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On the extreme eigenvalues of regular graphs

Sebastian M. Cioabă[1]

Department of Mathematics, Queen's University, Kingston, Ont., Canada, K7L 3N6

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To the memory of Dom de Caen

Abstract

In this paper, we present an elementary proof of a theorem of Serre concerning the greatest eigenvalues of k-regular graphs. We also prove an analogue of Serre's theorem regarding the least eigenvalues of k-regular graphs: given $\varepsilon > 0$, there exist a positive constant $c = c(\varepsilon, k)$ and a non-negative integer $g = g(\varepsilon, k)$ such that for any k-regular graph X with no odd cycles of length less than g, the number of eigenvalues μ of X such that $\mu \leqslant -(2-\varepsilon)\sqrt{k-1}$ is at least $c|X|$. This implies a result of Winnie Li.

Keywords: Alon–Boppana theorem; Eigenvalues of graphs

1. Preliminaries

Let X be a graph and let v_0 be a vertex of X. A *closed walk* in X of length $r \geqslant 0$ starting at v_0 is a sequence $v_0, v_1, \dots, v_r$ of vertices of X such that $v_r = v_0$ and v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant r$. For $r \geqslant 0$, let $\Phi_r(X)$ denote the number of closed walks of length r in X. A *cycle* of length r in X is a subgraph of X whose vertices can be labeled $v_0, \dots, v_r$ such that $v_0, \dots, v_r$ is a closed walk in X and $v_i \neq v_j$ for all i, j with $0 \leqslant i < j \leqslant r$. The *girth*, denoted $\mathrm{girth}(X)$, of X is the length of a smallest cycle in X if such a cycle exists and ∞ otherwise; the *oddgirth*, denoted $\mathrm{oddg}(X)$, of X is the length of a smallest odd cycle in X if such a cycle exists and ∞ otherwise. The adjacency matrix of X is the matrix $A = A(X)$ of order $|X|$, where the (u, v) entry is 1 if the vertices u and v are adjacent and 0 otherwise. It is a well known fact that $\Phi_r(X) = \mathrm{Tr}(A^r)$, for any $r \geqslant 0$. The eigenvalues of X are the eigenvalues of A. If X is k-regular, then it is easy to see that k is an eigenvalue of X with multiplicity equal to the number of components of X and that any eigenvalue

E-mail address: sebi@mast.queensu.ca.
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λ of X satisfies $|\lambda| \leqslant k$. For $l \geqslant 1$, we denote by $\lambda_l(X)$ the lth greatest eigenvalue of X and by $\mu_l(X)$ the lth least eigenvalue of X.

2. An elementary proof of Serre's theorem

Serre has proved the following theorem (see [4,5,7,15]) using Chebyschev polynomials. See also [2] for related results. In this section, we present an elementary proof of Serre's result.

Theorem 1. *For each $\varepsilon > 0$, there exists a positive constant $c = c(\varepsilon, k)$ such that for any k-regular graph X, the number of eigenvalues λ of X with $\lambda \geqslant (2-\varepsilon)\sqrt{k-1}$ is at least $c|X|$.*

For the proof of this theorem we require the next lemma which can be deduced from McKay's work [11, Lemma 2.1]. For the sake of completeness, we include a short proof here.

Lemma 2. *Let v_0 be a vertex of a k-regular graph X. Then the number of closed walks of length $2s$ in X starting at v_0 is greater than or equal to $\frac{1}{s+1}\binom{2s}{s}k(k-1)^{s-1}$.*

Proof. The number of closed walks of length $2s$ in X starting at v_0 is at least the number of closed walks of length $2s$ starting at a vertex u_0 in the infinite k-regular tree. To each closed walk in the infinite k-regular tree, there corresponds a sequence of non-negative integers $\delta_1, \ldots, \delta_{2s}$, where δ_i is the distance from u_0 after i steps. The number of such sequences is the sth Catalan number $\frac{1}{s+1}\binom{2s}{s}$. For each sequence of distances, there are at least $k(k-1)^{s-1}$ closed walks of length $2s$ since for each step away from u_0 there are $k-1$ choices (k if the walk is at u_0). □

By Stirling's bound on $s!$ or by a simple induction argument it is easy to see that $\binom{2s}{s} \geqslant \frac{4^s}{s+1}$, for any $s \geqslant 1$. Hence, for any k-regular graph X and for any $s \geqslant 1$, we have by Lemma 2

$$\operatorname{Tr}(A^{2s}) \geqslant |X| \frac{1}{s+1}\binom{2s}{s}k(k-1)^{s-1} > |X| \frac{1}{(s+1)^2}(2\sqrt{k-1})^{2s}. \tag{1}$$

Proof of Theorem 1. Let X be k-regular graph of order n with eigenvalues $k = \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant -k$. Given $\varepsilon > 0$, let m be the number of eigenvalues λ of X with $\lambda \geqslant (2-\varepsilon)\sqrt{k-1}$. Then $n-m$ of the eigenvalues of X are less than $(2-\varepsilon)\sqrt{k-1}$. Thus

$$\begin{aligned}
\operatorname{Tr}(kI + A)^{2s} &= \sum_{i=1}^{n} (k + \lambda_i)^{2s} \\
&< (n-m)(k + (2-\varepsilon)\sqrt{k-1})^{2s} + m(2k)^{2s} \\
&= m((2k)^{2s} - (k + (2-\varepsilon)\sqrt{k-1})^{2s}) + n(k + (2-\varepsilon)\sqrt{k-1})^{2s}.
\end{aligned}$$

On the other hand, the binomial expansion and relation (1) give

$$\operatorname{Tr}(kI + A)^{2s} = \sum_{i=0}^{2s} \binom{2s}{i} k^i \operatorname{Tr}(A^{2s-i})$$

$$\begin{aligned}
&\geqslant \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} \operatorname{Tr}(A^{2s-2j}) \\
&> \frac{n}{(s+1)^2} \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \\
&= \frac{n}{2(s+1)^2} \left((k+2\sqrt{k-1})^{2s} + (k-2\sqrt{k-1})^{2s}\right) \\
&> \frac{n}{2(s+1)^2} (k+2\sqrt{k-1})^{2s}.
\end{aligned}$$

Thus,

$$\frac{m}{n} > \frac{\frac{1}{2(s+1)^2}(k+2\sqrt{k-1})^{2s} - (k+(2-\varepsilon)\sqrt{k-1})^{2s}}{(2k)^{2s} - (k+(2-\varepsilon)\sqrt{k-1})^{2s}}$$

for any $s \geqslant 1$. Since

$$\begin{aligned}
\lim_{s\to\infty} \left(\frac{(k+2\sqrt{k-1})^{2s}}{2(s+1)^2}\right)^{\frac{1}{2s}} &= k+2\sqrt{k-1} \\
&> k+(2-\varepsilon)\sqrt{k-1} = \lim_{s\to\infty} \left(2(k+(2-\varepsilon)\sqrt{k-1})^{2s}\right)^{\frac{1}{2s}}
\end{aligned}$$

it follows that there exists $s_0 = s_0(\varepsilon, k)$ such that for all $s \geqslant s_0$

$$\frac{(k+2\sqrt{k-1})^{2s}}{2(s+1)^2} - (k+(2-\varepsilon)\sqrt{k-1})^{2s} > (k+(2-\varepsilon)\sqrt{k-1})^{2s}.$$

Hence, if

$$c(\varepsilon, k) = \frac{(k+(2-\varepsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0} - (k+(2-\varepsilon)\sqrt{k-1})^{2s_0}}$$

then $c(\varepsilon, k) > 0$ and $m > c(\varepsilon, k)n$. □

The proofs of Serre's theorem given in [4,5,7] do not allow an easy estimation of the constant $c(\varepsilon, k)$ in terms of ε and k. We relegate the detailed analysis of the constant obtained by those arguments to a future work [3]. We should mention that Serre's theorem can be also deduced from the work of Friedman [6] or Nilli [13]. Friedman's results imply an estimate of $\left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\sqrt{\varepsilon}}\right)}$ for the proportion of the eigenvalues that are at least $(2-\varepsilon)\sqrt{k-1}$. Nilli's work provides a bound of $\left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\arccos(1-\varepsilon)}\right)}$. Their methods provide better bounds on $c(\varepsilon, k)$ than ours. From our proof of Serre's theorem, we obtain that a proportion of $\left(\frac{1}{2}\right)^{O\left(\frac{\sqrt{k}}{\varepsilon}\log\left(\frac{\sqrt{k}}{\varepsilon}\right)\right)}$ of the eigenvalues are at least $(2-\varepsilon)\sqrt{k-1}$. This is because in Theorem 1 we pick s_0 such that $\frac{s_0}{\log s_0} = \Theta\left(\frac{\sqrt{k}}{\varepsilon}\right)$.

Theorem 1 has the following consequence regarding the asymptotics of the greatest eigenvalues of k-regular graphs.

Corollary 3. *Let $(X_i)_{i\geqslant 0}$ be a sequence of k-regular graphs such that $\lim_{i\to\infty}|X_i|=\infty$. Then for each $l\geqslant 1$,*

$$\liminf_{i\to\infty}\lambda_l(X_i)\geqslant 2\sqrt{k-1}.$$

This corollary has also been proved directly by Serre in an appendix to [8] using the eigenvalue distribution theorem in [16]. When $l=2$, we obtain the asymptotic version of the Alon–Boppana theorem (see [1,10,12,14] for more details).

3. Analogous theorems for the least eigenvalues of regular graphs

The analogous result to Theorem 1 for the least eigenvalues of a k-regular graph is not true. For example, the eigenvalues of line graphs are all at least -2. However, by adding an extra condition to the hypothesis of Theorem 1, we can prove an analogue of Serre's theorem for the least eigenvalues of a k-regular graph.

Theorem 4. *For any $\varepsilon>0$, there exist a positive constant $c=c(\varepsilon,k)$ and a non-negative integer $g=g(\varepsilon,k)$ such that for any k-regular graph X with* $\operatorname{oddg}(X)>g$, *the number of eigenvalues μ of X with $\mu\leqslant-(2-\varepsilon)\sqrt{k-1}$ is at least $c|X|$.*

Proof. Let X be a k-regular graph of order n with eigenvalues $-k\leqslant\mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_n=k$. Given $\varepsilon>0$, let m be the number of eigenvalues μ of X with $\mu\leqslant-(2-\varepsilon)\sqrt{k-1}$. Then $n-m$ of the eigenvalues of X are greater than $-(2-\varepsilon)\sqrt{k-1}$. Thus

$$\begin{aligned}\operatorname{Tr}(kI-A)^{2s}&=\sum_{i=1}^{n}(k-\mu_i)^{2s}<(n-m)\big(k+(2-\varepsilon)\sqrt{k-1}\big)^{2s}+m(2k)^{2s}\\&=m\big((2k)^{2s}-\big(k+(2-\varepsilon)\sqrt{k-1}\big)^{2s}\big)+n\big(k+(2-\varepsilon)\sqrt{k-1}\big)^{2s}.\end{aligned}$$

In the previous section, we proved that there exists $s_0=s_0(\varepsilon,k)$ such that for all $s\geqslant s_0$

$$\frac{(k+2\sqrt{k-1})^{2s_0}}{2(s_0+1)^2}-\big(k+(2-\varepsilon)\sqrt{k-1}\big)^{2s_0}>\big(k+(2-\varepsilon)\sqrt{k-1}\big)^{2s_0}.$$

Let $g(\varepsilon,k)=2s_0$. If $\operatorname{oddg}(X)>2s_0$, then for $0\leqslant j\leqslant s_0-1$, the number of closed walks of length $2s_0-2j-1$ in X is 0. Hence, $\operatorname{Tr}(A^{2s_0-2j-1})=0$, for $0\leqslant j\leqslant s_0-1$. Using also 1, we obtain

$$\begin{aligned}\operatorname{Tr}(kI-A)^{2s_0}&=\sum_{j=0}^{s_0}\binom{2s_0}{2j}k^{2j}\operatorname{Tr}(A^{2s_0-2j})-\sum_{j=0}^{s_0-1}\binom{2s_0}{2j+1}k^{2j+1}\operatorname{Tr}(A^{2s_0-2j-1})\\&=\sum_{j=0}^{s_0}\binom{2s_0}{2j}k^{2j}\operatorname{Tr}(A^{2s_0-2j})>\frac{n}{(s_0+1)^2}\sum_{j=0}^{2s_0}\binom{2s_0}{2j}k^{2j}(2\sqrt{k-1})^{2s_0-2j}\\&>\frac{n}{2(s_0+1)^2}(k+2\sqrt{k-1})^{2s_0}.\end{aligned}$$

From the previous inequalities, it follows that if

$$c(\varepsilon, k) = \frac{(k + (2-\varepsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0} - (k + (2-\varepsilon)\sqrt{k-1})^{2s_0}}$$

then $c(\varepsilon, k) > 0$ and $m > c(\varepsilon, k)n$. □

The next result is an immediate consequence of Theorem 4.

Corollary 5. *Let $(X_i)_{i \geqslant 0}$ be a sequence of k-regular graphs such that $\lim_{i\to\infty} \operatorname{oddg}(X_i) = \infty$. Then for each $l \geqslant 1$*

$$\limsup_{i\to\infty} \mu_l(X_i) \leqslant -2\sqrt{k-1}.$$

When $l = 1$, we get the main result from [8]. Also, Corollary 5 holds when $l = 1$ and $\lim_{i\to\infty} \operatorname{girth}(X_i) = \infty$. This special case of Corollary 5 was proved directly in [9] using orthogonal polynomials and is also a consequence of the eigenvalue distribution theorem from [11].

A theorem stronger than Corollary 5 has been proved by Serre in [8] using the eigenvalue distribution results from [16]. We now present an elementary proof of this theorem. For $r \geqslant 0$, let $c_r(X)$ be the number of cycles of length r in a graph X.

Theorem 6. *Let $(X_i)_{i \geqslant 0}$ be a sequence of k-regular graphs such that $\lim_{i\to\infty} |X_i| = \infty$. If $\lim_{i\to\infty} \frac{c_{2r+1}(X_i)}{|X_i|} = 0$ for each $r \geqslant 1$, then for each $l \geqslant 1$*

$$\limsup_{i\to\infty} \mu_l(X_i) \leqslant -2\sqrt{k-1}.$$

Proof. Let $l \geqslant 1$. For a graph X and $r \geqslant 1$, let $n_{2r+1}(X)$ denote the number of vertices v_0 in the graph X such that the subgraph of X induced by the vertices at distance at most r from v_0 is bipartite. Thus, $|X| - n_{2r+1}(X)$ is the number of vertices u_0 of X such that the subgraph of X induced by the vertices at distance at most r from u_0 contains at least one odd cycle. Since each such vertex is no further than r from each of the vertices of an odd cycle of length at most $2r + 1$, it follows that

$$|X| - n_{2r+1}(X) \leqslant \sum_{l=1}^{r-1} \alpha_{l,r} c_{2l+1}(X),$$

where $0 \leqslant \alpha_{l,r} \leqslant 3(2l+1)(k-1)^r$. Thus, we have the following inequalities

$$1 - \sum_{l=1}^{r-1} \alpha_{l,r} \frac{c_{2l+1}(X_i)}{|X_i|} \leqslant \frac{n_{2r+1}(X_i)}{|X_i|} \leqslant 1$$

for all $r \geqslant 1$, $i \geqslant 0$. Hence, for each $r \geqslant 1$

$$\lim_{i\to\infty} \frac{n_{2r+1}(X_i)}{|X_i|} = 1. \tag{2}$$

For $i \geqslant 0$, let $A_i = A(X_i)$. Then, for $i \geqslant 0$ and $r \geqslant 1$, we have

$$\mathrm{Tr}(A_i^{2r+1}) = n_{2r+1}(X_i) \cdot 0 + (|X_i| - n_{2r+1}(X_i))\theta_{2r+1}(X_i), \tag{3}$$

where $0 \leqslant \theta_{2r+1}(X_i) \leqslant k^{2r+1}$. From 2 and 3, we obtain that for each $r \geqslant 1$

$$\lim_{i\to\infty} \frac{\mathrm{Tr}(A_i^{2r+1})}{|X_i|} = 0. \tag{4}$$

By using relation 1, it follows that for each $r \geqslant 1$

$$\liminf_{i\to\infty} \frac{\mathrm{Tr}(A_i^{2r})}{|X_i|} \geqslant \frac{(2\sqrt{k-1})^{2r}}{(r+1)^2}. \tag{5}$$

Now for each $i \geqslant 0$, we have

$$\mathrm{Tr}(kI - A_i)^{2s} = \sum_{j=1}^{|X_i|} (k - \lambda_j(X_i))^{2s} \leqslant (|X_i| - l)(k - \mu_l(X_i))^{2s} + l(2k)^{2s}.$$

Once again, the binomial expansion gives us

$$\mathrm{Tr}(kI - A_i)^{2s} = \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \mathrm{Tr}(A_i^{2s-j}).$$

From the previous two relations, we get that

$$(k - \mu_l(X_i))^{2s} + \frac{4^s l k^{2s}}{|X_i| - l} \geqslant \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \frac{\mathrm{Tr}(A_i^{2s-j})}{|X_i| - l}.$$

Using relations 4 and 5, it follows that

$$\begin{aligned} k - \limsup_{i\to\infty} \mu_l(X_i) &\geqslant \left(\sum_{j=0}^{s} \binom{2s}{2j} k^{2j} \frac{(2\sqrt{k-1})^{2s-2j}}{(s-j+1)^2} \right)^{\frac{1}{2s}} \\ &> \left(\frac{1}{(s+1)^2} \sum_{j=0}^{s} \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \right)^{\frac{1}{2s}} \\ &> \left(\frac{1}{2(s+1)^2} \right)^{\frac{1}{2s}} (k + 2\sqrt{k-1}) \end{aligned}$$

for any $s \geqslant 1$. By taking the limit as $s \to \infty$, we get

$$k - \limsup_{i\to\infty} \mu_l(X_i) \geqslant k + 2\sqrt{k-1},$$

which implies the inequality stated in the theorem. □

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