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EDGE-CONNECTIVITY, EIGENVALUES, AND MATCHINGS IN REGULAR GRAPHS*

SUIL O† AND SEBASTIAN M. CIOABĂ‡

Abstract. In this paper, we study the relationship between eigenvalues and the existence of certain subgraphs in regular graphs. We give a condition on an appropriate eigenvalue that guarantees a lower bound for the matching number of a t-edge-connected d-regular graph when $t \leq d-2$. This work extends some classical results of von Baebler [Comment. Math. Helv., 10 (1937), pp. 275–287] and Berge [*Théorie des Graphes et Ses Applications*, Collection Universitaire de Mathematiques II, Dunod, Paris, 1958] and more recent work of Cioabă, Gregory, and Haemers [J. Combin. Theory Ser. B, 99 (2009), pp. 287–297]. We also study the relationships between the eigenvalues of a d-regular t-edge-connected graph G and the maximum number of pairwise disjoint connected subgraphs in G that are each joined to the rest of the graph by exactly t edges.

Key words. graph eigenvalues, matching, connectivity, minimum edge cut

AMS subject classifications. 05C40, 15A18, 05C50, 05C70

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1. Introduction. The eigenvalues of a d-regular graph G are closely related to many important properties of G (see [5, 11, 15]). In particular, the second largest eigenvalue of G is closely related to the edge-distribution of G. When S and T are subsets of vertices of G, denote by $[S,T]$ the set of edges with one endpoint in S and one endpoint in T, and let $e(S,T) = |[S,T]|$. If G is a d-regular n-vertex graph with second largest eigenvalue λ_2 and S a nonempty proper subset of $V(G)$, then

$$e(S, V(G) \setminus S) \geq \frac{(d-\lambda_2)|S|(n-|S|)}{n}.$$

A set M of edges of a graph G is a *matching* if each vertex of G is contained in at most one edge of M. The *matching number* $\alpha'(G)$ is the maximum size of a matching in G. A graph is *t-edge-connected* if the removal of any $t-1$ edges does not disconnect it. The *eigenvalues* of G are the eigenvalues of its adjacency matrix. The *adjacency matrix* of G has its rows and columns indexed by the vertices of G, and the (i,j)-th entry of A is 1 if i and j are adjacent and 0 otherwise. If G has n vertices, then let $\lambda_1(G), \ldots, \lambda_n(G)$ be its eigenvalues indexed in nonincreasing order. It is well known that if G is a connected d-regular graph, then $\lambda_1 = d > \lambda_2$ and $\lambda_n \geq -d$ with equality if and only if G is bipartite.

In this paper, we show that many other eigenvalues of a regular graph G are related to the existence of various substructures in G. A well-known result in graph theory due to von Baebler [1] (for d odd) and Berge [2] states that any d-regular $(d-1)$-edge-connected graph contains a perfect matching. This extends the work of Petersen [22], who showed that a 3-regular graph without cut-edges contains a perfect matching.

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†Department of Mathematics, University of Illinois, Urbana, IL 61801 (suilo2@math.uiuc.edu). This author's research was partially supported by the Korean Research Foundation (MOEHRD, Basic Research Promotion Fund), grant KRF-2005-C00003.
‡Department of Mathematical Sciences, University of Delaware, Newark, DE 19716 (cioaba@math.udel.edu). This author's research was supported by a startup grant from the Department of Mathematical Sciences of the University of Delaware.

Recently, Cioabă, Gregory, and Haemers [10] found the best possible conditions on the eigenvalues of a d-regular graph that guarantee the existence of a perfect matching. Their work improved previous results of various authors (see [4, 7, 9, 15]). In this paper, we determine connections between the eigenvalues of a t-edge-connected d-regular graph and its matching number when $t \leq d-2$. Our work can be seen both as an extension of the work of von Baebler and Berge (for lower values of the edge-connectivity) and as an extension of the results of Cioabă, Gregory, and Haemers (for higher values of the edge-connectivity).

Our main result in this direction is the following theorem, whose proof is contained in section 2.

THEOREM 1.1. *Denote by* θ *the greatest solution of* $x^3 - x^2 - 6x + 2 = 0$*, and let*

$$\rho(d) = \begin{cases} \theta & \text{if } d = 3, \\ \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \geq 4 \text{ is even}, \\ \frac{d-3+\sqrt{(d+1)^2+16}}{2} & \text{if } d \geq 5 \text{ is odd}. \end{cases} \tag{1}$$

Let $p \geq 3$ *be an integer. If* G *is a* t*-edge-connected* d*-regular graph such that* $\lambda_p(G) < \rho(d)$*, then*

$$\alpha'(G) > \begin{cases} \frac{n-p+\lfloor \frac{tp}{d} \rfloor}{2} & \text{when } d \equiv t \pmod 2, \\ \frac{n-p+\lfloor \frac{(t+1)p}{d} \rfloor}{2} & \text{when } d \equiv t+1 \pmod 2. \end{cases}$$

In section 5, we present examples that show that our result is best possible when $t = d-2$; for each d and for infinitely many values of p, we construct d-regular graph H with edge-connectivity $d-2$ having $\lambda_p(H) = \rho(d)$ and $\alpha'(H) = \frac{n-p+\lfloor \frac{tp}{d} \rfloor}{2}$.

Henning and Yeo [12] determined the minimum value of the matching number of a connected d-regular graph of order n. O and West [19] extended their results and obtained a relationship between the matching number of a connected d-regular graph G and the number of balloons in G. A balloon is a maximal 2-edge-connected subgraph that is joined to the rest of G by exactly one cut-edge. Let $b(G)$ denote the number of balloons of G. Obviously $b(G) = 0$ when d is even as G contains no cut-edges in this case. If b denotes the maximum possible number of balloons in a d-regular graph with n vertices, O and West proved $\alpha'(G) \geq \frac{n}{2} - \frac{(d-1)b}{2d}$ when d is odd, and they showed that this inequality implies the result of Henning and Yeo from [12].

The number of balloons is also related to other combinatorial invariants such as the total domination number of G (see [19]) and the number of cut-edges in G. If $c(G)$ denotes the number of cut-edges of G, then $b(G) \leq c(G)$ when $c(G) \geq 2$. Motivated by the connections between balloons and these combinatorial invariants, we study the relationship between the number of balloons in G and its eigenvalues. Our result is the following theorem, whose proof is contained in section 3.

THEOREM 1.2. *When* d *is an odd integer with* $d \geq 3$*, let* $\theta(d)$ *denote the largest solution of*

$$x^3 - (d-2)x^2 - 2dx + d - 1 = 0.$$

If k *is an integer with* $k \geq 3$ *and* G *is a connected* d*-regular graph such that* $\lambda_k(G) < \theta(d)$*, then* $b(G) \leq k-1$.

In section 3, we also show for each d that this result is best possible for infinitely many values of k by presenting examples of d-regular graphs H having $\lambda_k(H) = \theta(d)$ and $b(G) = k$.

Note that for $k = 2$, the following result was proved in [8].

THEOREM 1.3 (see [8]). *Let d be an odd integer with $d \geq 3$, and denote by $\gamma(d)$ the largest root of*

$$x^3 - (d-3)x^2 - (3d-2)x - 2 = 0.$$

If G is a d-regular graph with $\lambda_2(G) < \gamma(d)$, then $b(G) = 0$ (equivalently G is 2-edge-connected).

Many authors have studied the number of cut-edges or, more generally, the number of smallest edge-cuts of a graph (see [6, 13, 16, 18]). In this paper, we determine a relationship between eigenvalues and a parameter closely related to the number of smallest edge-cuts. If G is a t-edge-connected graph, then let $b_t(G)$ denote the maximum number of pairwise disjoint connected subgraphs $H_1, \ldots, H_l$ of G with the property that $e(V(H_i), V(G) \setminus V(H_i)) = t$ for $1 \leq i \leq l$. If $c_t(G)$ denotes the number of edge-cuts of size t of G, then $b_t(G) \leq c_t(G)$ when $c_t(G) \geq 2$. Our main result in this direction is the following theorem, whose proof is contained in section 4.

THEOREM 1.4. *Let d and t be two integers of the same parity with $d > t \geq 1$. If $p \geq 3$ is an integer and G is a t-edge-connected d-regular graph with*

$$\lambda_p(G) < \begin{cases} \frac{d-2+\sqrt{(d+2)^2-4t}}{2} & \text{if } d \text{ and } t \text{ are even,} \\ \frac{d-3+\sqrt{(d+3)^2-4t}}{2} & \text{if } d \text{ and } t \text{ are odd,} \end{cases} \tag{2}$$

then $b_t(G) \leq p - 1$.

In section 4, we also show for each d that this result is best possible for infinitely many values of p by presenting examples of d-regular t-edge-connected graphs H having $\lambda_p(H)$ equal to the right-hand side of (2) and $b_t(G) = p$.

We remark that, for $t = 2$ and $p = 2$, the following result was proved in [8].

THEOREM 1.5 (see [8]). *If G is a d-regular graph with $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $b_2(G) = 0$ (or equivalently G is 3-edge-connected).*

The main tool in our arguments is eigenvalue interlacing (see [5, 11, 14]). Let $\lambda_j(M)$ be the j-th largest eigenvalue of a matrix M.

LEMMA 1.6 (Interlacing Theorem). *If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A with order $m \times m$, then, for $1 \leq i \leq m$,*

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A).$$

This theorem implies that if H is an induced subgraph of a graph G, then the eigenvalues of H interlace the eigenvalues of G.

Consider now a partition $V(G) = V_1 \cup \cdots \cup V_s$ of the vertex set of G into s nonempty subsets. For $1 \leq i, j \leq s$, let $b_{i,j}$ denote the average number of neighbors in V_j of the vertices in V_i. The quotient matrix of this partition is the $s \times s$ matrix whose (i, j)-th entry equals $b_{i,j}$. The eigenvalues of the quotient matrix interlace the eigenvalues of G. This partition is *equitable* if, for each $1 \leq i, j \leq s$, any vertex $v \in V_i$ has exactly $b_{i,j}$ neighbors in V_j. In this case, the eigenvalues of the quotient matrix are eigenvalues of G, and the spectral radius of the quotient matrix equals the spectral radius of G (see [5, 11] for more details).

2. Proof of Theorem 1.1. The deficiency def(S) of a vertex set S in G is defined by def$(S) = o(G - S) - |S|$, where $o(H)$ is the number of components of H having an odd number of vertices. Tutte [24] proved that a graph G has a perfect matching if and only if $\mathrm{def}(S) \leq 0$ for all $S \in V(G)$. The equivalent Berge–Tutte formula (see Berge [3] and also [17, 20]) states that

$$\alpha'(G) = \frac{1}{2}(n - \max_{S \subseteq V} \mathrm{def}(S)). \tag{3}$$

LEMMA 2.1. *Let G be a t-edge-connected d-regular graph with n vertices, and let r be an integer with $r \geq 2$. If $\alpha'(G) \leq \frac{n-r}{2}$, then*

$$\rho(d) \leq \begin{cases} \lambda_{\lceil \frac{rd}{d-t} \rceil} & \text{if } d \equiv t \pmod 2, \\ \lambda_{\lceil \frac{rd}{d-(t+1)} \rceil} & \text{if } d \equiv t+1 \pmod 2. \end{cases} \tag{4}$$

Proof. Let S be a subset of G with maximum deficiency. Let $\mathcal{O}_1, \dots, \mathcal{O}_q$ be the odd components of $G - S$. Because $\alpha'(G)$ is at most $\frac{n-r}{2}$, we have $q \geq |S| + r$. Let $n_i = |V(\mathcal{O}_i)|, e_i = |E(\mathcal{O}_i)|$, and $t_i = e(S, \mathcal{O}_i)$ for $1 \leq i \leq q$. By the degree-sum formula, d and t_i have the same parity. Because G is t-edge-connected, we have $t_i \geq t$ when $d \equiv t \pmod 2$ and $t_i \geq t+1$ when $d \equiv t+1 \pmod 2$.

We will prove the lemma for $d \equiv t \pmod 2$. The proof of the other case is similar and will be omitted.

By counting the edges between S and $V(G) \setminus S$, we have $d|S| \geq e(S, V(G) \setminus S) \geq \sum_{i=1}^{q} t_i \geq qt \geq (|S|+r)t$. Thus, $(d-t)|S| \geq rt$, which implies that $|S| \geq \frac{rt}{d-t}$. Because $q \geq |S| + r$, we obtain $q \geq \frac{rt}{d-t} + r = \frac{rd}{d-t}$. Thus, $q \geq \lceil \frac{rd}{d-t} \rceil$.

We claim that there are at least $\lceil \frac{rd}{d-t} \rceil$ indices i such that $t_i < d$. Otherwise, there are at most $\lceil \frac{rd}{d-t} \rceil - 1$ values of i such that $t_i < d$, which means that there are at least $q - (\lceil \frac{rd}{d-t} \rceil - 1)$ values of i such that $t_i \geq d$. Because G is t-edge-connected, we have $t_i \geq t$ for each $1 \leq i \leq q$. These facts imply

$$\begin{aligned} d|S| \geq \sum_{i=1}^{q} t_i &\geq d\left[q - \left(\left\lceil \frac{rd}{d-t} \right\rceil - 1\right)\right] + t\left(\left\lceil \frac{rd}{d-t} \right\rceil - 1\right) \\ &= dq - (d-t)\left(\left\lceil \frac{rd}{d-t} \right\rceil - 1\right) \\ &> dq - (d-t)\frac{rd}{d-t} = d(q-r) \\ &\geq d|S|, \end{aligned}$$

which is a contradiction. Here we used the inequality $x > \lceil x \rceil - 1$ for any real number x.

Let $p = \lceil \frac{rd}{d-t} \rceil$. Without loss of generality, assume that $t_i < d$ for $1 \leq i \leq p$. By Theorem 2 in [10] (see also Lemma 4.2), we obtain $\lambda_1(\mathcal{O}_i) \geq \rho(d)$ for $1 \leq i \leq p$. This fact and Lemma 1.6 imply

$$\lambda_p(G) \geq \lambda_p(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p) \geq \min_{1 \leq i \leq p} \lambda_1(\mathcal{O}_i) \geq \rho(d),$$

which finishes the proof. ∎

We are now ready to present the proof of Theorem 1.1.

Proof. For $x \in \{t, t+1\}$, we have $p = \lceil \frac{rd}{d-x} \rceil$ if and only if $r = \lfloor \frac{(d-x)p}{d} \rfloor$. This and Lemma 2.1 imply the desired result. ∎

COROLLARY 2.2 (Cioabă, Gregory, and Haemers [10]). *Let p be an integer with $p \geq 3$. If G is a connected d-regular graph of order n such that $\lambda_p(G) < \rho(d)$, then $\alpha'(G) > \frac{n-p+1}{2}$.*

Proof. Take $t = 1$ in Theorem 1.1. □

COROLLARY 2.3. *Let d and t be two integers with $d \geq 3$ and $t \leq d-2$. If G is a t-edge-connected d-regular graph of order n such that*

$$\rho(d) > \begin{cases} \lambda_{\lceil \frac{2d}{d-t} \rceil} & \text{if } d \equiv t \mod (2), \\ \lambda_{\lceil \frac{2d}{d-(t+1)} \rceil} & \text{if } d \equiv t+1 \mod (2), \end{cases}$$

then $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.

Proof. Take $r = 2$ in Lemma 2.1. □

When $t \in \{d-2, d-3\}$, the previous result states that a t-edge-connected d-regular graph with $\lambda_d(G) < \rho(d)$ must have $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$. In particular, because every connected 4-regular graph is 2-edge-connected, we deduce that if the fourth largest eigenvalue of a 4-regular graph is less than $\rho(4)$, then the matching number of the graph is $\lfloor \frac{n}{2} \rfloor$. The result of Cioabă, Gregory, and Haemers from [10] states, for all d, that $\lambda_3(G) < \rho(d)$ implies $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$; our result improves this when $d = 4$. In Figure 1, we present an example of a 2-edge-connected 4-regular graph H_1 having $\lambda_4(H_1) < \rho(4) = 1 + \sqrt{7} = \lambda_3(H_1)$; our result applies here, but the earlier result does not.

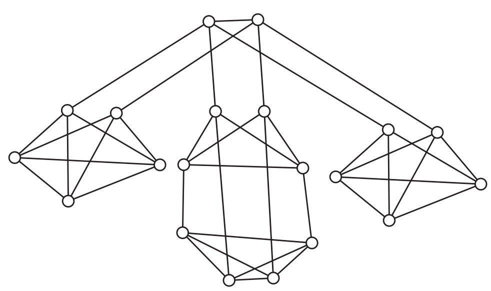

FIG. 1. *2-edge-connected 4-regular graph with $\lambda_4 = 2 < \rho(4) = 1 + \sqrt{7} = \lambda_3$.*

In Figure 2, we present a 2-edge-connected 5-regular graph satisfying $\lambda_5(H_2) = 2 < \rho(5) = 1 + \sqrt{13} = \lambda_3(H_2)$. The existence of the perfect matching in these graphs will follow by taking edge-connectivity into account and by using Corollary 2.3, but it cannot be deduced using the results from [10]. Similar examples can be constructed for larger values of d.

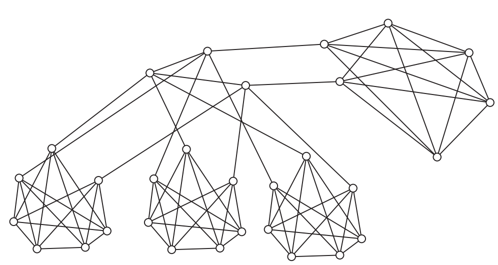

FIG. 2. *2-edge-connected 5-regular graph with $\lambda_5 = 2 < \rho(5) = 1 + \sqrt{13} = \lambda_3$.*

COROLLARY 2.4. *If G is a d-regular $(d-2)$-edge-connected graph such that $\lambda_p(G) < \rho(d)$, then*

$$\alpha'(G) > \frac{n + \lfloor -\frac{2p}{d} \rfloor}{2}.$$

Proof. Take $t = d-2$ in Theorem 1.1. ∎

In section 5, we will show for each d that this result is best possible for infinitely many values of p by presenting examples of d-regular $(d-2)$-edge-connected graphs H with $\lambda_p(H) = \rho(d)$ and $\alpha'(G) = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$.

3. Proof of Theorem 1.2. In this section, we prove Theorem 1.2, which relates the number of balloons of a regular graph to its eigenvalues.

When $d \geq 1$ is an odd integer, let B_d denote the unique graph on $d+2$ vertices having one vertex of degree $d-1$ and $d+1$ vertices of degree d. Equivalently B_d is the complement of the disconnected graph on $d+2$ vertices with $\frac{d-1}{2}$ components equal to K_2 and one component equal to P_3, the path on three vertices.

LEMMA 3.1. *If $\theta(d)$ denotes the largest solution of*

$$x^3 - (d-1)x^2 - 2dx + d - 1 = 0,$$

then the spectral radius of B_d is $\theta(d)$.

Proof. Partition the vertex set of B_d in three parts: the vertex of degree $d-1$, its neighbors, and the remaining two vertices. This partition is equitable, and its quotient matrix is

$$\begin{bmatrix} 0 & d-1 & 0 \\ 1 & d-3 & 2 \\ 0 & d-1 & 1 \end{bmatrix}.$$

The characteristic polynomial of this quotient matrix equals $P(x) = x^3 - (d-1)x^2 - 2dx + d - 1$. Because the partition is equitable, we conclude that the spectral radius of B_d equals the largest root of this polynomial (see [11]). This finishes the proof. ∎

A result from [9] and a simple manipulation yield the following bounds:

$$d - \frac{1}{d+2} + \frac{1}{(d+2)^2} < \theta(d) < d - \frac{1}{d+2} + \frac{1}{d(d+2)}. \tag{5}$$

LEMMA 3.2. *If H is a connected graph of order m with $m-1$ vertices of odd degree d and one vertex of degree $d-1$, then $\lambda_1(H) \geq \theta(d)$ with equality if and only if $H = B_d$.*

Proof. If $m = d+2$, then $H = B_d$. We will show that $\lambda_1(H) > \theta(d)$ for any connected graph H if $m > d+2$. Because the sum of the degrees of H is $dm - 1$ and d is odd, we conclude that m is odd.

The average degree of H is $\frac{(m-1)d+1(d-1)}{m} = d - \frac{1}{m}$. If $m \geq d+4$, then (5) implies that

$$\lambda_1(H) > d - \frac{1}{m} \geq d - \frac{1}{d+4} > d - \frac{d-1}{d^2+2d} > \theta(d)$$

for $d \geq 5$. Thus, the lemma is proved for $d \geq 5$ and $m \geq d+4$. If $d \geq 5$ and $m < d+4$, then $m = d+2$ because n is odd. This finishes the case $d \geq 5$.

When $d = 3$, we have $\lambda_1(B_3) = 2.855\cdots < 2.86$. If $m \geq 9$, then $\lambda_1(H) > 3 - \frac{1}{9} > 2.89 > \lambda_1(B_3)$. Because m is odd, the only remaining case is $m = 7$. In this case, each graph H has one vertex u of degree 2 and six vertices of degree 3. We have two cases depending on whether the neighbors of u are adjacent.

If the neighbors of u are adjacent, then H is the graph pictured in Figure 3, and its spectral radius is greater than 2.91, which exceeds $\lambda_1(B_3)$.

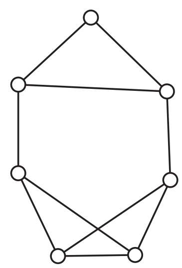

FIG. 3. *The neighbors of the vertex of degree* 2 *are adjacent.*

If the neighbors of u are not adjacent, then H is one of the three graphs pictured in Figure 4. The spectral radii of these graphs are greater than 2.9, which exceeds $\lambda_1(B_3)$. ∎

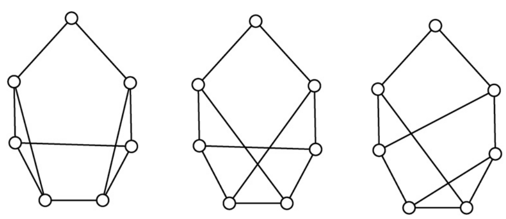

FIG. 4. *The neighbors of the vertex of degree* 2 *are not adjacent.*

Note that every balloon of a d-regular graph satisfies the conditions of the previous lemma.

Proof of Theorem 1.2. We prove the contrapositive: For $k \geq 3$ if G contains k balloons, then $\lambda_k(G) \geq \theta(d)$.

Let $H_1, \ldots, H_k$ be k balloons of G. These balloons are pairwise disjoint, and $H_1 \cup \cdots \cup H_k$ is an induced subgraph of G. Note that $H_1 \cup H_2$ might not be an induced subgraph of G when G contains exactly two balloons. The previous lemma and Lemma 1.6 imply that

$$\lambda_k(G) \geq \lambda_k(H_1 \cup \cdots \cup H_k) \geq \min_{1 \leq i \leq k}(\lambda_1(H_i)) \geq \theta(d). \qquad ∎$$

Fixing d, for some values of k, we construct examples of d-regular graphs H containing exactly k balloons and having $\lambda_k(H) = \theta(d)$. The vertex of B_d having degree $d-1$ is called the neck of B_d.

Let $m \geq d-1$ be an even integer, and let $k = 2m$. Consider a connected $(d-2)$-regular graph X on m vertices. For each vertex x of X, create two new vertices x_1 and x_2 adjacent to x. Identify each of x_1 and x_2 with the neck of a copy of B_d. The resulting graph G_1 is connected, is d-regular, has $m+2m(d+2)$ vertices, and contains k balloons.

Removing the m vertices of X yields a disconnected graph with $2m$ components, each isomorphic to B_d. Lemma 1.6 implies

$$\lambda_{k-m}(G_1) \geq \lambda_{k-m}(2mB_d) = \theta(d) \geq \lambda_k(G_1) \geq \lambda_k(2mB_d) = \theta(d),$$

which means $\lambda_k(G_1) = \theta(d)$.

For $d = 3$, we can construct a 3-regular graph G_k with $\lambda_k(H) = \theta(3)$ and $b(G_k) = k$ for any $k \geq 3$. Consider a tree of order n whose degrees are 1 or 3. If k is the number of leaves, then $2(n-1) = k + 3(n-k)$; thus $n = 2k - 2$. Identify each leaf with the neck of a copy of B_3. If G_k is the resulting graph, then G_k is 3-regular and $b(G_k) = k$. Remove the vertices adjacent to a neck of a balloon. There are $t \leq n - k = k - 2$ such vertices. By Lemma 1.6, we obtain

$$\theta(3) = \lambda_2(kB_3) \geq \lambda_{k-t}(kB_3) \geq \lambda_k(G_k) \geq \lambda_k(kB_3) = \theta(3),$$

and thus $\lambda_k(G_k) = \theta(3)$.

Each of these examples can be transformed into an infinite family of examples by replacing one balloon B_d or B_3 by a larger balloon.

4. Proof of Theorem 1.4. In this section, we give a proof of Theorem 1.4. We use the following notation. Let C_m denote the cycle on m vertices. If m is even, we denote by M_m the 1-regular graph on m vertices, i.e., a perfect matching on m vertices. Also $\overline{G}$ will denote the complement of a graph G. If H_1 and H_2 are vertex disjoint graphs, their join $H_1 \vee H_2$ is the graph obtained from H_1 and H_2 by joining each vertex of H_1 with each vertex of H_2.

Let d and t be two integers of the same parity with $d > t \geq 1$. Define the graph $B_{d,t}$ as follows:

$$B_{d,t} = \begin{cases} K_{d+1-t} \vee \overline{M_t} & \text{if } d \text{ and } t \text{ are even,} \\ \overline{M_{d+2-t}} \vee \overline{C_t} & \text{if } d \text{ and } t \text{ are odd.} \end{cases} \tag{6}$$

LEMMA 4.1. *The spectral radius of $B_{d,t}$ equals*

$$\lambda_1(B_{d,t}) = \begin{cases} \frac{d-2+\sqrt{(d+2)^2-4t}}{2} < d - \frac{t}{d+2} & \text{if } d \text{ and } t \text{ are even,} \\ \frac{d-3+\sqrt{(d+3)^2-4t}}{2} < d - \frac{t}{d+3} & \text{if } d \text{ and } t \text{ are odd.} \end{cases} \tag{7}$$

Proof. We prove this lemma only in the case when d and t are even. The proof of the other case is similar and is omitted.

Let V_1 and V_2 be the subsets of vertices of degree d and degree $d-1$ in $B_{d,t}$, respectively. From the definition of $B_{d,t}$, it is easy to see that $|V_1| = d + 1 - t$ and $|V_2| = t$. The partition $V(B_{d,t}) = V_1 \cup V_2$ is equitable, and its quotient matrix is

$$\begin{bmatrix} d-t & t \\ d-t+1 & t-2 \end{bmatrix}.$$

The characteristic polynomial of the quotient matrix is $x^2-(d-2)x-2d+t$. Because the partition is equitable, we deduce that the spectral radius of $B_{d,t}$ equals the largest root of this polynomial, which is $\frac{d-2+\sqrt{(d+2)^2-4t}}{2}$. The inequality $\frac{d-2+\sqrt{(d+2)^2-4t}}{2} < d-\frac{t}{d+2}$ follows easily using the inequality $\sqrt{x^2+a} < x + \frac{a}{2x}$ for $x > 0$. This finishes the proof. ∎

Note that $\lambda_1(B_{d,d-2})$ equals $\rho(d)$ from Theorem 1.1. The following result extends Theorem 2 from [10].

LEMMA 4.2. *Let d and t be two integers of the same parity with $d > t \geq 1$. If H is a graph of order m such that $\Delta(H) = d$ and $2e(H) = dm - t$, then $\lambda_1(H) \geq \lambda_1(B_{d,t})$. Equality occurs if $H = B_{d,t}$ when d and t are even and if $H = \overline{M_{d+2-t}} \vee \overline{C_{t_1} \cup \cdots \cup C_{t_l}}$, where $t_1 + \cdots + t_l = t$, when d and t are odd.*

Proof. The average degree of H is $\frac{dm-t}{m} = d - \frac{t}{m}$. This implies $\lambda_1(H) \geq d - \frac{t}{m}$.

Let $V(H) = V_1 \cup V_2$ be a partition of the vertex set of H. Let $n_i = |V_i|$, and denote by e_i the number of edges of the subgraph induced by V_i for $1 \leq i \leq 2$. Let $e_{12} = e(V_1, V_2)$. The quotient matrix of this partition is

$$\tilde{A} = \begin{bmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} \\ \frac{e_{12}}{n_2} & \frac{2e_2}{n_2} \end{bmatrix}. \tag{8}$$

Eigenvalue interlacing (see [5, 11] or the last part of section 1) implies

$$\lambda_1(H) \geq \lambda_1(\tilde{A}) = \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{e_{12}^2}{n_1 n_2}} \tag{9}$$

with equality if the partition $V(H) = V_1 \cup V_2$ is equitable.

If $d > t \geq 1$ are both even and $m \geq d + 2$, then by Lemma 4.1, we obtain $\lambda_1(H) \geq d - \frac{t}{m} \geq d - \frac{t}{d+2} > \lambda_1(B_{d,t})$. Thus, the only remaining case is $m = d + 1$.

If $m = d+1$, then H is a graph obtained by deleting $\frac{t}{2}$ distinct edges from K_{d+1}. The graph H has at most t vertices of degree less than $d - 1$, and thus it contains at least $d + 1 - t$ vertices of degree d. Let V_1 be a set of $d + 1 - t$ vertices of degree d, and let $V_2 = V(H) \setminus V_1$. By using (9), we obtain $\lambda_1(H) \geq \lambda_1(B_{d,t})$ with equality if $H = B_{d,t}$.

If $d > t \geq 1$ are both odd and $m \geq d + 3$, then by Lemma 4.1, we obtain $\lambda_1(H) \geq d - \frac{t}{m} \geq d - \frac{t}{d+3} > \lambda_1(B_{d,t})$.

If $m = d+2$, then H is a graph of maximum degree d obtained by deleting $\frac{d+2+t}{2}$ distinct edges from K_{d+2}. The graph H has at most t vertices of degree $d - 1$, and thus it contains at least $d + 2 - t$ vertices of degree d. Let V_1 be a set of $d + 2 - t$ vertices of degree d, and let $V_2 = V(H) \setminus V_1$. Let $2r$ denote the maximum number of vertices of V_1 which induce a 1-regular graph in $\overline{H}$.

Case 1. $r = (d + 2 - t)/2$.

In this case, $e_1 = \frac{(d+2-t)d}{2}$, $e_{12} = (d + 2 - t)t$, and $e_2 = \frac{t(t-3)}{2}$. We use inequality (9) to obtain $\lambda_1(H) \geq \lambda_1(B_{d,t})$. Equality occurs if the partition $V(H) = V_1 \cup V_2$ is equitable. This happens when $H = \overline{M_{d+2-t}} \vee \overline{C_{t_1} \cup \cdots \cup C_{t_l}}$, where $t_1 + \cdots + t_l = t$.

Case 2. $r = 0$.

In this case, $e_1 = \binom{d+2-t}{2}$, and each vertex of V_1 is adjacent to all but one vertex of V_2. Thus, $e_{12} = n_1(n_2 - 1) = (d + 2 - t)(t - 1)$. This and $2e = d(d + 2) - t$ imply that $e_2 = e - e_1 - e_{12} = \frac{d(d+2)-t}{2} - \frac{(d+2-t)(d+1-t)}{2} - (d + 2 - t)(t - 1) = \frac{d+t^2-4t+2}{2}$.

In this case, the characteristic polynomial of $\tilde{A}$ is the following:

$$P_{\tilde{A}}(x) = x^2 - \frac{dt + d + 2 - 3t}{t} x + \frac{d^2 + 2d + t^2 - 3dt - t}{t}.$$

We obtain

$$P_{\tilde{A}}(x) = (x^2 + (3 - d)x + t - 3d) + \frac{d(d + 2) - t - (d + 2)x}{t}.$$

Thus,

$$\begin{aligned} P_{\tilde{A}}(\lambda_1(B_{d,t})) &= \frac{d+2}{t}\left(d - \frac{t}{d+2} - \lambda_1(B_{d,t})\right) \\ &= \frac{d+2}{t}\left(d - \frac{t}{d+2} - \frac{d-3+\sqrt{(d+3)^2-4t}}{2}\right) \\ &< 0, \end{aligned}$$

where the last inequality follows by straightforward calculations as $t < d+2$. We conclude that the largest root of $\tilde{A}$ is larger than $\lambda_1(B_{d,t})$. By eigenvalue interlacing, this means $\lambda_1(H) > \lambda_1(B_{d,t})$.

Case 3. $1 \le r \le (d-t)/2$.

Consider the partition of $V(H)$ into three parts: the $2r$ vertices inducing an $\overline{M_{2r}}$ in H, the other $d+2-t-2r$ vertices of degree d in V_1, and the remaining t vertices. The quotient matrix of this partition is

$$A_3 = \begin{bmatrix} 2r-2 & d+2-t-2r & t \\ 2r & d+1-t-2r & t-1 \\ 2r & \frac{(d+2-t-2r)(t-1)}{t} & t-1-\frac{2r+3t-d-2}{t} \end{bmatrix}.$$

Dividing the characteristic polynomial $P_{A_3}(x)$ of A_3 by $x^2+(3-d)x+t-3d$, we obtain

(10)

$$P_{A_3}(x) = (x^2+(3-d)x+t-3d)\left(x + \frac{2t+2r-d-2}{t}\right) + \frac{(d+2-t-2r)(x-d)}{t}.$$

Plugging in $x = \lambda_1(B_{d,t})$, we get $P_{A_3}(\lambda_1(B_{d,t})) = \frac{(d+2-t-2r)(\lambda_1(B_{d,t})-d)}{t}$. This expression is negative because $r \le \frac{d-t}{2}$ and $\lambda_1(B_{d,t}) < d$. Thus, the largest root of $P_{A_3}(x)$ is greater than $\lambda_1(B_{d,t})$. Eigenvalue interlacing (see [5, 11]) implies $\lambda_1(H) > \lambda_1(B_{d,t})$ and finishes the proof. ☐

We can now prove Theorem 1.4.

Proof of Theorem 1.4. We can assume that G has edge-connectivity t. If $b_t(G) \ge p$, there exist at least p disjoint connected subgraphs $X_1, \dots, X_p$ of G that satisfy the conditions of Lemma 4.2. Because $p \ge 3$, $X_1 \cup \cdots \cup X_p$ is an induced subgraph of G. Lemma 4.2 and Lemma 1.6 imply

$$\lambda_p(G) \ge \lambda_p(X_1 \cup \cdots \cup X_p) \ge \min_{1\le i\le p} \lambda_1(X_i) \ge \lambda_1(B_{d,t}). \qquad ☐$$

We present now a construction for fixed d showing that Theorem 1.4 is best possible for infinitely many values of p.

Let t be an integer of the same parity as d with $d > t \ge 1$. Also let $p \ge 3$ be a positive integer such that $\frac{pt}{d}$ is also an integer. Let Y be a t-edge-connected bipartite graph with color classes P and Q such that $|P| = p$ and $|Q| = \frac{pt}{d} < k$. Assume also that each vertex in the color class P has degree t and that each vertex in Q has degree d. See [21] for a proof of the existence of such graphs.

For each vertex $x \in P$, consider its neighbors $x_1, \dots, x_t \in Q$. Remove x, and add t new vertices $y_1, \dots, y_t$ such that y_i is adjacent to x_i for each $1 \le i \le t$. Identify $y_1, \dots, y_t$ with the t vertices of degree $d-1$ from a copy of $B_{d,t}$. The resulting graph H is d-regular and t-edge-connected (see [21] for a short proof of this fact) and has $q + p(d-2+\epsilon)$ vertices, where $\epsilon = 3$ if d is even and $\epsilon = 4$ if d is odd.

Removing Q from H creates a disconnected graph $H - Q$ having p components $B_{d,t}$. Lemma 1.6 implies

$$\lambda_1(B_{d,t}) = \lambda_{p-\frac{pt}{d}}(H - Q) \geq \lambda_p(H) \geq \lambda_p(H - Q) = \lambda_1(B_{d,t}). \tag{11}$$

Thus, $\lambda_p(H) = \lambda_1(B_{d,t})$, and also $b_t(H) = p$.

5. Examples showing that Theorem 1.1 is best possible. In this section, we show that Theorem 1.1 is best possible when $t = d-2$ by presenting examples of d-regular graphs H of edge-connectivity $d-2$ with $\lambda_p(H) = \rho(d)$ and $\alpha'(H) = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$ for infinitely many values of p.

Let d and s be two integers with $d \geq 3$ and $s \geq 1$. Let $p = (d-2)s$ and $q = ds$. Our construction consists of the graphs H presented at the end of the previous section in the special case $t = d - 2$.

Removing the vertices of Q creates p disjoint copies $B_{d,d-2}$. Because $B_{d,d-2}$ has an odd number of vertices, it follows that $o(H - Q) = p$. Using (3), the matching number of H will be at least $\frac{n-p+q}{2} = \frac{n-ds+(d-2)s}{2} = \frac{n-2s}{2}$. It is actually easy to see that $\alpha'(H) = \frac{n-2s}{2} = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$. By Lemma 1.6, we obtain $\lambda_p(H) = \lambda_1(B_{d,d-2}) = \rho(d)$ as claimed.

In Figure 5, we illustrate this construction when $d = 4$ and $Y = K_{4,2}$.

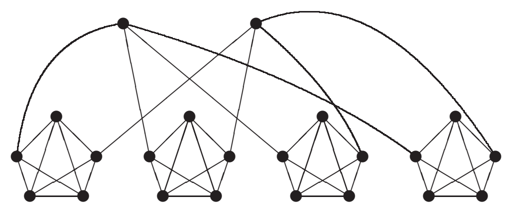

FIG. 5. *4-regular graph with edge-connectivity* 2 *and* $\lambda_4 = \rho(4)$.

Plesnik [23] (see also exercise 30, section 7 in Lovász [17]) proved that a graph obtained by removing $d - 1$ edges from a d-regular $(d - 1)$-edge-connected graph contains a perfect matching. This implies that a d-regular $(d - 1)$-edge-connected graph is matching covered, i.e., each edge is contained in a perfect matching. It would be interesting to determine sharp relations between this property and the eigenvalues of a graph.

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