Expander Graphs and Gaps between Primes*

Sebastian M. Cioabă† and M. Ram Murty ‡

May 1, 2007

Abstract

The explicit construction of infinite families of d-regular graphs which are Ramanujan is known only in the case $d-1$ is a prime power. In this paper, we consider the case when $d-1$ is not a prime power. The main result is that by perturbing known Ramanujan graphs and using results about gaps between consecutive primes, we are able to construct infinite families of "almost" Ramanujan graphs for almost every value of d. More precisely, for any fixed $\epsilon > 0$ and for almost every value of d (in the sense of natural density), there are infinitely many d-regular graphs such that all the non-trivial eigenvalues of the adjacency matrices of these graphs have absolute value less than $(2+\epsilon)\sqrt{d-1}$.

1 Introduction

Our graph notation is standard (see [42]). For a graph X on n vertices, we denote by $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ the eigenvalues of its adjacency matrix. Let $\lambda(X) = \max_{\lambda_i \neq k} |\lambda_i|$. Following Alon (see [25]), a graph X is called an (n, k, λ)-graph if it is k-regular and $\lambda(X) \leq \lambda$.

*MSC Numbers: 05C50, 15A18, 11N05, 11F30

†Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, U.S.A, scioaba@math.ucsd.edu. Research partially supported by an NSERC post-doctoral fellowship.

‡Department of Mathematics, Queen's University, Kingston, Ontario K7L 3N6, Canada, murty@mast.queensu.ca. Research partially supported by an NSERC grant

Given $k \geq 3$, it is of great interest to construct infinite families of (n, k, λ)-graphs with $\lambda < k$ as small as possible. These graphs are called expanders. By the Alon and Boppana theorem [1], it is easy to see $\lambda = 2\sqrt{k-1}$ is best possible. See [1, 11, 23] for related results.

A k-regular graph X is Ramanujan if it is an $(n, k, 2\sqrt{k-1})$-graph. These graphs are examples of expander graphs and their explicit construction has applications to communication theory (see Bien [4]). When $k-1$ is a prime or a prime power, explicit constructions of infinite families of Ramanujan graphs were derived by Lubotzky-Phillips and Sarnak [29] and Margulis [31](in the prime case) and Morgenstern [33] (in the prime power case). In both cases, the explicit construction turns out to be a consequence of the Ramanujan conjecture for eigenvalues of certain Hecke operators. This also explains the appellation for such k-regular graphs. The Ramanujan conjecture (in its full generality) is still an open problem in the theory of automorphic forms. However, in the two cases needed above in the construction, it is a theorem due to Deligne [18] (in the prime case) and Drinfeld [19] (in the prime power case). We refer the reader to Li [27] for more details.

In this paper, we would like to address the question of constructing Ramanujan graphs when $k-1$ is not a prime power. In this context, no explicit constructions of infinite families are known, though there is the important non-constructive work of Friedman [20].

Our goal is to begin with the infinite families of Ramanujan graphs described above and perturb them in an explicit way to obtain what we call almost Ramanujan graphs. Thus, when $k-1$ is not a prime power, the question of how close it is to a prime power becomes important. It turns out that when gaps between consecutive primes are small, the almost Ramanujan graphs are easily constructed.

The study of gaps between primes has a large history. Recently, a major advance was made in the theory by Goldston, Pintz and Yildirim [21]. If p_n denotes the n-th prime, they proved that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0 \tag{1}$$

Applied to our context, this means that almost Ramanujan graphs can be constructed by perturbing known Ramanujan graphs for infinitely many values of k in the case that $k-1$ is not a prime power.

There is also some classical work of Selberg [41] (see Theorem 3.2 below) that can be applied to our setting. Generally speaking, Selberg's theorem shows that assuming the Riemann hypothesis, there is always a prime between x and $x + (\log x)^{1+\epsilon}$ for almost all values of x (in the sense of Lebesgue measure).

The main result of this paper is the following theorem.

Theorem 1.1. *Let $\epsilon > 0$. Then for almost all d, one can explicitly construct infinite families of $(n, d, (2+\epsilon)\sqrt{d-1})$-graphs.*

The proof of this result is contained in Section 4. In Section 2, we present some results connecting the existence of perfect matchings to the eigenvalues of regular graphs and describe some methods for explicitly finding perfect matchings in regular graphs. In Section 3, we present some results regarding the gaps between consecutive primes. In Section 4, we combine the results from the previous two sections to prove Theorem 1.1.

2 Eigenvalues and perfect matchings

For a real symmetric matrix A, we denote by $\lambda_i(A)$ its i-th largest eigenvalue. For $\alpha \in \mathbb{R}$, let $\mathcal{E}(A, \alpha) = \{x \in \mathbb{R}^n : Ax = \alpha x\}$. Obviously, $\mathcal{E}(A, \alpha) = \{0\}$ unless $\alpha = \lambda_i(A)$ for some i. The next theorem is due to H.Weyl (see [24], page 181). It follows from the Courant-Fisher theorem.

Theorem 2.1 (Weyl). *For any real symmetric matrices A and B of order n and for any $1 \le i \le n$, the following inequalities hold:*

$$\lambda_n(B) \le \lambda_i(A+B) - \lambda_i(A) \le \lambda_1(B) \tag{2}$$

A *matching* P in a graph X is a set of mutually disjoint edges. The vertices incident to the edges in P are *saturated* by P. We call P a *perfect matching* if all the vertices of G are saturated by P, i.e. P is a 1-regular graph with vertex set $V(X)$. A *factor* of X is a spanning subgraph of X. A *t-factor* is a spanning t-regular subgraph. Thus, a perfect matching is the same as a 1-factor.

For each $S \subseteq V(X)$, denote by $N(S)$ the set of vertices adjacent to at least one vertex in S. The following theorem is well-known (see West [42], page 110).

Theorem 2.2 (P.Hall, 1935). *A bipartite graph X with partite sets A and B has a matching that saturates A if and only if $|N(S)| \ge |S|$, for each $S \subseteq A$.*

The following corollary to Hall's theorem is also known as the Marriage Theorem. It was originally proved by Frobenius in 1917.

Corollary 2.3. *For $k > 0$, every k-regular bipartite graph has a perfect matching.*

For the existence of perfect matchings in general (not necessarily bipartite) graphs, the following characterization was found by Tutte in 1947. An *odd component* of a graph is a component of odd order. The number of odd components of a graph G will be denoted by $\mathrm{odd}(G)$.

Theorem 2.4 (Tutte, 1947)**.** *A graph X contains a perfect matching if and only if*

$$\mathrm{odd}(X \setminus S) \leq |S|,$$

for each $S \subseteq V(G)$.

Brouwer and Haemers [9] used Tutte's theorem to prove the following eigenvalue condition that is sufficient for the existence of a perfect matching in a regular graph. In [14], the authors found a best possible upper bound for the third largest eigenvalue of a regular graph X that is sufficient to guarantee that X contains a perfect matching.

Theorem 2.5 (Cioabă-Gregory-Haemers [14])**.** *If X is a connected, k-regular graph on n vertices (n even) and*

$$\lambda_3(X) \leq \begin{cases} 2.85577, & \text{if } k = 3 \\ \frac{k-2+\sqrt{k^2+12}}{2}, & \text{if } k \text{ even} \\ \frac{k-3+\sqrt{(k+1)^2+16}}{2}, & \text{if } k \text{ odd,} \end{cases} \tag{3}$$

then X has a perfect matching.

This result is best possible in the sense that there exist k-regular graphs with an even number of vertices that contain no perfect matchings and have λ_3 equal to the right hand side (3). The previous theorem improves results from [9, 12, 13].

We should note here that the converse of previous theorem is not true, i.e. the existence of perfect matchings in X does not imply $k - \lambda_3(X) > \epsilon > 0$. The Cayley graph of $\mathbb{Z}_{2n}$ with generating set $S = \{\pm 1, n\}$ is a 3-regular graph that contains 3 disjoint perfect matchings and satisfies $\lambda_3(X) = 2\cos\frac{4\pi}{n} + 1$. The difference $k - \lambda_3 = 2 - 2\cos\frac{4\pi}{n}$ tends to 0 as n gets large.

The adjacency matrix of a perfect matching on $2n$ vertices is a permutation matrix. Its eigenvalues are 1 and -1, each with multiplicity n. Notice that the vertex set may be indexed so that $\mathcal{E}(A(P), 1) = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R}^n \right\}$ and $\mathcal{E}(A(P), -1) = \left\{ \begin{pmatrix} -x \\ x \end{pmatrix} : x \in \mathbb{R}^n \right\}$

If A is the adjacency matrix of a k-regular graph X and B is the adjacency matrix of a perfect matching P on $V(X)$, then from Theorem 2.1 we obtain that

$$|\lambda_i(X \cup P) - \lambda_i(X)| \leq 1, \tag{4}$$

for each integer i with $1 \leq i \leq n$. Note that $X \cup P$ might have multiple edges. Using (4), we immediately obtain the following lemma.

Lemma 2.6. *Let X be a k-regular graph on n vertices (assume n even) and P be a perfect matching on $V(X)$ such that $E(X) \cap E(P) = \varnothing$. If X is an (n, k, λ)-graph, then $X \cup P$ is an $(n, k+1, \lambda+1)$-graph.*

Of course, if we extend the definition of (n, k, λ)-graphs to (n, k, λ)-multigraphs, then the previous theorem is true without the assumption that $E(X) \cap E(P) = \varnothing$.

Using Theorem 2.1 and Theorem 2.5 we can prove the following lemma.

Lemma 2.7. *Let X be an (n, k, λ)-graph such that n is even and $k - \lambda > 2$. Then X contains at least one perfect matching and for each perfect matching P of X, $X \setminus P$ is an $(n, k-1, \lambda+1)$-graph.*

Proof. If $k - \lambda > 2$, then $k - \lambda_3(X) > 2$. By Theorem 2.5, we deduce that X has a perfect matching P. Then by Theorem 2.1, we obtain

$$|\lambda_i(X \setminus P) - \lambda_i(X)| \leq 1,$$

for each $i \neq 1$. It follows that $\lambda(X \setminus P) \leq \lambda(X) + 1$. Since $\lambda(X) < k - 2$, the previous inequality implies that $\lambda(X \setminus P) < k - 1$. Since $G \setminus P$ is $(k-1)$-regular and $\lambda(X \setminus P) < k - 1$, we deduce that $X \setminus P$ is connected. Hence, $X \setminus P$ is an $(n, k-1, \lambda+1)$-graph. □

Again, notice that this theorem is true if (n, k, λ)-graph is replaced by (n, k, λ)-multigraph throughout.

Lemma 2.8. *If X is an (n, k, λ)-graph and $n > k + \lambda$, then its complement $\overline{X}$ is an $(n, n-k-1, \lambda+1)$-graph.*

Proof. If $P_X(x) = \det(xI - A(G))$ is the characteristic polynomial of X, then it follows (see Biggs [5], page 20) that

$$(x + k + 1)P_{\overline{X}}(x) = (-1)^n(x - n + k + 1)P_X(-x - 1)$$

Thus, if $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of X, then the eigenvalues of $\overline{X}$ are $n-k-1, -1-\lambda_2, \ldots, -1-\lambda_n$. Since $n - k > \lambda$, it follows that $n-k-1 > -1-\lambda_i$, for each i with $2 \leq i \leq n$. Hence, $\overline{X}$ is $(n-k-1)$-regular and the multiplicity of the eigenvalue $n-k-1$ is 1. This implies that $\overline{X}$ is an $(n, n-k-1, \lambda+1)$-graph. □

The following result is an easy consequence of the previous results and of Theorem 2.5.

Corollary 2.9. *If n is even and X is an (n, k, λ)-graph with $n \geq k + \lambda + 3$, then the complement of X contains at least one perfect matching.*

We show now how one can explicitly find perfect matchings in complements of regular graphs.

In general, if X is a k-regular graph, then we can find perfect matchings in its complement using the following procedure.

If X is a k-regular bipartite graph with partite sets A and B of equal size, then consider the bipartite graph X^c with partite sets A and B with $xy \in E(X^c)$ if and only if $xy \notin E(G)$. Then X^c is $(n-k)$-regular. By Hall's theorem, it follows that X^c contains a matching P that saturates A, i.e., a perfect matching. Actually, X^c contains $(n-k)!$ perfect matchings. This implies that $X \cup P$ is a bipartite $(k+1)$-regular graph with partite sets A and B. The best known algorithm for finding such a matching P in X^c is due to Rizzi [39] and it has complexity $\mathrm{O}(n(\log n)^2)$.

If we start with a non-bipartite k-regular graph X, then we can use Lemma 2.8 to check whether or not we can find a perfect matching in the complement of G. The best known algorithm for finding a maximum matching in a k-regular graph on n vertices is due to Micali and Vazirani [34] and has complexity $\mathrm{O}(kn^{\frac{3}{2}})$.

3 Gaps between primes

Denote by p_m the m-th prime number and let $\Delta_m = p_{m+1} - p_m$. Let $\pi(x)$ be the number of primes that are less than x. The Prime Number Theorem (see Ribenboim [38] for example) states that $\pi(x) \sim \frac{x}{\log x}$ or equivalently, $p_m \sim m \log m$, as $m \to \infty$. This implies that

$$\frac{\Delta_1 + \Delta_2 + \cdots + \Delta_m}{m} = \frac{p_{m+1} - 2}{m} \sim \log m \sim \log p_m,$$

as $m \to \infty$. Thus, the average order of the difference $p_{m+1} - p_m$ is $\log p_m$.

Problems concerning the maximum and the minimum order of Δ_m are very difficult. One of the most famous open problems in mathematics is the twin primes problem which asks whether or not there are infinitely many m's such that $\Delta_m = 2$.

Crámer proved in [16] (see also [15]) the following result concerning the maximum order of Δ_m.

Theorem 3.1. *If the Riemann hypothesis is true, then there is a positive constant c such that*

$$\pi(x + c\sqrt{x}\log x) - \pi(x) > \sqrt{x},$$

for each $x \geq 2$. Thus,

$$\Delta_m = \mathrm{O}(\sqrt{p_m}\log p_m).$$

as $m \to +\infty$.

Based on probability arguments, Crámer conjectured in 1937 that $\Delta_m = \mathrm{O}((\log p_m)^2)$.

In 1943, Selberg proved the following.

Theorem 3.2. *Let $\Phi(x)$ be a positive and increasing function such that $\frac{\Phi(x)}{x}$ is decreasing for $x > 0$. Assume that $\frac{\Phi(x)}{x} \to 0$ and $\frac{\Phi(x)}{\log x} \to \infty$ as $x \to \infty$. Then assuming the Riemann hypothesis is true, we have for almost all $x > 0$*

$$\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}$$

This implies that for almost all $x > 0$, there is a prime between $[x - k(x)\log^2 x, x]$, where $k(x)$ is a function tending arbitrarily slowly to infinity.

For our purposes, it suffices to show that, given $\epsilon > 0$, then almost always $\Delta_m \leq \epsilon\sqrt{p_m}$. Let $B_\epsilon(x)$ denote the set of primes $p_m \leq x$ such that the interval $(p_m, p_m + \epsilon\sqrt{p_m})$ contains no primes and let $b_\epsilon(x) = |B_\epsilon(x)|$. Consider the following function

$$S(x) = \sum_{p_{m+1} \leq x} \Delta_m$$

Obviously,

$$S(x) = \sum_{p_{m+1} \leq x} (p_{m+1} - p_m) = p_{n+1} - 2 < Ax \log x,$$

where p_{n+1} is the largest prime less than or equal to x and A is some positive constant. On the other hand,

$$\begin{aligned} S(x) &\geq \sum_{\substack{\frac{x}{2} \leq p_m < p_{m+1} \leq x \\ p_m \in B(x)}} \Delta_m \\ &\geq \epsilon \sum_{\substack{\frac{x}{2} \leq p_m < p_{m+1} \leq x \\ p_m \in B(x)}} \sqrt{p_m} \\ &> \epsilon \left(b(x) - b\left(\frac{x}{2}\right)\right) \sqrt{C\frac{x}{2} \log \frac{x}{2}} \\ &> \epsilon \left(b(x) - b\left(\frac{x}{2}\right)\right) C' \sqrt{x \log x}, \end{aligned}$$

where C and C' are some positive constants. From the previous two relations, we obtain that for each positive x

$$b_\epsilon(x) - b_\epsilon\left(\frac{x}{2}\right) \leq A_\epsilon \sqrt{x \log x},$$

where A_ϵ is some positive constant. By iteration, this implies

$$b_\epsilon(x) \leq A'_\epsilon \sqrt{x} \log^{\frac{3}{2}} x. \tag{5}$$

Hence, for each positive x

$$\frac{b_\epsilon(x)}{\pi(x)} \leq D_\epsilon \frac{\log^{\frac{5}{2}} x}{\sqrt{x}}, \tag{6}$$

where D_ϵ is a positive constant.

This inequality states that $\Delta_m \leq \epsilon\sqrt{p_m}$ for almost all m.

Using more complicated arguments, Crámer proved the following stronger result in [17].

Theorem 3.3 (Crámer [17]). *Let $h(x)$ be the number of primes $p_n \leq x$ such that $p_{n+1} - p_n > p_n^k$, where $k \in (0, \frac{1}{2}]$ is a constant. Then*

$$h(x) = \mathrm{O}(x^{1-\frac{3}{2}k+\epsilon}),$$

for each $\epsilon > 0$.

Hence, $b_1(x) = O(x^{\frac{1}{4}+\epsilon})$ for each $\epsilon > 0$ which is better than the bound (5).

In [36], Peck shows that for each $\epsilon > 0$,

$$\sum_{p_n \leq x, \Delta_n > n^{\frac{1}{2}}} \Delta_n = \mathrm{O}(x^{\frac{25}{36}+\epsilon})$$

This implies an even better bound on $b_1(x)$:

$$b_1(x) = \mathrm{O}(x^{\frac{7}{36}+\epsilon} \log x)$$

for each $\epsilon > 0$.

To our knowledge, Baker, Harman and Pintz [3] obtained the best unconditional result on the maximum value of Δ_m. In [3], they proved that

$$\Delta_m \leq p_m^{0.525} \tag{7}$$

if p_m is large enough. The hope is (cf. Ribenboim [38]) to prove unconditionally that $\Delta_m = \mathrm{O}(p_m^{\frac{1}{2}+\epsilon})$ for each $\epsilon > 0$.

4 New expanders from old

Constructing explicit families of expanders turns out to be a very difficult problem. The problem is, given a degree $k \geq 3$, construct an infinite family of $(|V(X_n)|, k, \lambda)$-graphs X_n, where $\lambda < k$. Using standard probabilistic arguments, one can prove the existence of infinite families of k-regular expanders. This is now a folklore result for $k \geq 3$ (see the monograph [28] for more details). The first explicit construction of an infinite family of expanders was given by Margulis in [30]. For an account of known expander constructions, see [23].

Friedman [20] proved that for any integer $k \geq 3$ and any $\epsilon > 0$, the probability that a random k-regular graph G satisfies $\lambda(G) \leq 2\sqrt{k-1} + \epsilon$

tends to 1 as the number of vertices of G tends to infinity. This proves a 20 years conjecture of Alon [1]. Note that Friedman's result is probabilistic, namely he proves that roughly speaking, *almost* all k-regular graphs are *almost Ramanujan* without explicitly constructing such graphs.

Our idea to construct expanders is to slightly modify known explicit expanders by adding or removing perfect matchings to or from their edge set. This idea appears in the works of Mohar [32] and it was pursued in a slightly different direction by Bollobás and Chung in [7]. In the next section, we describe their approach.

In the previous section, we proved that, given $\epsilon > 0$, then for almost all primes p_m, we have $\Delta_m \leq \epsilon\sqrt{p_m}$. This will imply the main result of our paper.

Theorem 1.1 will follow from the following result.

Theorem 4.1. *Let $d > k \geq 2$ be two integers. Let X be a k-regular graph on n vertices (assume n even) and $P_1, \ldots, P_{d-k}$ be a family of perfect matchings on $V(X)$ such that $E(X) \cap \left(\cap_{i=1}^{d-k} E(P_i)\right) = \varnothing$. If X is an (n, k, λ)-graph, then $X \cup \left(\cup_{i=1}^{d-k} P_i\right)$ is an $(n, d, d-k+\lambda)$-graph.*

Proof. The result follows by applying Lemma 2.6 to X and P_1 and to $X \cup (\cup_{j=1}^{i} P_j)$ and P_{i+1} for $1 \leq i \leq d-k-1$. □

We present now the proof of Theorem 1.1.

Proof. Let $d \geq 3$ be an integer such that $d-1$ is not a prime. Let $m \geq 1$ be the integer such that $p_m < d-1 < p_{m+1}$. The results of Lubotzky, Phillips and Sarnak [29] provide us with an infinite family of graphs $(X_{p_m,n})$ such that $X_{p_m,n}$ is a $(|V(X_{p_m,n})|, p_m+1, 2\sqrt{p_m})$-graph for each n. Each $X_{p_m,n}$ has $n(n^2-1)$ or $\frac{n(n^2-1)}{2}$ vertices, depending on whether or not n is a square in $\mathbb{F}_{p_m}$. Since n is always odd in the construction from [29], we find that each $X_{p_m,n}$ has an even number of vertices.

By adding any $d - p_m - 1$ perfect matchings to each $X_{p_m,n}$, we obtain a new family of (multi)graphs (X'_n). If we require that the (multi)graphs X'_n have no repeated edges, then when adding perfect matchings, we need to make sure the edge set of the perfect matching to be added and the edge set of our current graph are disjoint. This can be done easily by applying Lemma 2.8 and its corollaries as well as the results from the end of Section 2.

By Theorem 4.1 , it follows that X'_n is a $(|V(X_{p_m,n})|, d, d-p_m-1+2\sqrt{p_m})$-graph. Since $d < p_{m+1}$, we deduce that X'_n is a $(|V(X_{p_m,n})|, d, \Delta_m - 2 + 2\sqrt{p_m})$-graph. Using the results in the previous section (inequality (6)), this proves the theorem. □

If X is a k-regular Ramanujan graph, then $\lambda(X) \leq 2\sqrt{k-1}$ and by Lemma 2.6, we obtain $\lambda(X \cup P) \leq 2\sqrt{k-1}+1$. This observation was used by De la Harpe and Musitelli in [22] where they construct an infinite family of 7-regular graphs with spectral graph $7 - \lambda_2 \geq 6 - 2\sqrt{5} = 1.52$. This falls short of the desired spectral gap for 7-regular Ramanujan graphs which is $7 - 2\sqrt{6} = 2.10$.

By removing perfect matchings from Ramanujan graphs, we can also obtain new families of graphs with small non-trivial eigenvalues. This result follows similarly to the proof of Theorem 4.1 by applying Lemma 2.7 instead of Lemma 2.6.

5 Diameter and perfect matchings

A very important problem in graph theory with connections to network optimization is constructing k-regular graphs on n vertices with small diameter. It is well known that any connected k-regular graph on n vertices has diameter at least $\log_{k-1} n$. The random k-regular graph has diameter $\log_{k-1} n + o(\log_{k-1} n)$ (as n tends to infinity) which is very close to the optimum value (see [7, 8]).

When searching for explicit k-regular graphs with small diameter, we consider first the k-regular graphs X with small $\lambda(X)$. This is because of the results connecting the diameter and the eigenvalues of a k-regular graph (see [10]). The best possible upper bound on the diameter of a k-regular graph that these theorems can provide, is $2\log_{k-1} n + O(1)$.

Bollobás and Chung [7] proved that the diameter of a k-regular expander on n vertices plus a random perfect matching is almost surely less than $\log_{k-1} n + \log_{k-1}\log(n) + O(1)$ as n goes to infinity. More precisely, they proved the following theorem.

Theorem 5.1 (Bollobás, Chung). *Let H be a graph with maximum degree k with the property that for each $x \in V(H)$, the i-th neighborhood $N_i(x) = \{y : d_H(x,y) = i\}$ of x satisfies the following*

$$|N_i(x)| \geq c_1 k(k-1)^{i-2} \tag{8}$$

for $i \le \left(\frac{1}{2}+\epsilon\right)\log_{k-1} n$*, where* c_1 *and* ϵ *are some fixed positive numbers. Let* G *be a graph obtained by adding a random perfect matching to* H*. Then with probability tending to* 1 *as* n *goes to infinity, the diameter* $\mathrm{diam}(G)$ *of* G *satisfies*

$$\log_k n - c \le \mathrm{diam}(G) \le \log_k n + \log_k \log_k n + c \tag{9}$$

where c *is a constant depending on* c_1 *and* ϵ*.*

It is also proved in [7] that (N, k, λ)-graphs with λ bounded away from k, satisfy property (8). This shows that we can achieve the best possible asymptotic diameter by a very small random perturbation of an explicit expander graph. The Ramanujan graphs have diameter less than $2\log_{k-1} n + O(1)$. The previous theorem implies that the k-regular Ramanujan graphs plus a random perfect matching have diameter $\log_{k-1} n + o(\log_{k-1} n)$ almost surely as $n \to +\infty$.

6 Concluding Remarks

In this paper, we showed how to construct almost Ramanujan graphs by adding perfect matchings to good expanders.

There are many other interesting questions regarding the connection between perfect matchings and expanders. Alon [2] asked whether it is true or not that (N, k, λ)-graphs with N even, $\lambda = O(\sqrt{k})$ and k large, are the disjoint union of k perfect matchings. By the results in Section 2, one can only show that (N, k, λ)-graphs with N even contain $\frac{k-\lambda}{2}$ disjoint perfect matchings.

Alon [2] also asked whether it is true or not that (N, k, λ)-graphs with N and k both even, $\lambda = O(\sqrt{k})$ and k large, are the disjoint union of $\frac{k}{2}$ Hamiltonian cycles. Krivelevich and Sudakov [26] (see also [25]) found sufficient conditions in terms of λ that imply the existence of a Hamiltonian cycle in an (N, k, λ)-graph. To answer Alon's questions, new ideas seem to be needed.

Friedman [20] showed that for each $\epsilon > 0$ almost all k-regular graphs have non-trivial eigenvalues at most $2\sqrt{k-1} + \epsilon$. Computations results of Novikoff [35] show that for n large, about 52% of all k-regular graphs are actually Ramanujan. By adding a random perfect matching to a cycle of even length, we have computed that about 70% of the 3-regular graphs obtained are Ramanujan.

References

[1] N. Alon, Eigenvalues and Expanders, *Combinatorica*, **6** (1986), 83-96.

[2] N. Alon, (n, d, λ)-graphs in Extremal Combinatorics, online at http://www.pims.math.ca/science/2002/aga/phenomenavideos/alon/

[3] R. C. Baker, G. Harman and J. Pintz, The difference between consecutive primes. II., *Proc. London Math. Soc.*, **83** (2001), no.3, 532-562.

[4] F. Bien, Construction of Telephone Networks by Group Representations, *Notices of the Amer.Math.Society*, **36** (1989), No.1, 5-22.

[5] N. Biggs, Algebraic Graph Theory, Cambridge University Press, Cambridge (1993), 2nd edition.

[6] Y. Bilu and N. Linial, Lifts, Discrepancy and Nearly Optimal Spectral Gaps, *Combinatorica*, **26** (2006), 495-519.

[7] B. Bollobás and F. Chung, The diameter of a cycle plus a random matching, *SIAM J. Discrete Math.*, **1** (1988), no.3, 328-333.

[8] B. Bollobás and W. F. de la Vega, The diameter of random graphs, *Combinatorica*, **2** (1982), 125-134.

[9] A. E. Brouwer and W. H. Haemers, Eigenvalues and perfect matchings, *Linear Algebra and its Applications*, **395** (2005), 155-162.

[10] F. Chung, Diameters and eigenvalues, *J. Amer. Math. Soc.*, **2** (1989), 187–200.

[11] S. M. Cioabă, Eigenvalues, Expanders and Gaps between Primes, Ph.D. Thesis, Queen's University at Kingston, (2005).

[12] S. M. Cioabă, Perfect matchings, eigenvalues and expansion, *C. R. Math. Acad. Sci. Soc. R. Can.* **27** (4) (2005), 101-104.

[13] S. M. Cioabă and D. A. Gregory, Large matchings from eigenvalues, *Linear Algebra and its Applications*, **422** (2007), 308-317.

[14] S. M. Cioabă, D. A. Gregory and W. Haemers, Perfect matchings in regular graphs from eigenvalues, submitted to J. Combin. Theory, Series B, available at `http://www.math.ucsd.edu/~scioaba`

[15] H. Crámer, Collected works, Springer-Verlag, Heidelberg, (1994).

[16] H. Crámer, Some theorems concerning prime numbers, *Arkiv för Mathematik, Astronom o. Fysik*, **5** 5(1920), vol. 5, 1-33.

[17] H. Crámer, On the distribution of primes, *Proc. Cambridge Phil. Soc.*, **20** (1921), 272-280.

[18] P. Deligne, La conjecture de Weil, *Inst. Hautes Etudes Sci. Publ. Math.*, (1974), 273-307.

[19] V. G. Drinfeld, The proof of Peterson's conjecture for GL(2) over global field of characteristic p, *Functional Anal. Appl.*, **22** (1988), 28-43.

[20] J. Friedman, A proof of Alon's second eigenvalue conjecture, *Memoirs of AMS*, to appear.

[21] D. A. Goldston, J. Pintz and C. Y. Yildirim, Small gaps between primes II, Preprint (2005).

[22] P. de la Harpe and A. Musitelli, Expanding graphs, Ramanujan graphs and 1-factor perturbations, available at `http://arxiv.org/abs/math.CO/0503330/`

[23] S. Hoory, N. Linial and A. Widgerson, Expanders Graphs and their Applications, *Bull. Amer. Math. Soc.*, **43** (2006), 439-561.

[24] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press (1990).

[25] M. Krivelevich and B. Sudakov, Pseudorandom graphs, in: More Sets, Graphs and Numbers, Bölyai Mathematical Studies **15**, Springer (2006), 199-262.

[26] M. Krivelevich and B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, *J. Graph Theory* **42** (1) (2003), 17-33.

[27] W.-C. Winnie Li, Number Theory with Applications, Series of University Mathematics, Vol.7, World Scientific, (1996).

[28] A. Lubotzky, Discrete Groups, Expanding Graphs and Invariant Measures, With an appendix by Jonathan D. Rogawski, Progress in Mathematics, 125, *Birkhaüser Verlag, Basel*, (1994).

[29] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan Graphs, *Combinatorica*, **8** (1988), no.3, 261-277.

[30] G. Margulis, Explicit group theoretic constructions of combinatorial schemes and their applications for the construction of expanders and concentrators, *J. of Problems of Information Transmission*, **9** (1975), 325-332.

[31] G. Margulis, Explicit constructions of concentrators, *J. of Problems of Information Transmission*, **24** (1988), 39-46.

[32] B. Mohar, Isoperimetric Numbers of Graphs, *Journal of Combinatorial Theory, Series B*, **47** (1989), 274-291.

[33] M. Morgenstern, Existence and explicit constructions of $q + 1$ regular Ramanujan graphs for every prime power q, *Journal of Combinatorial Theory, Series B* , **62** (1994), no.1, 44-62.

[34] S. Micali and V. V. Vazirani, An $O(\sqrt{|V|}|E|)$ algorithm for finding maximum matchings in general graphs, *21st Annual Symposium on Foundations of Computer Science*, New York (1980), 17-27.

[35] T. Novikoff, Asymptotic Behaviour of the Random 3-Regular Bipartite graph, http://www.math.princeton.edu/mathlab/book/

[36] A. S. Peck, Differences between consecutive primes, *Proc. London Math. Soc.*, **76** (1998), no.1, 33-69.

[37] M. S. Pinsker, On the complexity of a concentrator, *7th Annual Teletraffic Conference Stockholm*, 318/1-318/4, (1973).

[38] P. Ribenboim, The Little Book of Bigger Primes, Springer, 2nd edition, (2004).

[39] R. Rizzi, Finding 1-factors in bipartite regular graphs and edge-coloring bipartite graphs, *SIAM J. Discrete Math.*, **15** (2002), no.3, 282-288.

[40] O. Reingold, S. Vadhan and A. Widgerson, Entropy Waves, the Zig-Zag Product, and New Constant-Degree Expanders, *Annals of Mathematics*, **155** (2002), no.1, 157-187.

[41] A. Selberg, On the normal density of primes in small intervals, and the difference between consecutive primes, *Archiv for Mathematik og Naturvidenskab*, **B(47)** (6) (1943), 87-105.

[42] D. B. West, Introduction to Graph Theory, Prentice Hall, New Jersey, 2nd edition, (2001).