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Note

Sums of powers of the degrees of a graph☆

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Abstract

For a graph G and k a real number, we consider the sum of the kth powers of the degrees of the vertices of G. We present some general bounds on this sum for various values of k.

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Keywords: Degree powers; Bounds

1. Introduction

Let $d(u)$ be the degree of the vertex u in the graph G and let $\Sigma_k(G)=\sum_{x\in V(G)} d^k(x)$. Obviously, $\Sigma_0(G)=|V(G)|$ and $\Sigma_1(G)=2|E(G)|$. In [8], Székely et al. showed that

$$\Sigma_k(G)\leqslant(\Sigma_{1/k}(G))^k, \tag{1}$$

for every integer $k\geqslant 1$.

De Caen proved the following inequality in [3]:

$$\Sigma_2(G)\leqslant e\left(\frac{2e}{n-1}+n-2\right), \tag{2}$$

where n is the order and e is the size of the graph G. This bound was generalized to hypergraphs by Bey in [1] and improved by Das in [2]. De Caen's inequality was also used by Li and Pan in [6] to provide an upper bound on the largest eigenvalue of the Laplacian of a graph. We discuss the Li–Pan bound in Section 6. The kth moment of the degree sequence of a graph G of order n is $\mu_k(G)=\Sigma_k(G)/n$. Sharp bounds for the moments of the degree sequences of monotone families of graphs were obtained by Füredi and Kündgen in [4].

In this paper, we prove a general upper bound on $\Sigma_k(G)$ for k real number. The main result is contained in Section 3. When $k=2$, we obtain the two results of Das that improve De Caen's bound.

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2. Preliminaries

We denote by $B_{n,t}$ the graph on n vertices with exactly t vertices of degree $n-1$ and the remaining of $n-t$ vertices forming an independent set. Notice that $B_{n,1}=K_{1,n-1}$ and $B_{n,n}=K_n$. The maximum degree of a graph G will be denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. We denote by $m_k(u)$ the average of the kth powers of the degrees of the vertices v adjacent to u and by $p_k(u)$ the average of the kth powers of the degrees of the vertices $w \neq u$ that are not adjacent to u. Then

$$\Sigma_k(G)=d^k(u)+d(u)m_k(u)+(n-1-d(u))p_k(u),$$

for each vertex u in G. It follows that

$$d^k(u)+(n-1)m_k(u)=\Sigma_k(G)+(n-1-d(u))(m_k(u)-p_k(u))$$

which implies

$$\frac{d^k(u)}{n-1}+m_k(u)=\frac{\Sigma_k(G)}{n-1}+\left(1-\frac{1}{n-1}d(u)\right)(m_k(u)-p_k(u)). \tag{3}$$

We will use Eq. (3) to obtain upper bounds on $\Sigma_k(G)$.

Lemma 1. *If G is a graph and k a real number, then*

$$\Sigma_{k+1}(G)=\sum_{u\in V(G)} d(u)m_k(u). \tag{4}$$

Proof. Using a double summation, we get

$$\begin{aligned}\sum_{u\in V(G)} d(u)m_k(u) &= \sum_{u\in V(G)}\sum_{v\sim u} d^k(v)\\ &= \sum_{v\in V(G)}\sum_{u\sim v} d^k(v)=\sum_{v\in V(G)} d^{k+1}(v)=\Sigma_{k+1}(G). \qquad \square\end{aligned}$$

Using Eq. (3), we obtain the following result.

Lemma 2. *Let G be a graph and k a positive number. Then*

$$\frac{d^k(u)}{n-1}+m_k(u)\leqslant\frac{\Sigma_k(G)}{n-1}+\left(1-\frac{1}{n-1}d(u)\right)(\Delta^k(G)-\delta^k(G)), \tag{5}$$

for each $u\in V(G)$. Equality holds if and only if either $d(u)=n-1$ or $d(v)=\Delta(G)$ for each v adjacent to u and $d(w)=\delta(G)$ for each $w\neq u$ non-adjacent to u.

Proof. Obviously, $1-d(u)/(n-1)\geqslant 0$ with equality if and only if $d(u)=n-1$. Because $k>0$, it follows that $m_k(u)-p_k(u)\leqslant \Delta^k(G)-\delta^k(G)$, for each $u\in V(G)$. These two relations together with Eq. (3) imply the desired inequality. Equality holds iff $d(u)=n-1$ or $m_k(u)=\Delta^k(G)$ and $p_k(u)=\delta^k(G)$. The second condition is equivalent to $d(v)=\Delta(G)$ for each v adjacent to u and $d(w)=\delta(G)$ for each $w\neq u$ non-adjacent to u. $\square$

3. Main results

Theorem 3. *If G is a connected graph and k a positive number, then*

$$\Sigma_{k+1}(G)\leqslant\frac{2e}{n}(\Sigma_k(G)+(n-1)(\Delta^k-\delta^k))-\frac{\Delta^k-\delta^k}{n}\Sigma_2(G), \tag{6}$$

where $\Delta=\Delta(G)$, $\delta=\delta(G)$ and $e=|E(G)|$. Equality holds if and only if $G=B_{n,t}$ for some $1\leqslant t\leqslant n$ or G is regular.

Proof. Using (5), we get

$$\frac{d^k(u)}{n-1} + m_k(u) \leqslant \frac{\Sigma_k(G)}{n-1} + (\Delta^k - \delta^k) - \frac{\Delta^k - \delta^k}{n-1} d(u), \tag{7}$$

for each $u \in V(G)$. Multiplying by $d(u)$, summing over all $u \in V(G)$ and using (4), we obtain

$$\Sigma_{k+1}(G) \leqslant \frac{2e}{n}(\Sigma_k(G) + (n-1)(\Delta^k - \delta^k)) - \frac{\Delta^k - \delta^k}{n}\Sigma_2(G).$$

Notice that we have equality in (6) if and only if we have equality in (7) for each $u \in V(G)$. It follows that equality holds if and only if for each $u \in V(G)$, we either have $d(u) = n-1$ or $d(v) = \Delta$ for v adjacent to u and $d(w) = \delta$ for $w \neq u$ non-adjacent to u. Obviously, if G is regular then equality holds.

Assume that equality holds and $\Delta > \delta$. We will first show that $\Delta = n-1$. Let $u \in V(G)$ such that $d(u) < n-1$. Then all the neighbors of u have degree Δ and all the vertices non-adjacent to u have degree δ. If all the vertices $v \neq u$ are adjacent to u, then $\Delta = n-1$. Otherwise, because G is connected, there is a vertex w adjacent to a neighbor v of u such that w is not adjacent to u. Thus, $d(w) = \delta$. If $\Delta = d(v) < n-1$, then all the neighbors of v (including w) have degree Δ, contradiction. Hence, $\Delta = n-1$.

Let t be the number of vertices u in G such that $d(u) = n-1$. Then the remaining $n-t$ vertices form an independent set. Suppose vw is an edge with $d(v) < n-1$ and $d(w) < n-1$. Since $d(v) < n-1$, it follows that $d(w) = \Delta = n-1$, contradiction. Hence, $G = B_{n,t}$ in this case. Finally, it is clear that each graph $B_{n,t}$ gives equality in (6). This completes the equality case analysis. □

Remark 4. By Chebyschev's inequality, we have

$$\Sigma_{k+1}(G) \geqslant \frac{2e}{n}\Sigma_k(G), \tag{8}$$

with equality if and only if G is regular. Thus, the difference between these two bounds on $\Sigma_{k+1}(G)$ is $(\Delta^k - \delta^k)(2e(n-1) - \Sigma_2(G))/n$ which gets smaller when the graph is almost regular.

Remark 5. If $k = 1$ in (6), we obtain the following inequality:

$$\Sigma_2(G) \leqslant \frac{2e(2e + (n-1)(\Delta - \delta))}{n + \Delta - \delta}. \tag{9}$$

This is Theorem 4.2 from [2]. Note that Theorem 4.2 of [2] states that equality holds in (9) if and only if G is regular or G is the disjoint union of $K_{\Delta+1}$ with $n - \Delta - 1$ isolated vertices. This is not correct since equality holds in (9) also for $G = B_{n,t}$ when $1 \leqslant t \leqslant n$. Note that inequality (9) implies De Caen's inequality (2). This is because the left-hand side of (9) gets larger as $\Delta - \delta$ increases and $\Delta - \delta \leqslant n-2$.

For $k = 1$ in Lemma 2, we obtain

$$\begin{aligned} d(u) + m_1(u) &\leqslant \frac{2e}{n-1} + (n - 2 - (\Delta(G) - \delta(G)))\frac{d(u)}{n-1} + (\Delta(G) - \delta(G)) \\ &\leqslant \frac{2e}{n-1} + (n - 2 - (\Delta(G) - \delta(G)))\frac{\Delta(G)}{n-1} + (\Delta(G) - \delta(G)), \end{aligned}$$

for each $u \in V(G)$. Multiplying by $d(u)$ and summing over all $u \in V(G)$, we obtain the following inequality:

$$\Sigma_2(G) \leqslant e\left(\frac{2e}{n-1} + (\Delta(G) - \delta(G))\left(1 - \frac{\Delta(G)}{n-1}\right) + \frac{n-2}{n-1}\Delta(G)\right). \tag{10}$$

Equality holds if and only if $\Delta(G) = n-1$ and $\delta(G) = 1$ or G is regular. Inequality (10) is Theorem 4.1 from [2]. It also implies De Caen's inequality (2) because $\Delta - \delta \leqslant n-2$.

Remark 6. The bounds (9) and (10) are incomparable. If $G = P_n$, the path on n vertices, then (10) is better than (9). If $G = W_n$, the wheel with n spokes, then (9) is better than (10).

For $k=2$ in Theorem 3, we obtain the following upper bound on the sum of the cubes of the degrees of the vertices in a connected graph.

Corollary 7. *If G is a connected graph, then*

$$\Sigma_3(G) \leqslant \frac{2e-(\Delta^2-\delta^2)}{n}\Sigma_2(G) + \frac{2e(n-1)(\Delta^2-\delta^2)}{n}. \tag{11}$$

Equality holds if and only if G is regular or $G = B_{n,t}$ for some t with $1 \leqslant t \leqslant n$.

Inequalities (11) and (1) (for $k=3$) are not comparable. For $G = W_n$ the wheel with n spokes, inequality (1) is better than (11). For $G = K_{n,n+1}$, inequality (11) is better than (1).

4. A general lower bound on $\Sigma_{1/2}(G)$

In [7], Linial and Rozenman showed that among all graphs G with n vertices and $\binom{k}{2}$ edges, the one that minimizes $\Sigma_{1/2}(G)$ is the union of K_k and $n-k$ isolated vertices. They conjectured that among all the graphs G with n vertices and e edges, $\binom{k-1}{2} < e < \binom{k}{2}$, the minimum $\Sigma_{1/2}(G)$ is attained precisely for the graph whose only non-trivial component is the graph obtained from K_{k-1} by adding one new vertex of degree $e - \binom{k-1}{2}$. This conjecture was proved by Ismailescu and Stefanica in [5].

We present a general bound on $\Sigma_{1/2}(G)$ when G is connected.

Theorem 8. *If G is a connected graph, then*

$$\Sigma_{1/2}(G) \geqslant \frac{\sqrt{8en + (\sqrt{\Delta}-\sqrt{\delta})^2(n-1-2e/n)^2} - (\sqrt{\Delta}-\sqrt{\delta})(n-1-2e/n)}{2}. \tag{12}$$

Equality holds iff G is regular.

Proof. If $k=\frac{1}{2}$ in (6), we get

$$\Sigma_{3/2}(G) \leqslant \frac{2e}{n}\Sigma_{1/2} + \frac{2e(n-1)}{n}(\sqrt{\Delta}-\sqrt{\delta}) - \frac{\sqrt{\Delta}-\sqrt{\delta}}{n}\Sigma_2(G),$$

for any connected graph G. Since $\Sigma_2(G) \geqslant (2e)^2/n$ (Cauchy–Schwarz) and $\Sigma_{3/2}(G)\Sigma_{1/2}(G) \geqslant (2e)^2$ (Cauchy–Schwarz), it follows that

$$(\Sigma_{1/2}(G))^2 + (\sqrt{\Delta}-\sqrt{\delta})\left(n-1-\frac{2e}{n}\right)\Sigma_{1/2}(G) - 2en \geqslant 0.$$

This implies the result stated in the theorem. □

The inequality (1) for $k=2$ can be regarded also as a general lower bound on $\Sigma_{1/2}(G)$. It states that $\Sigma_{1/2}(G) \geqslant \sqrt{\sum_{v\in V(G)} d^2(v)}$, for any graph G. If G is d-regular on n vertices, inequality (12) is always equality, whereas the inequality (1) gives a lower bound of $d\sqrt{n} < n\sqrt{d} = \Sigma_{1/2}(G)$.

5. A general upper bound on Σ_{-1}

By the Cauchy–Schwarz inequality, we have

$$\Sigma_{-1}(G) \geqslant \frac{n^2}{\Sigma_1(G)} = \frac{n^2}{2e},$$

for any graph G. Taking $k=-1$ in (3), we get

$$d^{-1}(u)+\frac{m_{-1}(u)}{n-1}\geqslant\frac{\Sigma_{-1}(G)}{n-1}+\left(1-\frac{d(u)}{n-1}\right)(\delta^{-1}-\Delta^{-1})$$

for any vertex $u\in V(G)$. Thus,

$$\frac{\Sigma_{-1}(G)}{n-1}\leqslant d^{-1}(u)+\frac{m_{-1}(u)}{n-1}+\frac{\Delta-\delta}{\Delta\delta}\left(1-\frac{d(u)}{n-1}\right),$$

for each vertex $u\in V(G)$. Multiplying by $d(u)$ and summing over all $u\in V(G)$, we obtain

$$\frac{2e}{n-1}\Sigma_{-1}(G)\leqslant n+\frac{\sum_{u\in V(G)}d(u)m_{-1}(u)}{n-1}+\frac{\Delta-\delta}{\Delta\delta}\sum_{u\in V(G)}d(u)\left(1-\frac{d(u)}{n-1}\right).$$

Using (4) with $k=-1$, we deduce that

$$\begin{aligned}\Sigma_{-1}(G)&\leqslant\frac{n^2}{2e}+(n-1)\frac{\Delta-\delta}{2e\Delta\delta}\sum_{u\in V(G)}d(u)\left(1-\frac{d(u)}{n-1}\right)\\&=\frac{n^2}{2e}+\left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(n-1-\frac{\Sigma_2(G)}{2e}\right)\\&\leqslant\frac{n^2}{2e}+\left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right).\end{aligned}$$

Hence, we have proved the following result.

Theorem 9. *If G is a connected graph, then*

$$\frac{n^2}{2e}\leqslant\sum_{u\in V(G)}\frac{1}{d(u)}\leqslant\frac{n^2}{2e}+\left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right),$$

with equality if and only if G is regular or $G=K_{1,n-1}$.

6. The Li–Pan inequality on the largest Laplacian eigenvalue of a graph

For a graph G, the Laplacian of G is $L(G)=D(G)-A(G)$, where $D=\mathrm{diag}(d(u))_{u\in V(G)}$ and $A(G)$ is the adjacency matrix of G. The eigenvalues of $L(G)$ are $\lambda_1(G)\geqslant\lambda_2(G)\geqslant\cdots\geqslant\lambda_n(G)=0$. Notice that $L(G)+L(G^c)=nI_n-J_n$, where G^c is the complement of G and J_n is the n by n matrix with all the entries 1. It follows easily that $\lambda_1(G)\leqslant n$, for any graph G on n vertices.

In [6], Li and Pan used De Caen's inequality (2) to prove the following upper bound on $\lambda_1(G)$ (see Theorem 3.1. in [6]).

Theorem 10 (*Li–Pan*). *Let be a G connected graph with n vertices and e edges. Then*

$$\lambda_1(G)\leqslant\frac{2e+\sqrt{(n-2)e(n(n-1)-2e)}}{n-1},\tag{13}$$

with equality if and only if G is one of K_n and $K_{1,n-1}$.

Inequality (13) can be deduced easily without applying de Caen's inequality. It follows from $\lambda_1(G)\leqslant n$ and $n\leqslant\frac{2e+\sqrt{(n-2)e(n(n-1)-2e)}}{n-1}$. We have already proved the first inequality. The inequality

$$n\leqslant\frac{2e+\sqrt{(n-2)e(n(n-1)-2e)}}{n-1}$$

is equivalent to $n-1\leqslant e$. This is true since G is connected.

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