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Note

On decompositions of complete hypergraphs

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ABSTRACT

We study the minimum number of complete r-partite r-uniform hypergraphs needed to partition the edges of the complete r-uniform hypergraph on n vertices and we improve previous results of Alon.

1. Introduction

Given an r-uniform hypergraph H, let $f_r(H)$ denote the minimum number of complete r-partite r-uniform hypergraphs needed to partition the edge set of H. When $r = 2$, the parameter $f_2(H)$ is also known as the biclique partition number of the graph H and has been well studied (see [2,9]). When $H = K_n$, this parameter equals $n - 1$ as shown by Graham and Pollak [5] (see also [2,10,12,14] for other proofs).

In this note, we are interested in extending the theorem of Graham and Pollak to complete r-uniform hypergraphs for larger values of r. This seems to be a difficult and interesting extremal problem with connections to other areas such as theoretical computer science (the complexity of computing bilinear forms and symmetric polynomials, see [6,8,11]) and linear algebra (tensor rank computation of high-dimensional arrays, see [8]).

To simplify our notation, let $f_r(n) = f_r(K_n^{(r)})$, where $K_n^{(r)}$ denotes the complete r-uniform hypergraph on n vertices. In [1], Alon showed that $f_3(n) = n - 2$ and that for $n \geqslant 2k \geqslant 4$,

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$$f_{2k}(n) \geqslant \frac{2\left(\binom{n}{k} - \binom{n}{k-1} - \binom{n}{k-3} - \cdots - \binom{n}{k+1-2\lceil\frac{k}{2}\rceil}\right)}{\binom{2k}{k}}. \tag{1}$$

It was also proved in [1] by a recursive construction that

$$f_r(n) \leqslant \sum_{i=0}^{r} f_i\big(\lfloor n/2\rfloor\big) f_{r-i}\big(\lceil n/2\rceil\big). \tag{2}$$

Using (2), Alon showed that for fixed $k \geqslant 2$, $f_{2k}(n) \leqslant \frac{n^k}{k!}(1+o(1))$ as $n \to \infty$.

In this note, we use known results on biclique decomposition of the Kneser graphs to improve and simplify (1) and (2). We show that

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leqslant f_{2k}(n) \leqslant \binom{n-k}{k}. \tag{3}$$

2. The proofs of our results

We follow Bollobás [3] for our hypergraph notation. We use $[n]$ to denote the set $\{1,\ldots,n\}$ and $[n]^{(r)}$ to denote the family of all r-subsets of $[n]$. If $X_1,\ldots,X_r$ are pairwise disjoint subsets of $[n]$, we denote by $\prod_{i=1}^{r} X_i$ the set of r-subsets Y of $[n]$ such that $|Y \cap X_i| = 1$ for each $i \in [r]$. The complete r-partite r-uniform hypergraph whose parts are $X_1,\ldots,X_r$ is the r-uniform hypergraph whose edge-set is $\prod_{i=1}^{r} X_i$. When $r=2$ for example, $\prod_{i=1}^{2} X_i$ is the edge-set of the complete bipartite subgraph (or biclique) of K_n whose colour classes are X_1 and X_2.

In this section, we present the proofs of our main results. We use known results about the biclique partition numbers of the Kneser graphs to prove the lower bound. Recall that the Kneser graph $K_{n:k}$ has vertex set $[n]^{(k)}$ with two k-subsets being adjacent if and only if they are disjoint.

Theorem 1. *For $n \geqslant 2k \geqslant 2$,*

$$f_{2k}(n) \geqslant \frac{2\binom{n-1}{k}}{\binom{2k}{k}}.$$

Proof. Let $m = f_{2k}(n)$ and consider a partition of the edge set of $K_n^{(2k)}$ into complete $2k$-partite $2k$-uniform hypergraphs $H_1,\ldots,H_m$. For $i \in [m]$, let $A_1^i,\ldots,A_{2k}^i$ denote the parts of H_i.

For each $i \in [m]$, consider the following $\frac{\binom{2k}{k}}{2}$ bicliques of the Kneser graph $K_{n:k}$. For each partition X, Y of $[2k]$ with $|X| = |Y| = k$, we construct the biclique of $K_{n:k}$ whose colour classes are $\prod_{j\in X} A_j^i$ and $\prod_{l\in Y} A_l^i$.

We claim that these $m \cdot \frac{\binom{2k}{k}}{2}$ bicliques partition the edge set of $K_{n:k}$. Let xy be an edge of $K_{n:k}$, where $x, y \in [n]^{(k)}$. This means $x \cap y = \emptyset$ and thus, $x \cup y$ is a $2k$-subset of $[n]$. Because $H_1,\ldots,H_m$ partition the edge set of $K_n^{(2k)}$, it follows that there is a unique $i \in [m]$ such that $x \cup y \in E(H_i)$. Now there exists a unique partition X, Y of $[2k]$ such that $|X| = |Y| = k$, $x \in \prod_{j\in X} A_j^i$ and $y \in \prod_{l\in Y} A_l^i$. This proves our claim and implies that

$$f_{2k}(n) \geqslant \frac{2f_2(K_{n:k})}{\binom{2k}{k}}.$$

We show that $f_2(K_{n:k}) = \binom{n-1}{k}$. This was proved by Vander Meulen [13] (see also [7]). For the sake of completeness, we give a short proof here. First, we can partition all the edges of $K_{n:k}$ by stars centered at vertices $x \in [n-1]^{(k)}$. This is possible because $[n-1]^{(k)}$ is a vertex cover of $K_{n:k}$ and implies $f_2(K_{n:k}) \leqslant \binom{n-1}{k}$. Recall that $f_2(G) \geqslant h(G)$, where $h(G)$ is the maximum of the number of positive, and of the number of negative eigenvalues of the adjacency matrix of G (see [5,7] for more details). When $G = K_{n:k}$, it is known (see [4] or [7]) that $h(G) = \binom{n-1}{k}$. This implies $f_2(K_{n:k}) \geqslant \binom{n-1}{k}$ and finishes the proof of the theorem. □

We obtain the upper bound for $f_{2k}(n)$ by a simple direct construction.

Theorem 2. *For each $n \geqslant 2k+1 \geqslant 3$, we have that*

$$f_{2k}(n-1) \leqslant f_{2k+1}(n) \leqslant \binom{n-k-1}{k}.$$

Proof. We prove first that $f_{2k+1}(n) \leqslant \binom{n-k-1}{k}$. For each k-tuple $1 < i_1 < i_2 < \cdots < i_k < n$ such that $i_{j+1} - i_j > 1$ for any $1 \leqslant j \leqslant k-1$, consider the complete $(2k+1)$-partite $(2k+1)$-uniform hypergraph $H_{i_1,\dots,i_k}$ whose parts are $\{1,\dots,i_1-1\}, \{i_1\}, \{i_1+1,\dots,i_2-1\}, \dots, \{i_{k-1}+1,\dots,i_k-1\}, \{i_k\}$ and $\{i_k+1,\dots,n\}$.

Note that the hypergraphs $H_{i_1,\dots,i_k}$ partition the edge set of $K_n^{(2k+1)}$. This is because any edge $j_1 j_2 \dots j_{2k+1}$ with $1 \leqslant j_1 < j_2 < \cdots < j_{2k+1} \leqslant n$ is contained in precisely one of these hypergraphs, namely $H_{j_2,j_4,\dots,j_{2k}}$.

Because $1 \leqslant i_1 - 1 < \cdots < i_k - k \leqslant n-k-1$, it follows that there are $\binom{n-k-1}{k}$ such hypergraphs $H_{i_1,\dots,i_k}$. This implies that $f_{2k+1}(n) \leqslant \binom{n-k-1}{k}$. The inequality $f_{2k}(n-1) \leqslant f_{2k+1}(n)$ is obvious and its proof is omitted (see [1, Lemma 2.1]). □

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