

Note

## Extreme eigenvalues of nonregular graphs

Sebastian M. Cioabă<sup>a,1</sup>, David A. Gregory<sup>b,2</sup>, Vladimir Nikiforov<sup>c</sup>

<sup>a</sup> Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA

<sup>b</sup> Department of Mathematics, Queen's University at Kingston, Ontario, K7L 3N6, Canada

<sup>c</sup> Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

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### Abstract

Let  $\lambda_1$  be the greatest eigenvalue and  $\lambda_n$  the least eigenvalue of the adjacency matrix of a connected graph  $G$  with  $n$  vertices,  $m$  edges and diameter  $D$ . We prove that if  $G$  is nonregular, then

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geq \frac{1}{n(D + 1)},$$

where  $\Delta$  is the maximum degree of  $G$ .

The inequality improves previous bounds of Stevanović and of Zhang. It also implies that a lower bound on  $\lambda_n$  obtained by Alon and Sudakov for (possibly regular) connected nonbipartite graphs also holds for connected nonregular graphs.

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### 1. Background

Let  $G$  be a simple graph with vertex set  $[n] = \{1, 2, \dots, n\}$ , maximum degree  $\Delta$  and minimum degree  $\delta$ . Let  $\lambda_1$  denote the largest eigenvalue of the adjacency matrix  $A$  of  $G$ . The Perron–Frobenius Theorem [2, p. 178] implies that  $\lambda_1$  has an eigenvector  $x$  with nonnegative entries

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*E-mail addresses:* [scioaba@math.ucsd.edu](mailto:scioaba@math.ucsd.edu) (S.M. Cioabă), [gregoryd@mast.queensu.ca](mailto:gregoryd@mast.queensu.ca) (D.A. Gregory), [vnikifrv@memphis.edu](mailto:vnikifrv@memphis.edu) (V. Nikiforov).

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which must be positive if  $G$  is connected. Then  $Ax = \lambda_1 x$  and, so  $\lambda_1 \sum_i x_i = \sum_j d_j x_j$  where  $d_1, d_2, \dots, d_n$  is the degree sequence of  $G$ . This implies the well-known result that

$$\delta \leq \lambda_1 \leq \Delta$$

and that, when  $G$  is connected, equality holds in either case if and only if  $G$  is regular.

Stevanović [3] proved that if a connected graph  $G$  is nonregular, then

$$\Delta - \lambda_1 > \frac{1}{2n(n\Delta - 1)\Delta^2}.$$

In the final remark of [3], Stevanović asked whether or not the power of  $\Delta$  could be improved. Recently, Zhang [4] obtained the finer bound

$$\Delta - \lambda_1 > \frac{(\sqrt{\Delta} - \sqrt{\delta})^2}{nD\Delta},$$

where  $D$  is the diameter of  $G$ . In this note, we prove the following stronger inequality for a connected nonregular graph  $G$ :

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}. \tag{1}$$

The Perron–Frobenius Theorem also implies that if  $\lambda_n$  is the least eigenvalue of a graph, then  $\lambda_1 \geq -\lambda_n$  with equality in the connected case if and only if the graph is bipartite [2, p. 178]. For graphs that are connected and nonbipartite (and possibly regular), Alon and Sudakov [1] proved that

$$\Delta + \lambda_n > \frac{1}{n(D + 1)}. \tag{2}$$

Because the lower bound in (1) is monotone increasing in  $n\Delta - 2m$ , it follows that, for a connected nonregular graph  $G$ ,

$$\Delta + \lambda_n \geq \Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geq \frac{1}{n(D + 1)}. \tag{3}$$

Thus, the bound (2) of Alon and Sudakov also holds for connected nonregular graphs and, in that case, is refined by the inequalities (3).

### 2. The proof

In this section, we present a proof of inequality (1).

**Theorem 1.** *If  $G$  is a connected nonregular graph with diameter  $D$ , then*

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}.$$

**Proof.** Let  $x$  be the unique unit positive eigenvector of  $A$  with eigenvalue  $\lambda_1$ . Let  $E$  denote the edge set of  $G$  and, for each  $i \in [n]$ , let  $d_i$  denote the degree of vertex  $i$ . Then

$$\lambda_1 = x^t Ax = 2 \sum_{ij \in E} x_i x_j = \sum_{ij \in E} (x_i^2 + x_j^2) - \sum_{ij \in E} (x_i - x_j)^2 = \sum_{i=1}^n d_i x_i^2 - \sum_{ij \in E} (x_i - x_j)^2.$$

This implies

$$\Delta - \lambda_1 = \sum_{ij \in E} (x_i - x_j)^2 + \sum_{i=1}^n (\Delta - d_i)x_i^2. \tag{4}$$

Choose vertices  $s, t \in [n]$  so that  $x_s = \max_i x_i$  and  $x_t = \min_i x_i$ . Then  $t \neq s$  because  $G$  is nonregular. If  $s = i_0, i_1, \dots, i_{k-1}, i_k = t$  are consecutive vertices of a shortest path from  $s$  to  $t$  in  $G$ , it follows from the Cauchy–Schwarz inequality that

$$\sum_{j=0}^{k-1} (x_{i_j} - x_{i_{j+1}})^2 \geq \frac{1}{k} \left( \sum_{j=0}^k (x_{i_j} - x_{i_{j+1}}) \right)^2 = \frac{1}{k} (x_s - x_t)^2 \geq \frac{1}{D} (x_s - x_t)^2.$$

Thus, from (4) we obtain

$$\Delta - \lambda_1 \geq \frac{(x_s - x_t)^2}{D} + (n\Delta - 2m)x_t^2,$$

where the right-hand side is a quadratic function of  $x_t$  that attains its minimum when  $x_t = \frac{x_s}{D(n\Delta - 2m) + 1}$ . It follows that

$$\Delta - \lambda_1 \geq \frac{(n\Delta - 2m)x_s^2}{D(n\Delta - 2m) + 1},$$

where  $x_s^2 > \frac{1}{n}$  since  $x^T x = 1$ . The statement in Theorem 1 now follows.  $\square$

A further refinement of Theorem 1 may be possible. For example, a computer search shows that  $\Delta - \lambda_1 > (\Delta - \delta)^{\frac{1}{2}}/nD$  for all connected nonregular graphs of order at most 8.

### 3. Examples

The lower bound on  $\Delta - \lambda_1$  in Theorem 1 is of order  $O(\frac{1}{nD})$ . The bipartite graphs in the following proposition suggest that for connected nonregular graphs with  $\Delta - \delta = 1$ , this order is best possible.

Let  $G_1, \dots, G_k$  be  $k$  disjoint copies of the complete bipartite graph  $K_{\Delta, \Delta}$ . Remove an edge, say  $v_{2i-1}v_{2i}$ , from each  $G_i$ , and join  $v_{2i}$  to  $v_{2i+1}$  for each  $i = 1, \dots, k - 1$ . Let  $G_{\Delta, k}$  denote the resulting chain of bipartite graphs. Clearly  $G_{\Delta, k}$  has  $n = 2k\Delta$  vertices, maximum degree  $\Delta$ , minimum degree  $\delta = \Delta - 1$  and diameter  $D = 4k - 1$ .

**Proposition 2.** For the graphs  $G_{\Delta, k}$ ,  $\Delta - \lambda_1 < \frac{4\pi^2}{nD}$ .

**Proof.** Let  $A$  be the adjacency matrix of  $G_{\Delta, k}$  and let  $U = \{v_1, v_{2k}\}$ , the two vertices of degree  $\delta = \Delta - 1$ . For each unit vector  $z$ , because  $\lambda_1 \geq z^T A z$ , the argument used to prove (4) gives

$$\Delta - \lambda_1 \leq \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in V} (\Delta - d_i)z_i^2 = \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in U} z_i^2. \tag{5}$$

Let  $y$  be the unit eigenvector corresponding to the greatest eigenvalue  $2 \cos \frac{\pi}{k+1}$  of the path  $P_k$  on  $k$  vertices. By (4),

$$\sum_{i=1}^{k-1} (y_i - y_{i+1})^2 + y_1^2 + y_k^2 = 2 - \lambda_1(P_k) < \frac{\pi^2}{(k+1)^2}. \tag{6}$$

For  $v \in V(G_{\Delta,k})$ , let  $z_v = y_i/\sqrt{2\Delta}$  when  $v \in G_i$ . Then  $z^T z = 1$ . Substituting  $z$  into (5), we obtain

$$\Delta - \lambda_1 \leq \sum_{i=1}^{k-1} \left( \frac{y_i - y_{i+1}}{\sqrt{2\Delta}} \right)^2 + \frac{y_1^2}{2\Delta} + \frac{y_k^2}{2\Delta} < \frac{\pi^2}{2\Delta(k+1)^2} < \frac{4\pi^2}{nD}. \quad \square$$

We noted above that Theorem 1 implies that there is a constant  $c$  such that  $\Delta - \lambda_1 > \frac{c}{nD}$  for all connected nonregular graphs. It is easy to check that the theorem implies that  $c \geq 2/3$ . Taking a graph formed from the binary tree on 7 vertices, together with a 4-cycle on the vertices of degree 1, we get  $c \leq nD(\Delta - \lambda_1) \approx 1.355$ . We conjecture that  $c \geq 1$  or, equivalently, that  $\Delta - \lambda_1 > 1/nD$  for all connected nonregular graphs.

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