Chapter 14
Some Applications of Eigenvalues of Graphs

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Abstract The main goal of spectral graph theory is to relate important structural properties of a graph to its eigenvalues. In this chapter, we survey some old and new applications of spectral methods in graph partitioning, ranking, epidemic spreading in networks and clustering.

Keywords Eigenvalues · Graph · Partition · Laplacian

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14.1 Introduction

The study of eigenvalues of graphs is an important part of combinatorics. Historically, the first relation between the spectrum and the structure of a graph was discovered in 1876 by Kirchhoff when he proved his famous matrix-tree theorem. The key principle dominating spectral graph theory is to relate important invariants of a graph to its spectrum. Often, such invariants such as chromatic number or independence number, for example, are difficult to compute so comparing them with expressions involving eigenvalues is very useful. In this chapter, we present some connections between the spectrum of a graph and its structure and some applications of these connections in fields such as graph partitioning, ranking, epidemic spreading in networks, and clustering. For other applications of eigenvalues of graphs we recommend the surveys [44] (expander graphs), [51] (pseudorandom graphs), or [61, 62] (spectral characterization of graphs).

To an undirected graph G of order n, one can associate the following matrices:

- The adjacency matrix $A = A(G)$.

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This is an n-by-n matrix whose rows and columns are indexed after the vertices of G. For each $u, v \in V(G)$, $A(u,v)$ equals the number of edges between u and v.

- The Laplacian matrix $L = L(G)$.
  It is also known as the combinatorial Laplacian of G and it equals $D - A$, where D is the diagonal matrix containing the degrees of the vertices of G and A is the adjacency matrix of G.
- The normalized Laplacian matrix $\mathcal{L} = \mathcal{L}(G)$.
  This equals $D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I_n - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$.

Given a real and symmetric matrix M of order n, we denote its eigenvalues by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$.

If G is an undirected graph, then all the previous matrices are symmetric and consequently, their eigenvalues are real numbers.

We use the following notation throughout this paper. The eigenvalues of the adjacency matrix $A(G)$ are indexed in nonincreasing order:

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \tag{14.1}$$

The eigenvalues of the combinatorial Laplacian matrix $L(G)$ are listed in nondecreasing order:

$$\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G) \tag{14.2}$$

The eigenvalues of the normalized Laplacian $\mathcal{L}(G)$ are listed in nondecreasing order:

$$\theta_1(G) \leq \theta_2(G) \leq \cdots \leq \theta_n(G) \tag{14.3}$$

If G is d-regular, the previous three matrices are related as follows:

$$\mathcal{L}(G) = \frac{1}{d} L(G) = I_n - \frac{1}{d} A(G). \tag{14.4}$$

This implies that the eigenvalues of these matrices satisfy the following equation:

$$\theta_i(G) = \frac{\mu_i(G)}{d} = 1 - \frac{\lambda_{n+1-i}(G)}{d}. \tag{14.5}$$

We list below some basic properties of eigenvalues of graphs. For more details on eigenvalues of graphs see the monographs of Cvetković et al. [24, 25] (for eigenvalues of the adjacency matrix), the survey of Mohar [55] (for eigenvalues of the Laplacian), the monograph of Godsil and Royle [38] (for eigenvalues of the adjacency matrix and of the Laplacian), or the book of Chung [18] (for eigenvalues of the normalized Laplacian). The close relation between eigenvalues and the edge distribution of a graph is outlined in the following section.

For any real and symmetric matrix M of order n, its eigenvalues are real and they can be described as follows.

Theorem 1 (Courant–Fisher). *Let M be a real and symmetric matrix of order n. Then*

$$\lambda_1(M) = \max_{x \in \mathbb{R}^n, x \neq \mathbf{0}} \frac{x^t M x}{x^t x}$$

For any $j \in \{2, \ldots, n\}$,

$$\begin{aligned}\lambda_j(M) &= \min_{u_1,\ldots,u_{j-1} \in \mathbb{R}^n} \max_{\substack{x \in \mathbb{R}^n, x \neq \mathbf{0} \\ x \perp u_1,\ldots,u_{j-1}}} \frac{x^t M x}{x^t x} \\ &= \max_{v_1,\ldots,v_{n-j} \in \mathbb{R}^n} \min_{\substack{x \in \mathbb{R}^n, x \neq \mathbf{0} \\ x \perp v_1,\ldots,v_{n-j}}} \frac{x^t M x}{x^t x}.\end{aligned}$$

14.2 Eigenvalues of the Adjacency Matrix

The eigenvalues of the adjacency matrix were studied in 1957 in a paper [23] by Collatz and Sinogowitz. In [23], the authors determined the eigenvalues of the following graphs:

- The complete graph K_n: spectrum $n-1$ and -1 with multiplicity $n-1$.
- The path P_n: spectrum

$$2\cos\frac{\pi j}{n+1},\ j \in \{1, \ldots, n\}. \tag{14.6}$$

- The cycle C_n: spectrum

$$2\cos\frac{2\pi j}{n},\ j \in \{1, \ldots, n\}. \tag{14.7}$$

Collatz and Sinogowitz also showed that the largest eigenvalue of the adjacency matrix of a graph G with n vertices satisfies the following inequalities:

$$2\cos\frac{\pi}{n+1} \leq \lambda_1(G) \leq n-1. \tag{14.8}$$

Equality holds in the first inequality if and only if $G = P_n$ and equality holds in the second inequality if and only if $G = K_n$.

A walk of length r in G is a sequence of vertices $u_0, u_1, \ldots, u_r$ such that u_i is adjacent to u_{i+1} for each $0 \leq i \leq r-1$. The previous walk is closed if $u_0 = u_r$.

The following lemma can be easily proved by induction.

Lemma 1. *The (u, v)-th entry of A^r equals the number of walks of length r which start at u and end at v.*

Let $W_r(G)$ denote the number of closed walks of length r in G. An easy consequence of the previous result is the following lemma which is the basis of many important results involving eigenvalues of the adjacency matrix. This is often used when studying the eigenvalues of random graphs.

Lemma 2. *For any integer* $r \geq 1$,

$$W_r(G) = \mathrm{tr}A^r = \sum_{i=1}^{n} \lambda_i^r. \tag{14.9}$$

A simple connection between the structure of a graph and its eigenvalues is given by the following result.

Lemma 3. *A graph G is bipartite if and only if the spectrum of its adjacency matrix is symmetric with respect to* 0.

Proof. The proof follows using the previous result. □

For regular graphs, we have more information regarding the extreme eigenvalues.

Lemma 4. *Let G be a connected d-regular graph on n vertices. Then*

(i) $d = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -d$.
(ii) *G is bipartite if and only if* $\lambda_n = -d$.

As mentioned earlier, there is a close connection between the eigenvalues of a graph and its structure. The following result, also known as the expander mixing lemma (cf. [44]), exemplifies this connection.

Theorem 2. *Let G be a connected d-regular graph and* $\lambda = \max(|\lambda_2|, |\lambda_n|)$. *If* $S, T \subset V(G)$, *then*

$$\left||E(S,T)| - \frac{d}{n}|S||T|\right| \leq \lambda \frac{\sqrt{|S|(n-|S|)|T|(n-|T|)}}{n}. \tag{14.10}$$

This result implies that if λ is small compared to d, then the edge distribution of G is close to the edge distribution of the random graph with the same edge density as G. The graphs with small λ are called expanders and are very important in many areas of mathematics and computer science (see the excellent survey of Hoory et al. [44] on expander graphs and their applications).

14.3 Eigenvalues of the Laplacian

The first application of the Laplacian of a graph is the matrix-tree theorem or Kirchhoff's theorem [49] (see [9], Chap. II for more details). If L is the Laplacian matrix of a graph G and $i \neq j$ are two vertices of G,then let $L_{(ij)}$ be the matrix obtained from L by deleting row i and column j.

Theorem 3 (Matrix-Tree Theorem). *If $i \neq j$ are two vertices of a connected graph G, then the number of spanning trees of G equals the absolute value of* $\det(L_{(ij)})$. *Also, the number of spanning trees of G equals $\frac{\mu_2 \ldots \mu_n}{n}$.*

We list now some simple properties of the eigenvalues of the Laplacian of a graph.

Lemma 5. *Let G be a graph. Then*

(i) The Laplacian matrix of G is a positive semidefinite matrix.
(ii) The smallest eigenvalue $\mu_1(G)$ of the Laplacian of G equals 0 and its multiplicity equals the number of components of G.
(iii) The graph G is connected if and only if $\mu_2(G) > 0$.

Proof. Orient the edges of G arbitrarily. Consider a signed incidence matrix N of G with respect to the orientation of the edges. The rows of N are indexed by the vertices of G, the columns of N are indexed by the edges of G, and the entries of N are defined as follows:

$$N(i,e) = \begin{cases} +1, \text{if } i \text{ is the head of } e \\ -1, \text{if } i \text{ is the tail of } e \\ 0, \text{otherwise.} \end{cases}$$

By a simple calculation, it follows that $L(G) = NN^t$. This implies that the Laplacian of G is a positive semidefinite matrix. Thus, all its eigenvalues are non-negative. Also, for any vector $x \in \mathbb{R}^n$,

$$x^t L x = x^t N N^t x = (N^t x)^t (N^t x) = \sum_{ij \in E(G)} (x_i - x_j)^2. \tag{14.11}$$

Let $H_1, \ldots, H_k$ denote the components of G. For each $i \in \{1, \ldots, k\}$, consider the vector v_i that is the characteristic vector of the component H_i. It is easy to see that v_i is an eigenvector of L corresponding to the eigenvalue 0. Also, $v_1, \ldots, v_k$ are linearly independent which implies that the multiplicity of 0 is at least k.

Let y be an arbitrary eigenvector corresponding to the eigenvalue 0. It follows that $y^t L y = 0$. Using (14.11), it follows that $\sum_{ij \in E(G)} (y_i - y_j)^2 = 0$ which means that the entries of y are constant on each component of G. This implies y is a linear combination of $v_1, \ldots, v_k$ which shows that the multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of components of G. The last part of the theorem follows easily. □

A very important property of the eigenvalues of the Laplacian is that they control the edge distribution in the graph. Chung [19] proved the following result.

Theorem 4. *Let G be a connected graph on n vertices with average degree d. Then for any subsets $S, T \subset V(G)$,*

$$||E(S,T)| - \frac{d}{n}|S||T|| \leq \frac{\max_{i \neq 0} |d - \mu_i|}{n} \sqrt{|S|(n - |S|)|T|(n - |T|)}. \tag{14.12}$$

14.4 Eigenvalues of the Normalized Laplacian

The normalized Laplacian was introduced by Chung [18]. We list now some simple properties of the eigenvalues of the normalized Laplacian of a graph. These properties are very similar to those of the Laplacian of G. The eigenvalues of the normalized Laplacian seem to relate better to parameters related to random walks on graphs (cf. [19]).

Lemma 6. *Let G be a graph. Then*

(i) The normalized Laplacian matrix of G is a positive semidefinite matrix.
(ii) The smallest eigenvalue $\theta_1(G)$ of the Laplacian of G equals 0 and its multiplicity equals the number of components of G.
(iii) The graph G is connected if and only if $\theta_2(G) > 0$.

In [19], Chung proved a matrix-tree theorem for the normalized Laplacian.

Theorem 5. *If G is a connected graph, the number of its spanning trees equals*

$$\frac{\prod_{i \in V(G)} d_i}{\sum_{i \in V(G)} d_i} \prod_{i \neq 1} \theta_i$$

The eigenvalues of the normalized Laplacian also influence the edge distribution in the graph. If $S \subset V(G)$, define vol(S) as $\sum_{i \in S} d_i$. Chung [19] proved the following result.

Theorem 6. *If S and T are subsets of vertices in a connected graph G, then*

$$||E(S,T) - \frac{\mathrm{vol(S)vol(T)}}{\mathrm{vol(G)}}| \leq \overline{\theta} \frac{\sqrt{\mathrm{vol(S)vol(\overline{S})vol(T)vol(\overline{T})}}}{\mathrm{vol(G)}}$$

where $\overline{\theta} = \max_{i \neq 0} |1 - \theta_i|$.

14.5 Graph Partitioning Using Eigenvalues and Eigenvectors

There are many examples of graph-theoretic questions which can be formulated as the problem of partitioning the vertices of a graph $G = (V, E)$ into a fixed number $k \geq 2$ of disjoint nonempty subsets $V_1, \ldots, V_k$ such that some objective function $f(V_1, \ldots, V_k)$ is maximized or minimized.

The famous MAX-CUT problem is concerned with determining a partition of G into two parts V_1 and V_2 such that the number of edges between these parts $e(V_1, V_2)$ is maximum.

Finding the edge-connectivity of G is equivalent to finding a partition of G into two parts V_1 and V_2 such that $e(V_1, V_2)$ is minimum. Finding the vertex connectivity of G means determining a partition of G into three parts V_1, V_2, V_3 such that $e(V_1, V_3) = 0$ and $|V_2|$ is minimum.

Given a subset of vertices S, let $\Phi(S) = \frac{e(S,S^c)}{\min(|S|,|S^c|)}$ and $\Psi(S) = \frac{|N(S)\backslash S|}{|S|}$. Determining the edge-expansion constant of a graph means finding the minimum of $\Phi(S)$ taken over all subsets S of $V(G)$. The vertex-expansion of G is the minimum of $\Psi(G)$ over all subsets of S with $|S| \leq \frac{|V(G)|}{2}$.

The idea of using eigenvalues to study graph-partitioning problems originated with Donath and Hoffman [29]. Their work was based on previous results of Hoffman and Wielandt [43]. A partition of the vertex set of a graph G into k nonempty subsets $V_1, \ldots, V_k$ is called a k-partition of G.

Theorem 7 ([29]). *Let $V_1, \ldots, V_k$ be a k-partition of a G such that $|V_i| = n_i$ for each $i \in \{1, \ldots, k\}$ and $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. Then*

$$\sum_{1 \leq i < j \leq k} e(V_i, V_j) \geq \frac{1}{2} \sum_{l=2}^{k} n_l \mu_l(G). \tag{14.13}$$

Proof. The proof uses the Hoffman–Wielandt inequality [43] which states that is A and B are two real and symmetric matrices of the same order m, then

$$\mathrm{tr}(\mathrm{AB^t}) \leq \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}(\mathrm{A}) \lambda_{\mathrm{i}}(\mathrm{B}) \tag{14.14}$$

Taking A to be the Laplacian of G and B the direct sum of all one matrices of order $n_1, n_2, \ldots, n_k$ yields the required result. □

Actually Donath and Hoffman proved some stronger results in [29]. They showed that the previous result is true when one replaces the Laplacian by any matrix of the form $F - A$ where F is a diagonal matrix whose entries sum up to twice the number of edges of G.

If the sizes n_i are not equal, then Donath and Hoffman prove the following improvement of the previous theorem.

Theorem 8. *Let $G = (V, E)$ be a graph and $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be a partition of G into k parts such that $|V_i| = n_i$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. Let $y_2 \geq \cdots \geq y_k$ be the roots of*

$$\left(\sum_i n_i\right) x^{k-1} - 2 \sum_{i<j} n_i n_j x^{k-2} + 3 \sum_{i<j<l} n_i n_j n_l x^{k-3} - \cdots = 0$$

Then

$$\sum_{1 \leq i < j \leq k} e(V_i, V_j) \geq \sum_{i=2}^{k} y_i \mu_i(G)$$

If k divides n, a k-partition $V_1, \dots, V_k$ of G is called balanced if $|V_i| = \frac{n}{k}$ for any $i \in \{1, \dots, k\}$. The k-section width $sw_k(G)$ of a graph G is defined as

$$sw_k(G) = \min \sum_{1 \le i < j \le k} e(V_i, V_j),$$

where the minimum is taken over all balanced k-partitions of G. The 2-section width is also called the bisection width. The calculation of the bisection width of a graph G is NP-hard [34], even when it is restricted to the class of d-regular graphs [16].

The result of Donath and Hoffman implies that for any graph G on n vertices,

$$sw_k(G) \ge \frac{n \sum_{l=2}^{k} \mu_l(G)}{2k} \tag{14.15}$$

In particular, for $k = 2$, the bisection width satisfies the inequality

$$sw_2(G) \ge \frac{n\mu_2(G)}{4} \tag{14.16}$$

Motivated by questions in parallel computation, Elsässer et al. [31] studied balanced k-partitions of graphs. The authors gave a new proof of inequality (14.15) and characterized the equality case. Note that Theorem 1 of [31] is a particular case of Corollary 4.3.18 from [45].

Theorem 9. *Let G be a connected graph on n vertices and let $V_1, \dots, V_k$ be a balanced k-partition of G of size $sw_k(G)$. Let $x_1, \dots, x_k$ be eigenvectors corresponding to the first k smallest eigenvalues of $L(G)$. If*

$$sw_k(G) = \frac{n \sum_{l=2}^{k} \mu_l(G)}{2k}$$

then

(i) For any $i \in \{1, \dots, k\}$, if $s, t \in V_i$, then $x_j(s) = x_j(t)$ for any $j \in \{1, \dots, k\}$.
(ii) For any $i \neq j \in \{1, \dots, k\}$ and any two vertices $s, t \in V_i$, the number of neighbours of s in V_j equals the number of neighbours of t in V_j.

The authors of [31] also provide examples of simple graphs for which the bound from Theorem 7 is far from optimal. We describe some of their examples below.

Given two graphs G and H, the Cartesian product $G \square H$ has vertex set $V(G) \times V(H)$ and its edges are defined as follows:

$$(a_1, b_1) \sim (a_2, b_2)$$

if and only if $a_1 \sim a_2$ and $b_1 = b_2$ or $a_1 = a_2$ and $b_1 \sim b_2$. The Laplacian matrix of $G \square H$ equals $L(G) \otimes I_{|V(H)|} + I_{|V(G)|} \otimes L(H)$. Thus, the eigenvalues of the Laplacian of $G \square H$ are of the form $\mu_i(G) + \mu_j(H)$ for $i \in \{1, \dots, |V(G)|\}$ and $j \in \{1, \dots, |V(H)|\}$ (see also [55]).

The $r \times r$ torus graph is the Cartesian product $C_r \square C_r$ of two cycles of length r. This graph is a 4-regular graph and thus, the eigenvalues of its Laplacian are related to the eigenvalues of its adjacency matrix as pointed out by (14.5). Using (14.5) and (14.7), a $\sqrt{n} \times \sqrt{n}$-torus has $\mu_2 = \mu_3 = \mu_4 = 2 - 2\cos\left(\frac{2\pi}{\sqrt{n}}\right)$. Thus, the right side of Theorem 7 yields a lower bound of $\frac{3\pi^2}{2}$ for sw_4. However, the 4-section width of the previous graph is $4\sqrt{n}$.

When $k = 2$, Bezrukov et al. [8] characterized the graphs for which equality is attained in (14.16).

Theorem 10. *Let $G = (V, E)$ be a connected graph on an even number of vertices. The following statements are equivalent:*

(i) $sw_2(G) = \frac{n\mu_2(G)}{4}$.
(ii) There is an eigenvector corresponding to $\mu_2(G)$ which has only -1 and $+1$ entries.
(iii) In any optimal bisection, $V(G) = V_0 \cup V_1$, any vertex is incident to exactly $\frac{\mu_2(G)}{2}$ edges.

There are many graphs whose bisection width equals $\frac{n\mu_2}{4}$. Such examples are the complete graphs K_n on an even number of vertices (they have $sw_2 = \frac{n^2}{4}$ and $\mu_2 = 4$), the Petersen graph (it has $n = 10, sw_2 = 5$, and $\mu_2 = 2$), and the d-dimensional hypercube Q_d (it has $n = 2^d, sw_2 = 2^{d-1}$, and $\mu_2 = 2$).

However, there are many graphs for which inequality (14.16) is weak. Guattery and Miller [39] constructed some examples of such graphs. We describe one of their examples below. For $k \geq 1$, the graph G_k has vertex set $\{1, \ldots, 4k\}$ and consists of two disjoint paths on $2k$ vertices (with vertex set $\{1, \ldots, 2k\}$ and $\{2k+1, \ldots, 4k\}$, respectively). Also, for any $1 \leq j \leq k$, the vertex $k + j$ is adjacent to the vertex $3k + j$. The graph G_k is planar and looks like a ladder with $2k$ steps from which the bottom k steps have been removed.

The graph G_k has $4k$ vertices. It has a bisection width 2 since removing the edges $(k, k+1)$ and $(3k, 3k+1)$ yields two disjoint components of order $2k$, namely the ones induced by $\{1, \ldots, k, 2k+1, \ldots, 3k\}$ and $\{k+1, \ldots, 2k, 3k+1, \ldots, 4k\}$.

Guattery and Miller [39] showed that $\mu_2(G) \leq 4\sin^2\left(\frac{\pi}{2k}\right)$ and that the spectral partition produces a bisection width of size k (the bisection width given by the spectral method is $\{1, \ldots, 2k\}$ and $\{2k+1, \ldots, 4k\}$).

In [8], the authors also improve the Donath–Hoffman lower bound on the bisection width for a graph with specific level structure. We briefly describe their results below. Let $V = V_0 \cup V_1$ be a bisection of a graph G with cut size σ. Let V_0^1 denote the subset of vertices in V_0 that are incident to a cut edge. For $i \geq 1$, let V_0^i denote the set of vertices in V_0 at distance $i-1$ from a vertex in V_0^1. One can define the sets V_1^i similarly. Also, let $E_\epsilon^i = E(V_\epsilon^i, V_\epsilon^{i+1})$ for $i \geq 1$ and $\epsilon \in \{0, 1\}$. If $g : \mathbb{N} \to \mathbb{N}$, then $LS(g, \sigma)$ denotes the class of graphs which have a bisection of cut size σ and a level structure as above such that $|E_\epsilon^i| \leq \sigma g(i)$ for $\epsilon \in \{0, 1\}$ and all $i \geq 1$.

Theorem 11. *If $G \in LS(g,\sigma)$, then there exists a function $\gamma : \mathbb{R}^+ \to \mathbb{R}$ with $\gamma(x) \to 0$ as $x \to \infty$ such that*

- *If $A := 1 + 2\sum_{i=2}^{\infty} \frac{1}{g(i-1)} < \infty$, then $\sigma \geq \frac{n\mu_2}{4} \cdot A\left(1+\gamma\left(\frac{n}{\sigma}\right)\right)$.*
- *If $g(i) = i+1$, then $\sigma \geq \frac{n\mu_2}{4} \cdot LambertW\left(\frac{4}{\mu_2}\right)\left(1+\gamma\left(\frac{n}{\sigma}\right)\right)$, where $LambertW(x)$ is the inverse function of xe^x.*
- *$g(i) = (i+1)^{\alpha}$ and $0 \leq \alpha < 1$, then $\sigma \geq \frac{n\mu_2^{\frac{1+\alpha}{2}}}{4} \cdot f(\alpha)\left(1+\gamma\left(\frac{n}{\sigma}\right)\right)$, where $f(\alpha) = \frac{1+\alpha}{2((1-\alpha)(3-\alpha))^{\frac{1+\alpha}{2}}}$.*

In [8], the authors show that there are graphs for which the bounds from the previous theorem are tight up to a constant factor. If G is a graph of maximum degree d, then $G \in LS(g, sw_2(G))$, where $g(i) = (d-1)^i$. This is because $\max(|V_0^i|, |V_1^i|) \leq sw_2(G)(d-1)^{i-1}$. In this case, the previous theorem implies that $sw_2(G) \geq \frac{n\mu_2}{4} \cdot \frac{d}{d-2}(1-o(1))$ as $\frac{n}{sw_2(G)} \to \infty$.

Recall that a connected d-regular graph is called Ramanujan if $|\lambda_i| \leq 2\sqrt{d-1}$ for each $\lambda_i \neq \pm d$. In [8], the authors use the previous theorem to improve the bisection width of Ramanujan graphs. The Donath–Hoffman bound implies a lower bound of $0.0042n$ for the bisection width of a 3-regular Ramanujan graph and of $0.133n$ for the bisection width of a 4-regular Ramanujan graph. They improve the previous bound to $0.082n$ and $0.176n$, respectively. In the opposite direction, Monien and Preis [56] gave upper bounds on the bisection width of $\left(\frac{1}{6}+\epsilon\right)n$ for 3-regular graphs and of $(0.4+\epsilon)n$ for 4-regular graphs, for any $\epsilon > 0$, when n is larger than some function of the chosen ϵ. For more recent results regarding the bisection width of random regular graphs, see [28].

Using some eigenvalue interlacing results of Haemers [40], Bollobás and Nikiforov [10] proved the following result which also implies inequality 14.15.

Theorem 12. *Let $V_1, \ldots, V_k$ be a k-partition of a graph G on n vertices. Then*

$$\sum_{l=0}^{k-1} \mu_{n-l}(G) \geq \sum_{1 \leq i < j \leq k} e(V_i, V_j)\left(\frac{1}{|V_i|} + \frac{1}{|V_j|}\right) \geq \sum_{l=2}^{k} \mu_l(G). \qquad (14.17)$$

Fiedler [32, 33] used the eigenvalues of the Laplacian in connection with the connectivity of a graph. From the Courant–Fisher theorem we know that

$$\mu_2(G) = \min_{\substack{x \in \mathbb{R}^n, x \neq 0 \\ x \perp 1}} \frac{\sum_{ij \in E(G)} (x_i - x_j)^2}{\sum_{i \in V(G)} x_i^2} \qquad (14.18)$$

In [32], Fiedler called $\mu_2(G)$ the algebraic connectivity of the graph G because of its connections with the usual vertex- and edge-connectivity of G. Recall that the vertex connectivity $k(G)$ of a connected graph G is the minimum number of vertices whose deletion disconnects G. By convention $k(K_n) = n-1$. The edge connectivity of a connected graph G is the minimum number of edges whose removal disconnects G. The following result is well known (see [67, 68] for more details).

Lemma 7 ([68]). *If G is a connected graph with minimum degree $\delta(G)$, then*

$$1 \leq k(G) \leq k'(G) \leq \delta(G)$$

Fiedler [32] proved the following important theorem.

Theorem 13 ([32]). *Let G be a connected graph. Then*

$$\mu_2(G) \leq k(G) \leq k'(G).$$

Proof. The proof follows after showing that deleting any vertex from a connected graph decreases the algebraic connectivity by at most 1. More precisely, if i is vertex of a connected graph G on n vertices and $H = G \setminus \{i\}$, then

$$\mu_2(G) \leq \mu_2(H) + 1$$

This can be proved using the Courant–Fisher theorem. □

Kirkland et al. [50] characterized the connected graphs for which $\mu_2(G) = k(G)$. If G_1 and G_2 are distinct graphs, then the join, $G_1 \vee G_2$ of G_1 and G_2 is the graph obtained from the union of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

Theorem 14 ([50]). *Let G be a connected graph on n vertices. Then $\mu_2(G) = k(G)$ if and only if G can be written as a join $G_1 \vee G_2$ where G_1 is a disconnected graph on $n - k(G)$ vertices and G_2 is a graph on $k(G)$ vertices with $\mu_2(G_2) \geq 2k(G) - n$.*

For regular graphs, Fiedler's results were improved by Krivelevich and Sudakov [51].

Theorem 15 ([51]). *Let G be a connected, d-regular graph. If $\mu_2(G) = d - \lambda_2(G) \geq 2$, then $k'(G) = d$. Also,*

$$k(G) \geq d - \frac{36\lambda^2(G)}{d}$$

Here, $\lambda(G) = \max |\lambda_i(G)|$ where the maximum is taken over all eigenvalues $\lambda_i(G) \neq \pm d$.

The first inequality of the previous theorem was recently improved by the author (see [22] for more details). In [51], Krivelevich and Sudakov show that the error term in the second inequality is tight up to a constant factor.

A graph G of order n is called strongly regular with parameters (n, d, a, b) if it is d-regular, any two adjacent vertices have exactly a common neighbours, and any two nonadjacent vertices have exactly b common neighbours. From the definition, it follows that if A is the adjacency matrix of an (n,d,a,b) strongly regular graph, then

$$A^2 = dI + aA + b(J - I - A)$$

where J is the all one matrix. This implies that the eigenvalues of an (n,d,a,b) strongly regular graph are

$$d, \frac{a-b \pm \sqrt{(a-b)^2 + 4(d-b)}}{2}$$

Strongly regular graphs are well studied and have many connections to finite geometry and algebra.

Brouwer and Mesner [15] showed that the vertex-connectivity of a strongly regular graph equals its degree. Brouwer and Haemers [12] showed that the edge-connectivity of a distance-regular graph equals its degree.

These results were recently improved by Brouwer and Koolen [14] who showed that the vertex-connectivity of a distance-regular graph equals its degree d and that the only disconnecting subsets of size d are the vertex neighborhoods.

In [32], Fiedler obtained other inequalities relating the connectivity of a graph to the eigenvalue of the Laplacian.

Theorem 16 ([32]). *Let G be a connected graph with n vertices and maximum degree $\Delta(G)$. Let $\omega = \frac{\pi}{n}$. Then*

$$\mu_2(G) \geq 2k'(G)(1 - \cos\omega)$$

and

$$\mu_2(G) \geq 2k'(G)(\cos\omega - \cos 2\omega) - 2\Delta(G)\cos\omega(1 - \cos\omega)$$

In another seminal paper [33], Fiedler studied the eigenvectors of the Laplacian of a graph.

Theorem 17 ([33]). *Let G be a connected graph, and let u_2 be an eigenvector corresponding to the eigenvalue $\mu_2(G)$. For any $\beta \in \mathbb{R}$, let $V_+(\beta) = \{i \in V(G) : u_2(i) \geq \beta\}$ and $V_-(\beta) = \{j \in V(G) : u_2(j) \leq \beta\}$. Then for any $\beta \geq 0$, the subgraph induced by $V_+(-\beta)$ is connected and the subgraph induced by $V_-(\beta)$ is connected as well.*

The entries of an eigenvector u_2 corresponding to $\mu_2(G)$ can be used to construct graph-partitioning algorithms. The basic idea of spectral partitioning is to find a splitting value β and partition the graph into $V_-(\beta) = \{i : u_2(i) \leq \beta\}$ and $V(G) \setminus V_-(\beta) = \{j : u_2(j) > \beta\}$. Choosing the value of β depends on the specific application. Some popular choices are the following.

- **Bisection width:** β is the median of $u_2(1), \ldots, u_2(n)$.
- **Edge expansion:** β is the value that minimizes $\Phi(S)$, where $S = V_-(\beta)$.
- **Vertex expansion:** β is the value that minimizes $\Psi(S)$, where $S = V_-(\beta)$.

Since the mid-1980s, many researchers (see [1,6,54,60] for example) have studied the connections between the expansion of a graph and its Laplacian eigenvalues. The following result is due to Mohar [54].

Theorem 18 ([54]). *If G is a connected graph, then*

$$\frac{\mu_2(G)}{2} \leq \Phi(G) \leq \sqrt{(2\Delta(G) - \mu_2(G))\mu_2(G)}. \tag{14.19}$$

In [18], Chung proved a similar result involving the eigenvalues of the normalized Laplacian and the expansion properties of a graph. Recall that if S is a subset of vertices of a graph G, vol(S) is defined to be $\mathrm{vol(S)} = \sum_{\mathrm{i} \in \mathrm{S}} \mathrm{d_i}$. The Cheeger ratio of S is defined to be $h_S = \frac{|E(S,\overline{S})|}{\min(\mathrm{volS}, \mathrm{vol}\overline{\mathrm{S}})}$. This is very similar to the definition of $\Phi(S)$. Also, note that $h_S = h_{\overline{S}}$. The Cheeger constant of G is

$$h_G = \min_{S \subset V(G)} h_S. \tag{14.20}$$

Recall that θ_i is the i-th smallest eigenvalue of the normalized Laplacian $I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$. Chung [18] proved the following result (see also [20] for a simpler proof).

Theorem 19. *If G is a connected graph, then*

$$2h_G \geq \theta_2 \geq \frac{s_G^2}{2} \geq \frac{h_G^2}{2}. \tag{14.21}$$

Here s_G is the minimum Cheeger ratio of subsets S_k, consisting of vertices with the largest k values in the eigenvector associated with θ_2, for all $1 \leq k \leq n-1$.

In a recent paper [21], Chung studied local cuts and local graph partitioning algorithms based on eigenvalues of the normalized Laplacian. More precisely, given a connected graph $G = (V, E)$ and a subset of vertices S, the local Cheeger constant c_S of S is defined as

$$c_S = \min_{T \subset S} \frac{|E(T, \overline{T})|}{\mathrm{vol(T)}}.$$

The closure S^* of S is formed by the vertices of S and the vertices adjacent to a vertex of S. A function $f : S^* \to \mathbb{R}$ satisfies the Dirichlet boundary condition if $f(i) = 0$ for all $i \in S^* \setminus S$. The Dirichlet eigenvalue θ_S of S is defined as

$$\theta_S = \min \frac{\sum_{ij \in E(G)} (f(i) - f(j))^2}{\sum_{i \in S} f^2(i) d_i},$$

where the minimum is taken over all nonzero functions $f : S^* \to \mathbb{R}$ satisfying the Dirichlet boundary condition.

Chung [21] proves the following local Cheeger inequality.

Theorem 20. *Let G be a connected graph and S be a subset of vertices of G such that $G[S]$ is connected. Then*

$$c_S \geq \theta_S \geq \frac{c_S^2}{2}.$$

The proof of this theorem yields a simple algorithm (which is based on the eigenvector corresponding to θ_S) for finding a local cut. The previous theorem will guarantee that this cut is within a quadratic of the optimum.

Spielman and Teng [58, 59] have proved that the spectral partitioning method works well for planar graphs.

Theorem 21 ([58]). *Let G be a connected planar graph on n vertices and maximum degree $\Delta(G)$. Then*

$$\mu_2(G) \leq \frac{8\Delta(G)}{n}.$$

Proof. From the Courant–Fisher theorem, we know that

$$\mu_2(G) = \min_{x \perp \mathbf{1}} \frac{\sum_{ij \in E(G)} (x_i - x_j)^2}{\sum_{i \in V(G)} x_j^2}.$$

It follows that

$$\mu_2(G) = \min \frac{\sum_{ij \in E(G)} ||v_i - v_j||^2}{\sum_{i \in V(G)} ||v_i||^2}, \tag{14.22}$$

where the minimum is taken over all vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $\sum_{i \in V(G)} v_i = \vec{0}$.

Spielman and Teng now use the following *kissing disk* theorem of Koebe, Andreev, and Thurston (see [58, 59] for more details).

Theorem 22 (Koebe–Andreev–Thurston). *If G is a planar graph with vertex set $\{1, \ldots, n\}$, then there exist a set of disks $\{D_1, \ldots, D_n\}$ in the plane such that (i, j) is an edge of G if and only if D_i touches D_j.*

A cap is the intersection of a half-space with a sphere and its boundary is a circle. Using a stereographic projection to map the kissing disk embedding of the graph G to a kissing cap embedding of G, Spielman and Teng show that we can represent the planar graph G by kissing caps on the unit sphere such that the centroid of the centers of the caps is the center of the sphere.

Let v_i be the center of the cap corresponding to vertex i. One can assume that $\sum_{i \in V(G)} v_i = \vec{0}$. Denote by r_i the radius of the cap corresponding to vertex i. If cap i touches cap j, then $||v_i - v_j|| < r_i + r_j$ by the triangle inequality. This implies that

$$\begin{aligned}\sum_{ij \in E(G)} ||v_i - v_j||^2 < \sum_{ij \in E(G)} (r_i + r_j)^2 \leq \sum_{ij \in E(G)} 2(r_i^2 + r_j^2) \\ \leq 2\Delta(G) \sum_{i \in V(G)} r_i^2.\end{aligned}$$

Since the caps do not overlap, it follows that the area of the unit sphere is larger than the sum of the areas of the caps which implies that

$$4\pi \geq \sum_{i \in V(G)} \pi r_i^2.$$

Using inequality (14.22), the desired result follows. □

The following theorem of Mihail [53] (see [54] for related results and see [18–20] for similar results for the normalized Laplacian) shows that one can use eigenvectors corresponding to μ_2 (or approximations of such eigenvectors) to find subsets S with small $\Phi(S)$.

Theorem 23. *Let G be a connected graph with maximum degree $\Delta(G)$, and let*

$$\Phi_G = \min_{S \subset V(G)} \Phi(S).$$

Then for any vector $v \in \mathbb{R}$ with $\mathbf{1} \perp v$, we have that

$$\Phi_G^2 \leq 2\Delta(G)\frac{v^t L(G) v}{v^t v}.$$

Also, there exists $\beta \in \mathbb{R}$ such that $\Phi(S) \leq \sqrt{2\Delta(G)\frac{v^t L(G) v}{v^t v}}$, where $S = V_-(\beta)$.

Combining the previous two results, one can deduce that the edge-expansion constant of a planar graph G on n vertices is at most $\frac{4\Delta(G)}{\sqrt{n}}$. Also, there is a polynomial time algorithm for finding a subset S such that $\Phi(S) \leq \frac{4\Delta(G)}{\sqrt{n}}$. By a classical result of Lipton and Tarjan [52], it follows that the previous result is tight up to a constant factor.

The genus of a graph G is the smallest g such that G can be embedded in a surface of genus g without any edge crossings. Planar graphs are graphs of genus 0.

Kelner [47] extended the previous results of Spielman and Teng to graphs of genus g.

Theorem 24. *Let G be a graph with n vertices, genus g, and bounded degree. Then*

$$\mu_2(G) \leq O\left(\frac{g}{n}\right) \tag{14.23}$$

where the constant in the O-notation depends on $\Delta(G)$.

Combining this result with Mihail's theorem yields a polynomial time algorithm for finding a subset S of a graph G of order n, genus g, and bounded maximum degree such that $\Phi(S) \leq O\left(\sqrt{gn}\right)$. Using a method described in the appendix of [59], one can use this algorithm to find a bisection of size $O(\sqrt{gn})$.

These results are tight up to a constant factor as shown by the examples found by Gilbert et al. [35]. The authors described a class of bounded degree graphs with no bisection of size less than $O(\sqrt{gn})$.

Clustering is the partitioning of data into groups of similar items. A clustering algorithm performs well if items that are similar are assigned to the same cluster and items that are not similar are assigned in different clusters. This situation can be modeled by a weighted graph in which the weight of an edge w_{ij} measures the similarity between the vertices i and j.

Kannan et al. [46] suggested the following measure of the quality of a clustering of a graph. Given a connected weighted graph $G = (V, E)$, a partition $V = V_1 \cup \cdots \cup V_k$ is called an (a, ϵ)-clustering if

- $\Phi_{G[V_i]} \geq a$ (the subgraph induced by each cluster has weighted edge expansion at least a).
- $\sum_{1 \leq i < j \leq k} |E(V_i, V_j)| \leq \epsilon |E(G)|$ (the weight of the intercluster edges is at most a times the weight of the edges in G).

The following optimization problem is studied in [46]: given a, find an (a, ϵ)-clustering that minimizes ϵ. In [46], the authors use a recursive algorithm based on the spectral partitioning method described above to find an approximate algorithm for the previous problem. They show that if G has an (a, ϵ)-clustering, then using the spectral partitioning method, one can find an $\left(\frac{a^2}{72 \log^2(n/\epsilon)}, 20\sqrt{\epsilon} \log(n/\epsilon)\right)$-clustering.

The idea of using eigenvectors for ranking goes back to Kendall [48] and Wei [66]. Brin and Page [11] introduced the notion of PageRank in their seminal paper on Web search. The Web pages are classified according to their importance scores given by PageRank which are computed from the graph structure of the Web. The PageRank importance of a Web page is determined by the PageRank importance of the Web pages linking to it.

The Web is regarded as a graph with nodes being Web pages and edges being hyperlinks. The basic idea of PageRank is that links from important vertices should weigh more than links from less important vertices.

Consider a connected graph G with adjacency matrix A and let D denote the diagonal matrix containing the degrees of the vertices of G. Define $W = D^{-1}A$. Thus,

$$W(i, j) = \begin{cases} \frac{1}{d_i}, \text{if } i \sim j \\ 0, \text{otherwise.} \end{cases}$$

The matrix W can be regarded as the transition probability matrix of a random walk on the vertices of G. The stationary distribution of this random walk is the row vector $\pi = \left(\frac{d_1}{\mathrm{vol(G)}}, \ldots, \frac{d_n}{\mathrm{vol(G)}}\right)$.

The PageRank vector pr(a, s) of a graph G is the unique solution of the equation

$$\mathrm{pr(a, s) = as + (1 - a)pr(a, s)W},$$

where $a \in (0, 1]$ is a jumping constant and s is a starting vector. The PageRank vector associated with search ranking has $s = \frac{1}{n}\mathbf{1}$. PageRank vectors whose starting vectors are concentrated on a small number of vertices are called personalized PageRank vectors and were introduced by Haveliwala [41]. The PageRank vector can be used to design graph partitioning algorithms (see Andersen et al. [7] and Chung [20]).

An important problem in communication networks is the following: given a connected graph G and a set of pairs of vertices (s_i, t_i), $1 \leq i \leq r$, find r edge-disjoint paths $Q_1, \ldots, Q_r$, where Q_i connects s_i to t_i. In [4], Alon and Capalbo used the connections between the edge distribution of a graph and its eigenvalues to prove the following result.

Theorem 25. *Let G be a connected d-regular graph and let $\lambda = \max(|\lambda_2|, |\lambda_n|)$. Assume that $d > 8\lambda$ and let $c > 0$ be a constant and $r := c\frac{nd \log(d/4\lambda)}{\log n}$. Given r pairs of vertices (s_i, t_i) such that no vertex of G appears more than $\frac{d}{3}$ times as s_i or t_i, there exists a polynomial time algorithm that finds r edge-disjoint paths Q_i such that Q_i joins s_i to t_i.*

The questions of finding paths of logarithmic length between each pair remains open.

14.6 Epidemic Spreading in Networks

It is well known that graphs can be used as abstract models of various networks that appear in computer science, biology, and sociology among others. The problem of virus propagation has been studied in these areas and various models have been proposed.

The susceptible–infective–susceptible (SIS) model assumes that each node of a network (graph) can be in one of two states: healthy but susceptible (S) to infection, or infected (I). An infected node can spread infection along the network to susceptible nodes. An infected node can be cured locally and it becomes susceptible again. A directed edge from node i to node j means that i can infect j. A rate of infection β is associated with each edge and a virus curing rate, δ, is associated with each infected node.

The epidemic threshold of a graph G is the value τ such that if $\frac{\beta}{\delta} < \tau$, then the viral outbreak dies out over time and if $\frac{\beta}{\delta} < \tau$, then the infection survives.

Recently, Wang et al. [65] found connections between the eigenvalues of a graph and the epidemic threshold in the SIS model.

Consider a connected network (graph) $G = (V, E)$. The model considered in [65] assumes discrete time. During each time interval, an infected node i tries to infect its neighbours with probability β. At the same time, the node i can be cured with probability δ.

Recall that $\lambda_1(G)$ denotes the largest eigenvalue of the adjacency matrix of G. The main result of [65] is the following theorem whose proof we sketch below.

Theorem 26. *The epidemic threshold of a graph G equals* $\frac{1}{\lambda_1(G)}$.

Proof. Let $p_{i,t}$ denote the probability that i is infected at time t and $q_{i,t}$ denote the probability that i will not be infected by its neighbours at time t.

A node i is healthy at time t if

- i was healthy at time $t-1$ and did not receive infections from its neighbours at t.
- i was infected before t, cured at t, and did not receive infections from its neighbours at t.
- i was infected before t, received and ignored infections from its neighbours at time t, and was cured at time t.

Assume that the probability that a curing event at node i takes place after infection from neighbours is 50%. This means that

$$1 - p_{i,t} = (1 - p_{i,t-1})q_{i,t} + \delta p_{i,t-1}q_{i,t} + \frac{1}{2}\delta p_{i,t-1}(1 - q_{i,t}).$$

Let P_t denote the column vector $(p_{1,t}, \ldots, p_{n,t})$. From the previous equation, one can obtain that

$$P_t = ((1 - \delta)I_n + \beta A(G))\, P_{t-1}. \tag{14.24}$$

For the infection to die off, the vector P_t should tend to zero as t gets large. This will happen when for each i, the i-th eigenvalue of $((1 - \delta)I_n + \beta A(G))^t$ tends to 0 as t gets large. It follows that $1 - \delta + \beta\lambda_1(G) < 1$ which means that $\tau = \frac{1}{\lambda_1(G)}$.□

Using the previous argument, it is shown in [65] that when $\frac{\beta}{\delta}$ is below the epidemic threshold, the number of infected nodes decays exponentially over time.

The result from [65] motivated further research. In [64], the authors studied the problem of minimizing the spectral radius of a connected graph of order n and diameter D. In [64], the authors solved this problem when $D \in \{1, 2, n-3, n-2, n-1\}$, but many questions remain open (see [63, 64] for more details).

Another model for epidemic spreading in networks is the susceptible-infective-removed (SIR) model. Consider again a graph G with n vertices. Each vertex can be in one of three possible states, susceptible (S), infective (I), or removed (R). Again, we assume discrete time. We assume that the initial set of infective vertices at time 0 is nonempty, and the rest of the vertices are susceptible at time 0.

Let $X_i(t)$ denote the indicator that the vertex i is infected at time t and $Y_i(t)$ the indicator that i is removed at time t. Each vertex that is infected tries to infect each of its neighbours; each infection attempt is successful with probability β independent of other infection attempts. Each infected node is removed at the end of the time slot. It follows that the probability that a vertex i becomes infected at the end of time t is $1 - \prod_{j \sim i}(1 - \beta X_i(t))$. The evolution stops when there are no more infective vertices in the graph. One would like to know how many vertices are removed at this time.

This model was studied by Draief et al. [30] who proved the following theorem.

Theorem 27. *Assume that $\beta\lambda_1(G) < 1$. Then, the total number of vertices removed $|Y(\infty)|$ satisfies the inequality*

$$E[|Y(\infty)|] \leq \frac{\sqrt{n|X(0)|}}{1 - \beta\lambda_1(G)}, \tag{14.25}$$

where $X(0)$ is the number of initial infective vertices.

14.7 Eigenvalues and Other Graph Invariants

Finding the chromatic number of a graph is also a graph partitioning problem. Among the first results connecting the eigenvalues of a graph to its chromatic and independence number were the following theorems due to Wilf [69], Delsarte [27], and Hoffman [42]. These are classical results with many applications in discrete mathematics and also more recent applications in quantum computing (see [26,36]). For extensions of these results and other applications, see Haemers [40], Nikiforov [57], or Godsil and Newman (see [37] and [36]).

Theorem 28 ([69]). *If G is a connected graph with chromatic number $\chi(G)$, then*

$$\chi(G) \leq 1 + \lambda_1(G).$$

Theorem 29 ([27,42]). *If G is a connected graph of order n, then*

$$\chi(G) \geq 1 + \frac{\lambda_1}{-\lambda_n}.$$

If G is d-regular and $\alpha(G)$ denotes the independence number of G, then

$$\alpha(G) \leq \frac{-n\lambda_n}{d - \lambda_n}.$$

Godsil and Newman [37] have obtained similar results for graphs containing loops and used these results to find bounds for the independence number of the Erdös–Rényi graphs.

We note here the results of Alon et al. [5] which provide inequalities in the opposite direction.

Theorem 30. *Let G be a connected d-regular graph and let $\lambda = \max(|\lambda_2|, |\lambda_n|)$. Then for any subset S of vertices of G, the subgraph $G[S]$ induced by S contains an independent set of size*

$$\alpha(G[S]) \geq \frac{n}{2(d - \lambda)} \ln\left(\frac{|S|(d - \lambda)}{n(\lambda + 1)} + 1\right).$$

Also, the chromatic number of G satisfies the inequality

$$\chi(G) \le \frac{6(d-\lambda)}{\ln\left(\frac{d-\lambda}{\lambda+1}+1\right)}.$$

For d-regular graphs with $\lambda = O(\sqrt{d})$, the previous result implies $\chi(G) = O(d/\ln d)$. As described in [5, 51], there are many graphs with this property.

As mentioned earlier, the MAX-CUT problem is an example of a graph partitioning problem. Alon [2] used the following result to find tight bounds for the maximum cut of several families of graphs such as triangle-free graphs. Given a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G.

Theorem 31. *If G is a d-regular graph of order n, then*

$$f(G) \le \frac{n(d-\lambda_n)}{4}.$$

Alon [2] showed that if G is a triangle-free graph with e edges, then it contains a bipartite subgraph with at least $\frac{e}{2} + ce^{\frac{4}{5}}$ edges and this result is tight up to the constant c. This result was extended by Alon et al. in [3] who showed that graphs with girth at least $r \ge 4$ contain a bipartite subgraph with at least $\frac{e}{2} + c'e^{\frac{r}{r+1}}$ edges and this result is tight up to a constant factor for $r = 4, 5$.

Butler and Chung [17] extended previous results of Krivelevich and Sudakov [51] and found an eigenvalue condition that implies the existence of a Hamiltonian cycle in a graph.

Theorem 32. *Let G be a connected graph with average degree d. If there exists a positive constant C such that*

$$|d-\mu_i| \ge C\frac{(\log\log n)^2}{\log n(\log\log\log n)}d$$

for $i > 1$ and n is sufficiently large, then G contains a Hamiltonian cycle.

Brouwer and Haemers [13] conjectured that any strongly regular graph (except the Petersen graph) is Hamiltonian. They have verified this conjecture for graphs with at most 99 vertices.

14.8 Conclusions

As our knowledge and technology advance, the complexity of the social, communication, and biological networks surrounding us is increasing rapidly. Many important combinatorial parameters of large networks are often hard to calculate or approximate. Eigenvalues provide an effective and efficient tool for studying properties of large graphs which arise in practice. In this chapter, we presented some applications of eigenvalues of graphs. Spectral graph theory is a very dynamic area that will continue to grow. We believe that more applications and tighter connections between graph eigenvalues and other graph invariants will be found in the future.

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References

1. Alon N (1986) Eigenvalues and expanders. Combinatorica 6:83–96
2. Alon N (1996) Bipartite subgraphs. Combinatorica 16:301–311
3. Alon N, Bollobás B, Krivelevich M, Sudakov B (2003) Maximum cuts and judicious partitions of graphs without short cycles. J Combin Theory B 88:329–346
4. Alon N, Capalbo M (2007) Finding disjoint paths in expanders deterministically and online. In: FOCS, pp 518–524
5. Alon N, Krivelevich M, Sudakov B (1999) List coloring of random and pseudo-random graphs. Combinatorica 19:453–472
6. Alon N, Milman V (1985) λ_1, Isoperimetric inequalities for graphs and superconcentrators. J Combin Theory B 38:73–88
7. Andersen R, Chung F, Lang K (2006) Local graph partitioning using pagerank vectors. In: Proceedings of the 47th annual IEEE symposium on foundation of computer science, pp 475–486
8. Bezrukov S, Elsässer R, Monien B, Preis R, Tillich J-P (2004) New spectral lower bounds on the bisection width of graphs. Theor Comput Sci 320:155–174
9. Bollobás B (1998) Modern graph theory. Graduate texts in mathematics, vol 184. Springer, New York
10. Bollobás B, Nikiforov V (2004) Graphs and Hermitian matrices: eigenvalue interlacing. Discrete Math 289:119–127
11. Brin S, Page L (1998) The anatomy of a large-scale hypertextual Web search engine. Comput Netw ISDN Syst 30:107–117
12. Brouwer A, Haemers W (2005) Eigenvalues and perfect matchings. Lin Algebra Appl 395:155–162
13. Brouwer A, Haemers W (2010) Spectra of graphs. Monograph in preparation +200pp, available at `http://homepages.cwi.nl/~aeb/math/ipm.pdf`
14. Brouwer A, Koolen J (2009) The vertex connectivity of a distance regular graph. Eur J Combinator 30:668–673
15. Brouwer A, Mesner DM (1985) The connectivity of strongly regular graphs. Eur J Combinator 6:215–216
16. Bui TN, Chaudhuri S, Leighton FT, Sipser M (1987) Graph bisection algorithms with good average case behaviour. Combinatorica 2:171–191
17. Butler S, Chung FRK (2010) Small spectral gap in the combinatorial Laplacian implies Hamiltonian, Annals of Combinatorics 13(4):403–412
18. Chung FRK (1997) Spectral graph theory. American Mathematical Society, Providence, RI
19. Chung FRK (2004) Discrete isoperimetric inequalities. Surveys in differential geometry IX. International Press, Somerville, MA, pp 53–82
20. Chung FRK (2007) Four proofs of the Cheeger inequality and graph paritition algorithms. In: Proceedings of ICCM, vol II

21. Chung FRK (2007) Random walks and local cuts in graphs. Lin Algebra Appl 423:22–32
22. Cioabă SM (2010) Eigenvalues and edge-connectivity of regular graphs. Lin Algebra Appl 432:458–470
23. Collatz L, Sinogowitz U (1957) Spektren endlicher Grafen. Abh Math Sem Univ Hamburg 21:63–77
24. Cvetković D, Doob M, Sachs H (1980) Spectra of graphs – theory and application. Academic, New York, 3rd edn, Johann Ambrosius Barth (1995)
25. Cvetković D, Rowlinson P, Simic S (1997) Eigenspaces of graphs. Cambridge University Press, Cambridge
26. de Klerk E, Pasechnik DV (2007) A note on the stability of an orthogonality graph. Eur J Combinator 28:1971–1979
27. Delsarte P (1973) An algebraic approach to the association schemes of coding theory. Philips Res Rep Suppl 10:1–97
28. Diáz J, Do N, Serna MJ, Wormald NC (2003) Bounds on the max and min bisection of random cubic and random 4-regular graphs. Theor Comput Sci 307:531–548
29. Donath WE, Hoffman AJ (1973) Lower bounds for the partitioning of the graphs. IBM J Res Dev 17:420–425
30. Draief M, Ganesh A, Massoulié L (2008) Threshold for virus spread on networks. Ann Appl Probab 18(2):359–378
31. Elsässer R, Lücking T, Monien B (2003) On spectral bounds for the k-partitioning of graphs. Theor Comput Syst 36:461–478
32. Fiedler M (1973) Algebraic connectivity of graphs. Czech Math J 23:298–305
33. Fiedler M (1975) A property of eigenvectors of non-negative symmetric matrices and its applications to graph theory. Czech Math J 85:619–633
34. Garey MR, Johnson DS, Stockmeyer L (1976) Some simplified NP-complete graph problems. Theor Comput Sci 1:237–267
35. Gilbert J, Hutchinson J, Tarjan R (1984) A separation theorem for graphs of bounded genus. J Algorithm 5:391–407
36. Godsil CD, Newman MW (2008) Coloring an orthogonality graph. SIAM J Discrete Math 22:683–692
37. Godsil CD, Newman MW (2008) Eigenvalue bounds for independent sets. J Combin Theory B 98:721–734
38. Godsil CD, Royle G (2001) Algebraic graph theory. Springer, Berlin
39. Guattery S, Miller GL (1998) On the quality of spectral separators. SIAM J Matrix Anal Appl 19:701–719
40. Haemers W (1995) Interlacing eigenvalues and graphs. Lin Algebra Appl 226/228:593–616
41. Haveliwala TH (2003) Topic-sensitive PageRank: a context-sensitive ranking algorithm for Web search. IEEE Trans Knowl Data Eng 15:784–796
42. Hoffman AJ (1970) On eigenvalues and colorings of graphs. In: Graph theory and its applications. (Proceedings of an advanced seminar, mathematical research center, University of Wisconsin, Madison, WI, 1969). Academic, New York, pp 79–91
43. Hoffman AJ, Wielandt H (1953) The variation of the spectrum of a normal matrix. Duke Math J 29:37–39
44. Hoory S, Linial N, Wigderson A (2006) Expander graphs and their applications. Bull AMS 43:439–561
45. Horn RA, Johnson CR (1985) Matrix analysis. Cambridge University Press, Cambridge
46. Kannan R, Vempala S, Vetta A (2004) On clusterigs: good, bad and spectral. J ACM 51:497–515
47. Kelner JA (2004) Spectral partitioning, eigenvalue bounds, and circle packings for graphs of bounded genus. In: Proceedings of the symposium on the theory of Computing (STOC)
48. Kendall MG (1955) Further contributions to the theory of paired comparisons. Biometrics 11:43–62
49. Kirchhoff G (1847) Über die Auflösung der Gleichungen auf welche man bei der Untersuchung der Linearen Vertheilung galvanischer Ströme geführt wird. Ann Phys Chem 72:497–508

50. Kirkland SJ, Molitierno JJ, Neumann M, Shader BL (2002) On graphs with equal algebraic and vertex connectivity. Lin Algebra Appl 341:45–56
51. Krivelevich M, Sudakov B (2006) Pseudo-random graphs. In: More sets, graphs and numbers. Bolyai Society Mathematical Studies, vol 15, pp 199–262
52. Lipton RJ, Tarjan RE (1979) A separator theorem for planar graphs. SIAM J Appl Math 36:177–189
53. Mihail M (1989) Conductance and convergence of Markov chains – a combinatorial treatment of expandes. In: Proceedings of the 30th annual symposium on foundations of computer science. IEEE Computer Society, Washington, DC, pp 526–531
54. Mohar B (1989) Isoperimetric numbers of graphs. J Combin Theory B 47:274–291
55. Mohar B (1997) Some applications of Laplace eigenvalues of graphs. In: Hahn G, Sabidussi G (eds) Graph symmetry. Kluwer, Dordrecht, pp 225–275
56. Monien B, Preis R (2001) Upper bounds on the bisection width of 3 and 4-regular graphs. In: Pultr A, Sgall J, Kolman P (eds) Mathematical foundations of computer science, vol 2136. Lecture Notes in Computer Science. Springer, pp 524–536
57. Nikiforov V (2001) Some inequalities for the largest eigenvalue of a graph. Combinator Probab Comput 11:179–189
58. Spielman D, Teng SH (1996) Spectral partitioning works: planar graphs and finite element meshes. In: Proceedings of the 37th annual symposium on foundations of computer science, pp 96–106
59. Spielman D, Teng SH (2007) Spectral partitioning works. Lin Algebra Appl 421:284–305
60. Tanner RM (1984) Explicit construction of concentrators from generalized N-gons. SIAM J Algebra Discr 5:287–294
61. van Dam E, Haemers W (2003) Which graphs are determined by their spectrum? Lin Algebra Appl 373:241–272
62. van Dam E, Haemers W (2009) Developments on spectral characterization of graphs. Discrete Math 309:576–586
63. van Dam E, Jamakovic A, Kooij RE, Van Mieghem P (2006) Robustness of networks against viruses: the role of spectral radius. In: Proceedings of the 13th annual symposium of the IEEE/CVT Benelux, Liége, Belgium, pp 35–38
64. van Dam E, Kooij RE (2007) The minimal spectral radius of graphs with give diameter. Lin Algebra Appl 423:408–419
65. Wang Y, Chakrabarti D, Wang C, Faloutsos C (2003) Epidemic spreading in real networks: an eigenvalue viewpoint. In: Proceedings of the symposium on reliable distributed computing, Florence, Italy, pp 25–34
66. Wei TH (1952) The algebraic foundations of ranking theory. Cambridge University Press, London
67. West DB (2001) Introduction to graph theory, 2nd edn. Prentice Hall, Englewood Cliffs, NJ
68. Whitney H (1932) Congruent graphs and the connectivity of graphs. Am J Math 54:150–168
69. Wilf H (1967) The eigenvalues of a graph and its chromatic number. J Lond Math Soc 42:330–332