

# Perfect matchings, eigenvalues and expansion

Sebastian M. Cioabă \*

Department of Mathematics  
Queen's University at Kingston  
Ontario, K7L 3N6, Canada

AMS Subject Classification Numbers: 05C50, 05C70

keywords: eigenvalues, matchings, expanders

October 3, 2005

## Abstract

In this note, we prove a sufficient condition for the existence of a perfect matching in a regular graph in terms of its eigenvalues and its expansion constant. We improve a recent result of Brouwer and Haemers.

French version: Dans cette note, nous prouvons un état suffisant pour l'existence d'un assortiment parfait dans un graphe régulier en termes de ses valeurs propres et son constante d'expansion. Nous améliorons un résultat récent de Brouwer et Haemers.

## 1 Preliminaries

Our graph notation is standard, see West [9]. For a graph  $G$  on  $n$  vertices, we denote by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  the eigenvalues of its adjacency matrix. If  $G$  is a connected  $k$ -regular graph, it is well known that  $\lambda_1 = k$  and we let  $\lambda(G) = \max_{\lambda_i \neq \pm k} |\lambda_i|$ . For  $S \subset V(G)$ , we let  $e(S, S^c)$  denote the number of edges with exactly one endpoint in  $S$ . We denote  $\text{odd}(G \setminus S)$  the number of odd components of  $G \setminus S$ . The expansion constant of a graph  $G$  is  $h(G) =$

---

\*Research partially supported by an Ontario Graduate Scholarship

$\min \frac{e(S, S^c)}{|S|}$ , where the minimum is taken over all  $S \subset V(G)$  with  $|S| \leq \frac{|V(G)|}{2}$ . For a  $k$ -regular graph  $G$ , it is known (see [1, 2, 8]) that

$$\frac{k - \lambda_2}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_2)}$$

In this note, we show that if  $G$  is a  $k$ -regular graph on  $n$  vertices,  $n$  is even and

$$h(G) > \begin{cases} \frac{k-2}{k+1} & \text{if } k \text{ is even} \\ \frac{k-2}{k+2} & \text{if } k \text{ is odd} \end{cases} \quad (1)$$

then  $G$  has a perfect matching.

Krivelevich and Sudakov [7] show that if  $G$  is a  $k$ -regular graph and  $k - \lambda(G) \geq 2$ , then  $G$  contains a perfect matching. Recently, Brouwer and Haemers [3] proved the following stronger result.

**Theorem 1.1** (Brouwer-Haemers, 2005). *A connected  $k$ -regular graph on  $n$  vertices with eigenvalues  $k = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ , and  $n$  even which satisfies*

$$k - \lambda_3 \geq \begin{cases} 1 - \frac{3}{k+1} & \text{if } k \text{ is even} \\ 1 - \frac{3}{k+2} & \text{if } k \text{ is odd,} \end{cases} \quad (2)$$

*has a perfect matching.*

In this note, we slightly improve Brouwer and Haemers' result when  $k$  is odd. We prove that if a  $k$ -regular graph  $G$  satisfies  $k - \lambda_3 \geq 1 - \frac{4}{k+2}$ , then  $G$  has a perfect matching.

The main tool in our proofs is the famous theorem of Tutte (see [9], page 137).

**Theorem 1.2** (Tutte, 1947). *A graph  $G$  has a perfect matching if and only if*

$$\text{odd}(G \setminus S) \leq |S|$$

*for each  $S \subset V(G)$ .*

## 2 Expansion and perfect matchings

In this section, we determine a lower bound on the expansion constant of a regular graph that implies the existence of a perfect matching.

**Theorem 2.1.** *Let  $G$  be a  $k$ -regular graph. If*

$$h(G) > \begin{cases} \frac{k-2}{k+1} & \text{if } k \text{ is even} \\ \frac{k-2}{k+2} & \text{if } k \text{ is odd} \end{cases} \quad (3)$$

*then  $G$  contains a perfect matching.*

*Proof.* Assume  $G$  has no perfect matching. By Tutte's theorem, it follows that there exists a subset  $S$  such that  $q = \text{odd}(G \setminus S) > |S| = s$ . Let  $G_1, \dots, G_q$  denote the odd components of  $G \setminus S$ . Denote by  $n_i$  and  $e_i$  the order and the size of  $G_i$  respectively. It is easy to see that  $\sum_{i=1}^q n_i + s$  is even. Since each  $n_i$  is odd, it follows that  $q + s$  is even. Because  $q > s$ , we deduce that  $q \geq s + 2$ .

For  $i \in [q]$ , denote by  $t_i$  the number of edges with one endpoint in  $G_i$  and another in  $S$ . Because  $G$  is connected, it follows that  $t_i \geq 1$  for each  $i \in [q]$ . Also, since vertices in  $G_i$  are adjacent only to vertices in  $G_i$  or  $S$ , we deduce that  $2e_i = kn_i - t_i = k(n_i - 1) + k - t_i$ . Because  $n_i$  is odd, it follows that  $k - t_i$  is even. Thus,  $t_i$  and  $k$  have the same parity for each  $i$ .

The sum of the degrees of the vertices in  $S$  is at least the number of edges between  $S$  and  $\cup_{i=1}^q G_i$ . Thus,  $ks \geq \sum_{i=1}^q t_i$ . Since  $q \geq s + 2$ , it follows that there are at least three  $t_i$ 's such that  $t_i < k$ . This implies there are at least three  $t_i$ 's satisfying  $t_i \leq k - 2$ . If  $t_i \leq k - 2$ , then  $n_i > 1$ . Assume  $t_i \leq k - 2$  for  $i \in [3]$ . If  $t_i \leq k - 2$ , then  $n_i(n_i - 1) \geq 2e_i = kn_i - t_i \geq kn_i - k + 2$ . Thus,  $n_i \geq k + \frac{2}{n_i - 1}$ . Hence,  $n_i \geq k + 1$ . Now, since  $k$  and  $n_i$  are both odd, we obtain  $n_i \geq k + 2$ .

Now  $n_1 + n_2 + n_3 < n$  implies that there is at least one  $i$  such that  $n_i < \frac{n}{2}$ . Assume  $n_1 < \frac{n}{2}$ . Then  $\frac{t_1}{n_1} \leq \frac{k-2}{k+1}$  if  $k$  is even and  $\frac{t_1}{n_1} \leq \frac{k-2}{k+2}$  if  $k$  is odd. This contradicts the assumption made on the expansion constant and thus, proves the theorem.  $\square$

In the next section, we provide examples that show that this result is best possible.

### 3 A slight improvement of Theorem 1.1

In this section, we present the proof of our improvement of Theorem 1.1.

**Theorem 3.1.** *If  $G$  is a  $k$ -regular graph,  $k$  is odd and*

$$k - \lambda_3 \geq 1 - \frac{4}{k+2} \quad (4)$$

*then  $G$  has a perfect matching.*

*Proof.* Assume that  $G$  has no perfect matching. As in the proof of Theorem 2.1, one shows that there exists  $S$  such that  $G \setminus S$  has at least three components of odd order  $G_i$ ,  $i \in [3]$  such that  $t_i \leq k - 2$  and  $n_i \geq k + 2$ .

Thus, the average degree  $\overline{d}_i$  of  $G_i$  satisfies the following inequality

$$\overline{d}_i = \frac{2e_i}{n_i} = k - \frac{t_i}{n_i} \geq k - \frac{k-2}{k+2} \quad (5)$$

Let  $\mu_i$  denote the largest eigenvalue of  $G_i$  for  $i \in [3]$ . Suppose  $\mu_1 \geq \mu_2 \geq \mu_3$ . Then, by interlacing in  $G_1 \cup G_2 \cup G_3$  (see [6], Chapter 9), it follows that  $\lambda_3 \geq \mu_3$ . Now,  $\mu_3 \geq \overline{d}_3$  with equality if and only if  $G_3$  is regular. But  $G_3$  is not regular since  $k > \overline{d}_3 = k - \frac{t_3}{n_3} > k - 1$ .

Thus,  $\lambda_3 \geq \mu_3 > k - \frac{k-2}{k+2}$ . This is a contradiction with (4) and proves the theorem.  $\square$

For  $k$  even, Brouwer and Haemers [3] construct examples of  $k$ -regular graphs with  $\lambda_3 = k - 1 + \frac{3}{k+1} + O(k^{-2})$  that contain no perfect matchings. This is done by taking  $k$  copies of  $G'$ , where  $G'$  is  $K_{k+1}$  from which a matching of size  $\frac{k-2}{2}$  is deleted. Add  $k - 2$  new vertices and join each of these vertices to a vertex of degree  $k - 1$  in each  $G'$ .

For  $k$  odd, Brouwer and Haemers [3] just mention that their construction is similar to the previous one, but slightly more complicated.

We describe below a construction for  $k$  odd. Let  $H'$  be a graph on  $k + 2$  vertices whose complement is the union of two disjoint edges and a cycle of length  $k - 2$ . Take  $k$  copies of  $H'$ . Add  $k - 2$  new vertices and join each of these vertices to a vertex of degree  $k - 1$  in each  $H'$ . This is a connected  $k$ -regular graph that has  $k^2 + 3k - 2$  vertices and no perfect matching. Also,

$$\lambda_3 = \lambda_1(H') = \frac{k - 3 + \sqrt{k^2 + 2k + 17}}{2} = k - 1 + \frac{4}{k+2} + O(k^{-2})$$

This shows that for  $n$  even and  $k$  odd, there are  $k$ -regular graphs with no perfect matching, for which  $\lambda_3$  gets arbitrarily close to the value from Theorem 3.1. These examples also show that Theorem 2.1 is best possible.

Note that the existence of perfect matchings in a  $k$ -regular graph does not even imply  $k - \lambda_2 > \epsilon > 0$ . The Cayley graph of  $\mathbb{Z}_{2n}$  with generating set  $S = \{\pm 1, n\}$  is a 3-regular graph that contains 3 disjoint perfect matchings and satisfies  $\lambda_2 = 2 \cos \frac{2\pi}{n} + 1$ . The difference  $k - \lambda_2 = 2 - 2 \cos \frac{2\pi}{n}$  tends to 0 as  $n$  gets large.

On the other hand, it follows from a theorem of Weyl that by adding perfect matchings to regular graphs, the eigenvalues of the new graphs increase by at most 1. This was used in [5] to construct new families of regular graphs with small non-trivial eigenvalues. See also [4] for related results regarding the eigenvalues of regular graphs.

## Acknowledgments

I thank David Gregory and Ram Murty for many useful comments.

## References

- [1] N. Alon, Eigenvalues and expanders, *Combinatorica*, **6** (1986), 83-96.
- [2] N. Alon and V. Milman,  $\lambda_1$ , isoperimetric inequalities for graphs, and superconcentrators, *J. Combin. Theory, Series B*, **38** (1985), 73-88.
- [3] A. Brouwer and W. Haemers, Eigenvalues and perfect matchings, *Linear Algebra and Its Applications*, **395** (2005), 155-162.
- [4] S. M. Cioabă, Eigenvalues, Expanders and Gaps between Primes, Ph.D. Thesis, Queen's University at Kingston, (2005).
- [5] S. M. Cioabă and R. Murty, Expander graphs and gaps between primes, submitted.
- [6] C. Godsil and G. Royle, Algebraic Graph Theory, Springer Verlag New York, (2001).
- [7] M. Krivelevich and B. Sudakov, Pseudo-random graphs, submitted.
- [8] M. R. Tanner, Explicit constructions from generalized  $n$ -gons, *SIAM Journal on Algebraic Discrete Methods*, **5** (1984), 287-293.

- [9] D. B. West, Introduction to graph theory, Second Edition, Prentice Hall (2001).