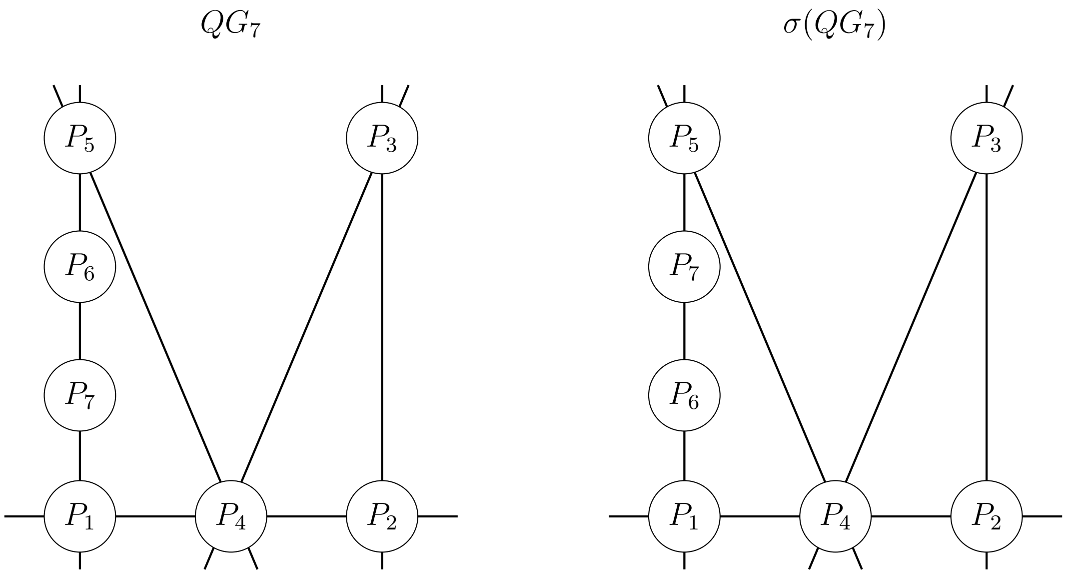
Therefore $V(\Gamma_{\rho(QG_k)}) = V(\Gamma_{QG_k})$. The edge set $E(\Gamma_{\rho(QG_k)})$ is given by joining $P_{\rho(i)} = P_j$ to all distinct lines in $\{P_{\rho(i-1)}P_{\rho(i)}, P_{\rho(i)}P_{\rho(i+1)}\} = \{P_{j+1}P_j, P_jP_{j-1}\}$ for $1 \le j \le k$. These are exactly the same edges that appear in Γ_{QG_k}. Thus, $\rho \in S(QG_k)$ and has order 2.

Now to demonstrate that D_k is in fact the Dihedral group, we show that $(\sigma\rho)^2 = 1$. Since $\sigma(i) = i + 1$ and $\rho(i) = k + 1 - i$ then $(\sigma\rho)(i) = \sigma(k + 1 - i) = k + 2 - i$. Therefore

$$(\sigma\rho)^2(i) = (\sigma\rho)(k + 2 - i) = k + 2 - (k + 2 - i) = i.$$

Therefore $\sigma^k = \rho^2 = (\sigma\rho)^2 = 1$, and so D_k is a Dihedral group of order $2k$. □

We now provide an example showing that it is possible to have $S(QG_k)$ be strictly larger than D_k. We refer to our previous example of QG_7, and we consider the permutation $\sigma = (67)$.



The reader should convince themselves that both $QG_7 = (P_1, P_2, P_3, P_4, P_5, P_6, P_7)$ and $\sigma(QG_7) = (P_1, P_2, P_3, P_4, P_5, P_7, P_6)$ have the same corresponding graph, namely the graph Γ_{QG_7} which we have drawn in the previous figure.

Lemma 2. *Let* Π *be a projective plane, and* $G_k = (P_1, \dots, P_k)$ *be a* k*-gon in* Π*. Then the group of symmetries of* G_k *is precisely* D_k*.*

Proof. In the following proof, all arithmetic is assumed to be taken modulo k. Let D_k represent the subgroup of $S(G_k)$ described above. Suppose $\tau \in S_k$ such that for each i, $1 \le i \le k$, $\tau(i + 1) = \tau(i) + 1$ or $\tau(i + 1) = \tau(i) - 1$. Then in fact, exactly one of the following must be true

- $\tau(i + 1) = \tau(i) + 1$ for all $1 \le i \le k$,
- $\tau(i + 1) = \tau(i) - 1$ for all $1 \le i \le k$.

Indeed, if this was not the case, then there would exist a j such that $\tau(j+1) = \tau(j)+1$ and $\tau(j+2) = \tau(j+1) - 1 = \tau(j) + 1 - 1 = \tau(j)$. Clearly this is a contradiction, as $\tau(j+2) = \tau(j)$ implies τ is not a bijection. Note that by definition of σ and ρ in Lemma 1, any permutation satisfying either condition above must in fact be an element of D_k.

Suppose then that $\tau \in S_k \setminus D_k$, which implies that there exists an i, $1 \leq i \leq k$ for which $\tau(i+1) \neq \tau(i) \pm 1$. For this i, let $\tau(i) = \ell$ and $\tau(i+1) = j$. Therefore in $\Gamma_{\tau(G_k)}$ we have $P_\ell \sim P_\ell P_j \sim P_j$. In Γ_{G_k} we know that

$$P_{\ell-1}P_\ell \sim P_\ell \sim P_\ell P_{\ell+1} \qquad \text{and} \qquad P_{j-1}P_j \sim P_j \sim P_j P_{j+1}.$$

If $\tau(G_k) \equiv G_k$, then we must have that $P_\ell P_j = P_{\ell-1}P_\ell$ or $P_\ell P_j = P_\ell P_{\ell+1}$ and $P_\ell P_j = P_{j-1}P_j$ or $P_\ell P_j = P_j P_{j+1}$. So we are left with four possibilities:

1. $P_{\ell-1}P_\ell = P_{j-1}P_j$.

2. $P_\ell P_{\ell+1} = P_j P_{j+1}$.

3. $P_{\ell-1}P_\ell = P_j P_{j+1}$.

4. $P_\ell P_{\ell+1} = P_{j-1}P_j$.

Each leads us to a contradiction, because all the lines of a k-gon are distinct. Cases 1 and 2 imply $\ell = j$, a contradiction. Case 3 implies $\ell - 1 = j$ and case 4 implies $\ell = j - 1$, both are contradictions since we assumed $j \neq \ell \pm 1$. Thus, if $\tau \in S_k \setminus D_k$, then $G_k \not\equiv \tau(G_k)$, and so $S(G_k) = D_k$. □

Remark: Since the order of $S(G_k)$ is precisely $2k$ for any k-gon, then this demonstrates the assertion we made in the beginning of this section that

$$\frac{1}{2k} c_k(\Pi) = c_{2k}(\Gamma_\Pi).$$

For positive integers x and k, let $x_{(k)} = x(x-1)\cdots(x-k+1)$. Let Q_k be the collection of all quasi k-gons in Π of order n. Clearly, we have that $|Q_k| = N_{(k)}$ where $N = n^2 + n + 1$. Define $Q_{k,j} = Q_{k,j}(\Pi) = \{QG_k \in Q_k : |\mathcal{L}_{QG_k}| = j\}$. The the sets $Q_{k,j}$ form a partition of Q_k and therefore

$$|Q_k| = |Q_{k,k}| + |Q_{k,k-1}| + \cdots + |Q_{k,1}| \tag{1}$$

where in fact $|Q_{k,k}| = c_k(\Pi)$. Given this fact and the remark above, we may re-write (1) in the form

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k}(|Q_k| - |Q_{k,k-1}| - \cdots - |Q_{k,1}|). \tag{2}$$

Our end goal is to obtain equality of the first and second terms in the lower and upper bounds of $c_{2k}(\Gamma_\Pi)$. We aim to do this by use of (2), combined with bounds which we will obtain for $|Q_{k,j}|$ where $j < k$. We consider several cases, and obtain bounds in each case independently. We count:

(a) The number of QG_k's with $|\mathcal{L}_{QG_k}| = k-1$ and such that there exists an m such that $P_mP_{m+1} = P_{m+1}P_{m+2}$. Let the set of all such quasi k-gons in Π be denoted by $A_k(\Pi)$.

(b) The number of QG_k's with $|\mathcal{L}_{QG_k}| = k-1$ and such that for all m, $P_mP_{m+1} \neq P_{m+1}P_{m+2}$. Let the set of all such quasi k-gons be denoted by $B_k(\Pi)$.

(c) The order of $Q_{k,j}$ for each $1 \leq j \leq k-2$.

Lemma 3. *Let $k \geq 4$ and $QG_k = (P_1, \dots, P_k)$ be a quasi k-gon with $|\mathcal{L}_{QG_k}| = k-1$. Suppose further that there exists an m such that $P_mP_{m+1} = P_{m+1}P_{m+2}$. Then $|S(QG_k)| = 2k$.*

Proof. Suppose QG_k is as defined above. This implies that P_m, P_{m+1}, P_{m+2} are collinear. Without loss of generality, we may assume $P_{m+1} = P_k$, since we may apply $\sigma = (12\dots k) \in S(QG_k)$ to QG_k, until we have moved P_{m+1} into the position of P_k and then relabel the points as $(P'_1, \dots, P'_k)$. We bring attention then to the fact that the lines P_iP_{i+1} are distinct for $1 \leq i \leq k-1$ and that the only lines in QG_k equal to one another are $P_{k-1}P_k = P_kP_1$.

From here, we follow in the foot steps of the proof of Lemma 2. Recall that if $\tau \in S_k \setminus D_k$, then there exists an i such that $\tau(i+1) \neq \tau(i) \pm 1$. For this i, let $\tau(i) = \ell$ and $\tau(i+1) = j$. This implies that in $\Gamma_{\tau(QG_k)}$ we have $P_\ell \sim P_\ell P_j \sim P_j$. On the other hand, in Γ_{QG_k} we know that

$$P_{\ell-1}P_\ell \sim P_\ell \sim P_\ell P_{\ell+1} \qquad \text{and} \qquad P_{j-1}P_j \sim P_j \sim P_jP_{j+1}.$$

If $\tau(QG_k) \equiv QG_k$, then we must have that $P_\ell P_j = P_{\ell-1}P_\ell$ or $P_\ell P_j = P_\ell P_{\ell+1}$ and $P_\ell P_j = P_{j-1}P_j$ or $P_\ell P_j = P_jP_{j+1}$.
So we are again left with the same four cases:

1. $P_{\ell-1}P_\ell = P_{j-1}P_j$
2. $P_\ell P_{\ell+1} = P_jP_{j+1}$
3. $P_{\ell-1}P_\ell = P_jP_{j+1}$
4. $P_\ell P_{\ell+1} = P_{j-1}P_j$

Case 1 implies that either $\ell = j$ or that $\ell = j-1 = k$. Case 2 implies that either $\ell = j$ or that $j = \ell - 1 = k$. All of these options are contradictions, since $j \neq \ell$ and $j \neq \ell \pm 1$.

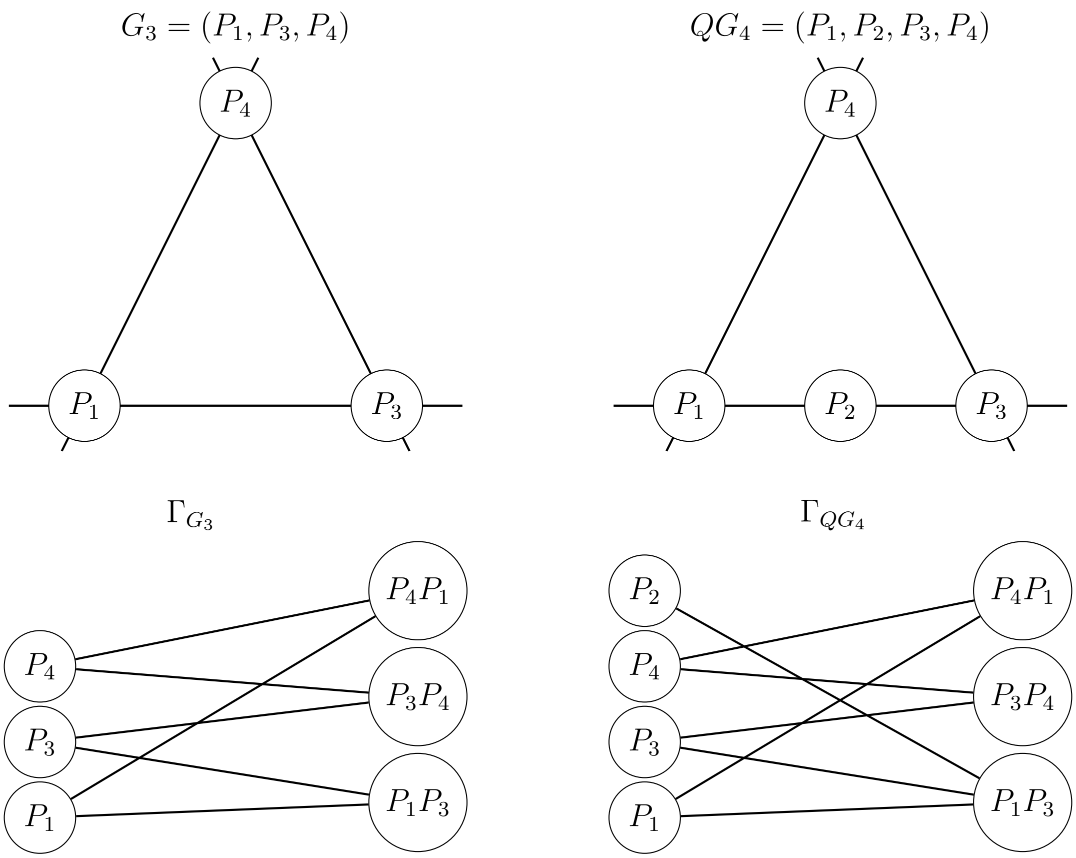
If we suppose case 3, then we obtain three possibilities. We may have $\ell - 1 = j$, a contradiction. We may have also consider $\ell = j = k$, again a contradiction. Therefore we are left with only one possibility, which is that $j = k-1$ and $\ell = 1$, in which case we obtain $P_kP_1 = P_{k-1}P_k$ which does not serve as a contradiction. Continuing on with this assumption, recall that $\tau(i) = \ell = 1$ and $\tau(i+1) = j = k-1$, so that in $\tau(QG_k)$, P_1 and P_{k-1} are consecutive. Note then, for some r, $\tau(r) = k$, so P_k is consecutive with $P_{\tau(r-1)}$ and $P_{\tau(r+1)}$ in $\tau(QG_k)$. Now, we must have that either $\tau(r-1) \neq 1, k-1$ or

$\tau(r+1) \neq 1, k-1$. If not, then in $\tau(QG_k)$ we would observe either $(\dots, P_1, P_k, P_{k-1}, \dots)$ or $(\dots, P_{k-1}, P_k, P_1, \dots)$, both cases which cannot happen since we have shown that in fact P_{k-1} and P_1 must appear as consecutive elements in $\tau(QG_k)$. Without loss of generality, suppose $\tau(r-1) \neq 1, k-1$. Then the line $P_{\tau(r-1)}P_k$ appears in $\tau(QG_k)$. We know that in QG_k, P_k has only one neighbor, namely $P_{k-1}P_k$, so then if $\tau(QG_k) \equiv QG_k$, then $P_{\tau(r-1)}P_k = P_{k-1}P_k$, which implies $P_{\tau(r-1)} \sim P_{k-1}P_k$ in $\Gamma_{\tau(QG_k)}$. This is a contradiction, since this edge does not appear in Γ_{QG_k}. Thus case 3 is also impossible. Case 4 follows in the same exact manner as case 3, except that the role of j and ℓ is swapped.

Thus, each of the four cases leads us to a contradiction, implying that if $\tau \in S_k \setminus D_k$, then $\tau(QG_k) \not\equiv QG_k$. Then $S(QG_k) = D_k$. □

By Lemma 3, each $QG_k \in \Pi$ is equivalent to exactly $2k$ quasi k-gons in $A_k(\Pi)$, and so $\frac{1}{2k}|A_k(\Pi)|$ counts the number of equivalence classes of such quasi k-gons.

We now comment on Γ_{QG_k} for $QG_k \in A_k(\Pi)$. Let $QG_k = (P_1, \dots, P_k) \in A_k(\Pi)$, meaning for some m, P_m, P_{m+1}, P_{m+2} are collinear. Then consider $G_{k-1} = (P_1, \dots, P_m, P_{m+2}, \dots, P_k)$ a $k-1$-gon. Note that $\mathcal{L}_{QG_k} = \mathcal{L}_{G_{k-1}}$ and that $\mathcal{P}_{QG_k} = \mathcal{P}_{G_{k-1}} \cup \{P_{m+1}\}$. Furthermore, if $P \sim \ell$ in $\Gamma_{G_{k-1}}$, then $P \sim \ell$ in Γ_{QG_k}. We use the following example to illustrate the relationship between a k-gon and a quasi $(k+1)$-gon of the form described.



Since equivalence is directly tied to equality of graphs, we note that $\frac{1}{2k}|A_k(\Pi)|$ is counting exactly the number of distinct subgraphs of Γ_Π that have the following form: A cycle of length $2k-2$ together with a distinct point vertex adjoined to exactly one line in the cycle. To be more rigorous, let C be a cycle of length $2k-2$ in Γ_Π. We know that we may choose a $(k-1)$-gon, $G_{k-1} = (P_1, \dots, P_{k-1})$ such that $\Gamma_{G_{k-1}} = C$. If $P \notin V(C)$ is a point vertex in Γ_Π that is a neighbor of a line $\ell \in V(C)$, or equivalently, is a neighbor of a line $P_m P_{m+1} \in \Gamma_{G_{k-1}}$ for some m, then $\Gamma_{G_{k-1}}$ together with the vertex P adjoined to $P_m P_{m+1}$, corresponds to the quasi k-gon $QG_k = (P_1, \dots, P_m, P, P_{m+1}, \dots, P_{k-1}) \in A_k(\Pi)$. So every graph of the form described, corresponds uniquely to an equivalence class of QG_k's belonging to $A_k(\Pi)$. Denote this set of graphs by $A_k(\Gamma_\Pi)$ so that $|A_k(\Gamma_\Pi)| = \frac{1}{2k}|A_k(\Pi)|$.

3 Bounds on $|A_k(\Gamma_\Pi)|, |B_k(\Pi)|$ and $|Q_{k,j}|$

Lemma 4. *Let Π be a projective plane of order n and $n \geq k \geq 4$. Then*

$$(n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi) \leq |A_k(\Gamma_\Pi)| \leq (n-1)(k-1)c_{2k-2}(\Gamma_\Pi).$$

Proof. We may obtain bounds on $|A_k(\Gamma_\Pi)|$ by counting the total number of cycles of length $2k-2$, and for each cycle adjoining a new point to one of the lines in the cycle. Let C be a cycle of length $2k-2$. Each line in $V(C)$ has $n+1$ neighbors with at least 2 of the neighbors in $V(C)$ and at most $k-1$ neighbors in $V(C)$. There are $k-1$ lines, and so we have that for each cycle we may create at least $(k-1)(n+1-(k-1)) = (k-1)(n-k+2)$ distinct subgraphs belonging to $A_k(\Gamma_\Pi)$, and at most $(k-1)(n-1)$. Clearly every graph in $A_k(\Gamma_\Pi)$ can be obtained this way. Furthermore, since every graph in $A_k(\Gamma_\Pi)$ corresponds to a unique cycle of length $2k-2$, then we have obtained each graph in $A_k(\Gamma_\Pi)$ exactly once. Thus $(n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi) \leq |A_k(\Gamma_\Pi)| \leq (n-1)(k-1)c_{2k-2}(\Gamma_\Pi)$. □

Let $QG_k = (P_1, P_2, \dots, P_k)$. We may associate with QG_k a sequence of k lines of the form $QG_k = (P_1P_2, P_2P_3, \dots, P_kP_1) = (\ell_1, \ell_2, \dots, \ell_k)$. Note the lines may not necessarily be distinct, and in the case that lines are repeated it is possible for distinct quasi k-gons to have the same line sequence. For the next two lemmas we will use this correspondence to obtain bounds on the remaining cases. We remind the reader that $Q_{k,j} = \{QG_k \in Q_k : |\mathcal{L}_{QG_k}| = j\}$ and $B_k(\Pi) = Q_{k,k-1} \setminus A_k(\Pi)$.

Lemma 5. *Let Π be a projective plane of order n and $k \geq 4$. Then*

$$|B_k(\Pi)| \leq (k-1)(k-2)N_{(k-1)}.$$

Proof. Let $QG_k = (P_1, P_2, \dots, P_k) \in B_k(\Pi)$ and let $(\ell_1, \ell_2, \dots, \ell_k)$ be its corresponding sequence of lines. By definition of $B_k(\Pi)$, we have that $|\mathcal{L}_{QG_k}| = k-1$ and for all i, $\ell_i \neq \ell_{i+1}$. This implies that we may describe each point $P \in \mathcal{P}_{QG_k}$ as an intersection of consecutive lines in the sequence $(\ell_1, \ell_2, \dots, \ell_k)$. More specifically, we have $P_i = \ell_{i-1} \cap \ell_i$ for all i. Therefore, distinct quasi k-gons must in fact have distinct line sequences.

We may then use the sequence of lines $(\ell_1, \ell_2, \dots, \ell_k)$ as an alternative description of QG_k. We now find an upper bound on the number of line sequences that correspond to quasi k-gons in $B_k(\Pi)$. In order to do this, we first consider all sequences of $(k-1)$ distinct lines. There are $N_{(k-1)}$ many such sequences. Then for each such sequence, we create sequences of k lines in the following way. Pick any line in the sequence of $(k-1)$

lines, which can be done in $k-1$ ways. Then insert it into the same sequence such that it will not appear next to itself, which can be done in at most $k-2$ ways. Therefore, the number of possible such sequences in total is at most $(k-1)(k-2)N_{(k-1)}$. Clearly, any line sequence corresponding to $QG_k \in B_k(\Pi)$ can be obtained this way, so that $(k-1)(k-2)N_{(k-1)}$ is an upper bound on $|B_k(\Pi)|$. □

Let $QG_k = (P_1, \ldots, P_k)$ with the corresponding sequence of lines $(\ell_1, \ell_2, \ldots, \ell_k)$ with exactly j lines being distinct. We will call a subsequence $(P_i, P_{i+1}, \ldots, P_{i+t})$ of consecutive points of QG_k a *maximal subsequence* if all its points lie on some line ℓ, but P_{i-1} and P_{i+t+1} do not lie on ℓ. Observe that if QG_k has a maximal subsequence of length $t+1$, then the corresponding line sequence of QG_k has a subsequence of t consecutive equal lines. We will also refer to this subsequence of lines as maximal.

Lemma 6. *Let k and j be fixed positive integers, with $k \geq 4$ and $k-2 \geq j \geq 2$. Let Π be a projective plane of order n, with $n \geq k$. Then*

$$|Q_{k,j}| \leq j^{k-j} k_{(j)} \binom{N}{j} (n-1)^{k-j} = O(n^{j+k}), \quad n \to \infty$$

and

$$|Q_{k,1}| = N(n+1)_{(k)}.$$

Proof. We follow similar line of reasoning as in the proof of Lemma 5 above. To obtain an upper bound for $|Q_{k,j}|$, we first obtain an upper bound on the number of distinct sequences of k lines made out of j distinct lines. We then proceed by placing an upper bound on the number of distinct $QG_k \in Q_{k,j}$ that can have the same corresponding sequence of lines.

We build the sequences in the following way. Begin with an empty sequence of k available positions. Choose any j disinct lines in Π. There are $\binom{N}{j}$ ways to do this. Then out of the k available posiitons in the empty sequence, choose j of them, and place the j chosen lines into these positions in any order. There are $k_{(j)}$ ways to do this. For every remaining empty position, place one of the chosen j lines to fill it. There are $k-j$ empty spots, and j possible choices for each, giving us j^{k-j} ways to fill all the remaining positions. So there are at most $j^{k-j}\binom{N}{j}k_{(j)}$ different sequences of lines with exactly j distinct lines appearing in the sequence.

Let $QG_k = (P_1, P_2, \ldots, P_k) \in Q_{k,j}$ and suppose that $(P_i, P_{i+1}, \ldots, P_{i+t})$ is a maximal subsequence of QG_k. As $j \geq 2$, we know that $t+1 < k$, meaning that this maximal subsequence is not equal to QG_k. Let $P_iP_{i+1} = \ell$ and consider $QG'_k = (P_1, \ldots, P_i, P'_{i+1}, \ldots, P'_{i+t-1}, P_{i+t}, \ldots, P_k)$ where $(P'_{i+1}, \ldots, P'_{i+t-1})$ are any sequence of distinct points from $\ell \setminus \{P_i, P_{i+t}\}$. Observe that QG_k and QG'_k have the same corresponding line sequence. There are $(n-1)_{t-1}$ different QG'_k that can be obtained in this way. In summary, for every maximal subsequence of length $t+1$ in QG_k, there are $(n-1)_{t-1}$ quasi k-gons that have the same line sequence as QG_k and only differ from QG_k by the interior points in the maximal subsequence.

Recall that in the line sequence of QG_k, each maximal subsequence of QG_k of length $t+1$ corresponds to a maximal subsequence of length t in the line sequence of QG_k.

Any two maximal subsequences contained in the line sequence are necessarily disjoint. Furthermore, we may uniquely partition the line sequence by maximal subsequences.

Let $L = (\ell_1, \ldots, \ell_k)$ be a sequence of k lines that corresponds to some $QG_k \in Q_{k,j}$. Partition L into its maximal subsequences. If the number of parts is r, then it is clear that $r \geq j \geq 2$. Let t_s be the length of each maximal subsequence, $1 \leq s \leq r$, then $t_1 + t_2 + \cdots + t_r = k$. For each maximal subsequence of length t_s, there are at most $(n-1)_{(t_s-1)}$ distinct quasi k-gons that have the same line sequence as QG_k and differ from QG_k only by the interior points of the corresponding maximal subsequence of $t_s + 1$ points of QG_k. Then the total number of QG_k's that have L as their line sequence is given by the product

$$\prod_{s=1}^{r} (n-1)_{(t_s-1)} \leq (n-1)^{(\sum_{s=1}^{r}(t_s-1))} = (n-1)^{k-r} \leq (n-1)^{k-j}.$$

When $j = 1$, this means that the sequence has only one distinct line. The number of such quasi k-gons is easy to count, as there are exactly N lines in Π and there are $(n+1)_{(k)}$ sequences of points that we can form from each line. $\square$

Theorem 3. *Let k be a fixed positive integer with $k \geq 4$. Let Π be a finite projective plane of order n, with $n \geq k$, and Γ_Π be its Levi graph. Then*

$$c_{2k}(\Gamma_\Pi) > \frac{1}{2k} N_{(k)} - \frac{1}{2}(n-1) N_{(k-1)} - \frac{(k-1)(k-2)}{2k} N_{(k-1)} - a_k n^{2k-2}.$$

where a_k is a constant dependent only on k.

Proof. Recall that

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k}(|Q_k| - |Q_{k,k-1}| - \cdots - |Q_{k,1}|).$$

Clearly $c_{2k-2}(\Pi) \leq |Q_{k-1}| = \frac{1}{2(k-1)} N_{(k-1)}$, which together with Lemma 4 implies that

$$\frac{1}{2k}|A_k(\Pi)| = |A_k(\Gamma_\Pi)| \leq (n-1)(k-1) c_{2k-2}(\Gamma_\Pi) \leq \frac{1}{2}(n-1) N_{(k-1)}.$$

Furthermore, as $A_k(\Pi) \dot\cup B_k(\Pi) = Q_{k,k-1}$, by applying the inequality above to $|A_k(\Pi)|$ and Lemma 5 to $|B_k(\Pi)|$, we have

$$\frac{1}{2k}|Q_{k,k-1}| = \frac{1}{2k}|A_k(\Pi)| + \frac{1}{2k}|B_k(\Pi)| \leq \frac{1}{2}(n-1) N_{(k-1)} + \frac{(k-1)(k-2)}{2k} N_{(k-1)}.$$

Finally, Lemma 6 implies that

$$\sum_{j=1}^{k-2} |Q_{k,j}| \leq a_k n^{2k-2}$$

for some constant a_k dependent only on k. Combining these results yields the claimed lower bound for $c_{2k}(\Gamma_\Pi)$. $\square$

Now we present an upper bound for $c_{2k}(\Gamma_\Pi)$.

Theorem 4. *Let* Π *be a finite projective plane of order* n *and* Γ_Π *be its Levi graph. If* k *is a fixed integer with* $n \geq k \geq 4$*, then*

$$c_{2k}(\Gamma_\Pi) \leq \frac{1}{2k} N_{(k)} - (n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi)$$

Proof. Observe that

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k}(|Q_k| - |Q_{k,k-1}| - \cdots - |Q_{k,1}|) \leq \frac{1}{2k}|Q_k| - \frac{1}{2k}|Q_{k,k-1}|.$$

As $|Q_{k,k-1}| = |A_k(\Pi)| + |B_k(\Pi)|$, then by Lemma 4,

$$\frac{1}{2k}|Q_{k,k-1}| \geq \frac{1}{2k}|A_k(\Pi)| = |A_k(\Gamma_\Pi)| \geq (n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi).$$

Recalling that $|Q_k| = N_{(k)}$, we obtain

$$c_{2k}(\Gamma_\Pi) \leq \frac{1}{2k} N_{(k)} - (n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi).$$

□

4 Proofs of Theorems 1 and 2

We now have all the tools we need to prove Theorem 1. For convenience we restate the theorem here.

Theorem 1. *Let* Π *be a projective plane of order* n *and* Γ_Π *be its Levi graph. Then for fixed* $k \geq 4$*,*

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k} n^{2k} + O(n^{2k-2}), \qquad n \to \infty$$

Proof. All we need to demonstrate, is that the coefficient of n^{2k-1} is 0. Theorem 3 states that

$$c_{2k}(\Gamma_\Pi) > \frac{1}{2k} N_{(k)} - \frac{1}{2}(n-1)N_{(k-1)} - \frac{(k-1)(k-2)}{2k} N_{(k-1)} - a_k n^{2k-2}.$$

Recalling that $N = n^2 + n + 1$, observe that the only terms that contribute to the coefficient of n^{2k-1} come from

$$\frac{1}{2k} N_{(k)} - \frac{1}{2}(n-1)N_{(k-1)}.$$

It is easy to see that both $\frac{1}{2k}N_{(k)}$ and $\frac{1}{2}(n-1)N_{(k-1)}$ have 1/2 as a coefficient of n^{2k-1}. This implies that the our lower bound has 0 as the coefficient of n^{2k-1}.

Now, Theorem 3 gave us that

$$c_{2k}(\Gamma_\Pi) \leq \frac{1}{2k} N_{(k)} - (n-k+2)(k-1)c_{2k-2}(\Gamma_\Pi).$$

By applying Theorem 4 to $c_{2k-2}(\Gamma_\Pi)$ we obtain an upper bound on $c_{2k}(\Gamma_\Pi)$ where the coefficient of n^{2k-1} is again 0. □

Corollary. Let $k \geq 4$, then

$$\mathrm{ex}(v, C_{2k}, \mathcal{C}_{\mathrm{odd}} \cup \{C_4\}) \geq \left(\frac{1}{2^{k+1}k} - o(1)\right) v^k, \qquad v \to \infty.$$

Proof. Theorem 1 implies that the stated result is in fact true when $v = 2(n^2+n+1)$ where n is a prime power with $n \geq k$. In [3], Baker, Harman, and Pintz show that for all sufficiently large v, there exists a prime in the interval $[v - v^{0.525}, v]$. A standard argument using this fact yields the corollary for all sufficiently large v. □

Theorem 5. *Let $k \geq 3$, then*

$$ex(v, C_{2k}, \mathcal{C}_{odd} \cup \{C_4\}) \leq \frac{1}{2k}\left(\frac{v}{2}\right)_{(k)}.$$

Proof. Let G be a bipartite graph containing no C_4 as a subgraph. As G is bipartite, we may partition $V(G)$ into two independent sets. Let A represent the smaller of the two parts, so that $|A| \leq v/2$. Let G^2 be the graph obtained from G in the following manner: $V(G^2) = A$ and if x, y are distinct vertices in A, then $x \sim y$ in G^2 when there exists $z \in V(G)$ such that $x \sim z \sim y$ in G. By definition, G^2 has no loops and since G has no C_4, then G^2 has no multiple edges, implying that G^2 is simple.

Suppose that C_{2k} is some cycle of length $2k$ in G. As G is bipartite, exactly k vertices of C_{2k} are in G, and as a result, every cycle of length $2k$ has a unique corresponding cycle of length k in G^2. Therefore, the number of k cycles in G^2, is an upper bound on the number of $2k$-cycles in G. As G^2 is simple, the number of k cycles in G^2 is no more than in a complete graph of order $|G^2|$. Thus the number of cycles of length $2k$ in G is no more than

$$\frac{1}{2k}|A|_{(k)} \leq \frac{1}{2k}\left(\frac{v}{2}\right)_{(k)}.$$

□

Finally, combining our results above, Theorem 2 falls out as a simple corollary.

Theorem 2. *Let $k \geq 4$ and $n \geq k$ be a prime power. If $v = 2(n^2+n+1)$, then*

$$\mathrm{ex}(v, C_{2k}, \mathcal{C}_{\mathrm{odd}} \cup \{C_4\}) = \left(\frac{1}{2^{k+1}k} - o(1)\right) v^k, \qquad v \to \infty.$$

Concluding Remarks

At this time, it is not known whether $c_{2k}(\Gamma_\Pi)$ is a polynomial in n for all k. For $k = 3, 4, \ldots, 10$, the exact value of $c_{2k}(\Gamma_\Pi)$ was determined in [22], [26], and it was shown that the function $c_{2k}(\Gamma_\Pi)$ is a polynomial of degree $2k$. Let $k = 3, 4, \ldots, 10$ and Π be a projective plane of order n, and

$$c_{2k}(\Gamma_\Pi) = a_{2k}n^{2k} + a_{2k-1}n^{2k-1} + \cdots + a_0.$$

The following table records the coefficient data for the first, second, third and fourth coefficients, namely the coefficients of $n^{2k}, n^{2k-1}, n^{2k-2}$ and n^{2k-3} respectively.

	a_{2k}	a_{2k-1}	a_{2k-2}	a_{2k-3}
$k=3$	1/6	1/3	1/3	1/6
$k=4$	1/8	0	-1/8	-1/8
$k=5$	1/10	0	0	-1/10
$k=6$	1/12	0	-1/2	0
$k=7$	1/14	0	-1	3/2
$k=8$	1/16	0	-3/2	3
$k=9$	1/18	0	-2	9/2
$k=10$	1/20	0	-5/2	6

Table 1: The 1st, 2nd, 3rd, and 4th coefficients in the formula for $c_{2k}(\Gamma_\Pi)$ up to $k=10$.

There is a pattern that can be observed for each coefficient in the table and for sufficiently large k. For a_{2k}, the pattern begins at $k=3$. For a_{2k-1}, the pattern begins at $k=4$. For a_{2k-2}, the patter begins at $k=5$, in which case we see a decrease by 1/2 every successive k after $k=5$. For a_{2k-3}, we notice an increment of 3/2 every successive k after $k=6$. In light of the information obtained from the table, we state the following conjecture.

Conjecture. *Let k be a fixed positive integer with $k \geq 6$. Let Π be a projective plane of order n and Γ_Π it's Levi graph, then*

$$c_{2k}(\Gamma_\Pi) = \frac{1}{2k}n^{2k} - \frac{1}{2}(k-5)n^{2k-2} + \frac{3}{2}(k-6)n^{2k-3} + O(n^{2k-4}), \quad n \to \infty.$$

We would like to conclude with the question posed at the beginning of this section:

Question. *Let k be a fixed positive integer with $k \geq 3$. Let Π be a projective plane of order n and Γ_Π it's Levi graph, is it the case that $c_{2k}(\Gamma_\Pi)$ is a polynomial in n?*

Acknowledgments

The author would like to thank Dr. Felix Lazebnik and Dr. Eric Moorehouse for posing the problem and for their valuable comments that improved both the results and the exposition of this paper.

References

[1] E. Aceves, D. Heywood, A. Klahr and O. Vega, *Cycles in projective spaces*, Journal of Geometry, 105, (2014), 111–117.

[2] N. Alon and C. Shikhelman, *Many T copies in H-free graphs*, Electronic Notes in Discrete Mathematics, 49, (2015), 683–689.

[3] R. C. Baker, G. Harman, and J. Pintz, *The Difference Between Consecutive Primes, II*, Proceedings of the London Mathematical Society, 83, (2001), 532–562.

[4] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, (1993).

[5] B. Bollobás, *Modern Graph Theory*, Springer Science and Business Media, (1998).

[6] B. Bollobás, *Extremal Graph Theory*, Academic Press London, (1978).

[7] R. Casse, *Projective Geometry: An Introduction*, Oxford University Press, (2006).

[8] S. De Winter, F. Lazebnik and J. Verstraëte, *An Extremal Characterization of Projective Planes*, Electronic Journal of Combinatorics, 15(1), (2008), 1–13.

[9] P. Erdös, *On some problems in graph theory, combinatorial analysis and combinatorial number theory*, Graph Theory and Combinatorics, (1983), 1–17.

[10] G. Fiorini and F. Lazebnik, *On a bound for the number of $C_8's$ in 4-cycle free bipartite graph*, Congressus Numerantium , 99, (1994), 191–197.

[11] Z. Füredi, *Turán type problems*, Survey's in Combinatorics, Cambridge University Press, (1991).

[12] Z. Füredi and M. Simonovits *The history of degenerate(bipartite) extremal graph problems*, Erdoös Cenetnnial, Bolyai Society Mathematical Studies, 25, (2013), pp169–264.

[13] D. Gerbner and C. Palmer, *Counting copies of a fixed subgraph in F-free graphs*, European Journal of Combinatorics, 82, (2019).

[14] D. Gerbner, E. Györi, A. Methuku and M. Vizer, *Generalized Turán Problems for Even Cycles*, Journal of Combinatorial Theory, Series B, 145, (2020), 169–213

[15] D. G. Glynn, *Rings of Geometries II*, Journal of Combinatorial Theory, Series A, 49(1), (1988), 26–66.

[16] A. Grzesik, *On the maximum number of five-cycles in a triangle free graph*, Journal of Combinatorial Theory, Series B, 102(5), (2012), 1061–1066.

[17] E. Györi, *On the number of C_5's in a triangle-free graph*, Combinatorica, 9, (1989), 101–102.

[18] H. Hatami, J. Hladký, D. Král, S. Norine and A. Razborov, *On the number of pentagons in triangle-free graphs*, Journal of Combinatorial Theory Series A, 120(3), (2013), 722–732.

[19] J. Verstraëte, *A survey of Turán problems for expansions*, In: Recent Trends in Combinatorics, (2016), 83–116.

[20] N. Kaplan, S. Kimport, R. Lawrence, L. Peilen and M. Weinreich, *Counting arcs in projective planes via Glynn's algorithm* , Journal of Geometry, 108, (2017), 1013–1029.

[21] F. Lazebnik, K. Mellinger and O. Vega, *Embedding Cycles in Finite Planes*, Electronic Journal of Combinatorics, 20(3), 2013.

[22] F. Lazebnik, K. Mellinger and O. Vega, *On the Number of k-Gons in Finite Projective Planes*, Note di Matematica, 29, (2009), 135–152.

[23] F. Lazebnik, S. Sun and Y. Wang, *Some Families of Graphs, Hypergraphs and Digraphs Defined by Systems of Equations: A Survey.*, Lecture Notes of Seminario Interdisciplinare di Matematica, 14, (2017), 105—142.

[24] D. Mubayi and J. Verstraëte, *A survey of Turán problems for expansions*, In: Recent Trends in Combinatorics, (2016), 117–143.

[25] J. Solymosi and C. Wong, *Cycles in graphs of fixed girth with large size*, European Journal of Combinatorics, 62, (2017), 124–131.

[26] A. N. Voropaev, *Counting k-gons in finite projective planes*, Siberian Electronic Mathematical Reports, 10, (2013), 241–270.