Graphs and Combinatorics (2012) 28:109–122
DOI 10.1007/s00373-011-1030-5

ORIGINAL PAPER

Covering Complete Hypergraphs with Cuts of Minimum Total Size

Sebastian M. Cioabă · André Kündgen

Received: 25 August 2009 / Revised: 7 January 2011 / Published online: 15 February 2011

Abstract A cut $[X, V - X]$ in a hypergraph with vertex-set V is the set of all edges that meet both X and $V - X$. Let $s_r(n)$ denote the minimum total size of any cover of the edges of the complete r-uniform hypergraph on n vertices K_n^r by cuts. We show that there is a number n_r such that for every $n > n_r$, $s_r(n)$ is uniquely achieved by a cover with $\lfloor \frac{n-1}{r-1} \rfloor$ cuts $[X_i, V - X_i]$ such that the X_i are pairwise disjoint sets of size at most $r - 1$. We show that $c_1 r 2^r < n_r < c_2 r^5 2^r$ for some positive absolute constants c_1 and c_2. Using known results for $s_2(n)$ we also determine $s_3(n)$ exactly for all n.

Keywords Hypergraph · Cut · Cover · 2-Colorable

1 Introduction

An r-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ (short r-graph) consists of a finite set of vertices V and a collection $\mathcal{E}$ of r-element subsets of V called the edges. Thus, graphs are 2-uniform hypergraphs. Let $e(\mathcal{H}) = |\mathcal{E}|$ denote the number of edges of $\mathcal{H}$. A *cut* $[X, V - X]$ in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is the set of all edges that intersect both X and $V - X$. A *cut cover* $\mathcal{C}$ of the hypergraph $\mathcal{H}$ is a collection of cuts such that each edge of $\mathcal{H}$ is contained in at least one of the cuts.

S. M. Cioabă (✉)
Department of Mathematical Sciences, University of Delaware,
Newark, DE 19716, USA
e-mail: cioaba@math.udel.edu

A. Kündgen
Department of Mathematics, California State University San Marcos,
San Marcos, CA 92096, USA
e-mail: akundgen@csusm.edu

Covering the edge set of a (complete) hypergraph by cuts is an important problem in extremal graph theory with connections to separating systems and hash functions (see [1–3,10,17,18] for more details). A useful observation is that a collection of m cuts $\{[X_i, V - X_i] : 1 \leq i \leq m\}$ of $\mathcal{H}$ corresponds to an assignment of binary m-tuples, b_v, to the vertices $v \in V$ where the i-th entry of b_v is 1 if and only if $v \in X_i$. The cuts form a cover if every edge contains two vertices v, w with $b_v \neq b_w$. The utility of this notion is illustrated by the following straight forward extension (see [16] for further results) of the similar result first published in the 1970s for graphs (see [6,7,15]). Recall that a hypergraph $\mathcal{H}$ is said to be *k-colorable* if there is a coloring of its vertex set by k colors such that no edge is monochromatic, and that $\chi(\mathcal{H})$ denotes the smallest k such that $\mathcal{H}$ is k-colorable.

Proposition 1 *The minimum number of cuts in a cover of* $\mathcal{H}$ *is* $\lceil \log_2 \chi(\mathcal{H}) \rceil$.

Proof The vectors b_v obtained from a cover of $\mathcal{H}$ with m cuts yield a 2^m-coloring, so that $m \geq \log_2 \chi(\mathcal{H})$. Conversely, if $m = \lceil \log_2 \chi(\mathcal{H}) \rceil$, then we can color the vertices v of $\mathcal{H}$ with binary m-tuples b_v, which can then be translated back to a cover with m cuts. □

In this paper we study a different measure of efficiency for a cut cover $\mathcal{C}$, namely minimizing the average number of times an edge is covered. Equivalently we want to minimize the *total size* $\|\mathcal{C}\|$ of $\mathcal{C}$, that is the sum of the *sizes* (number of edges) of the cuts in $\mathcal{C}$. Let $\mathrm{cs}(\mathcal{H})$ denote the minimum possible total size of a cut cover of $\mathcal{H}$, and call a cut cover achieving this value *optimal*. The problem of optimally covering the edges of 2-graphs with cuts has been studied in [4,12]. The important special case of the complete graph K_n has been independently solved in at least four different contexts (see [8,9,11,13]): For $n \neq 4, 8$, an optimal cover consists of $n - 1$ star-cuts $[\{x\}, V \setminus \{x\}]$ and has total size $\mathrm{cs}(K_n) = (n-1)^2$. For $n = 4, 8$ we have $\mathrm{cs}(K_4) = 8$, attained uniquely by using two balanced cuts, and $\mathrm{cs}(K_8) = 48$ attained only by using three balanced cuts. A cut $[X, V - X]$ is called *balanced* if $|X| = |V - X|$.

Determining $\mathrm{cs}(\mathcal{H})$ is a hard problem. Trivially $\mathrm{cs}(\mathcal{H}) \geq e(\mathcal{H})$, and generalizing the elementary result for graphs found in [4], we obtain the cases of equality:

Proposition 2 *A hypergraph* $\mathcal{H}$ *is* 2*-colorable if and only if* $\mathrm{cs}(\mathcal{H}) = e(\mathcal{H})$.

Proof If $\mathcal{H}$ is 2-colorable, then one cut suffices to cover $\mathcal{H}$. Conversely, suppose that $\mathrm{cs}(\mathcal{H}) = e(\mathcal{H})$, and let b_v denote the binary vectors associated with an optimal cover. Assign color 1 to v when b_v has an odd number of ones, and color 2 otherwise. A monochromatic edge would have been covered at least twice, contradicting $\mathrm{cs}(\mathcal{H}) = e(\mathcal{H})$. □

Unfortunately this result shows that even deciding when equality holds in the trivial lower bound is in general very difficult since deciding if a hypergraph is 2-colorable is NP-complete, even for 3-uniform hypergraphs (see [5,14]).

The following upper bound on the average number of times an edge is cut in an optimal cover of an n-vertex r-graph $\mathcal{H}$

$$\frac{\mathrm{cs}(\mathcal{H})}{e(\mathcal{H})} \leq \frac{\mathrm{cs}(K_n^r)}{\binom{n}{r}} \tag{1}$$

follows easily by assigning each vertex of $\mathcal{H}$ to a random vertex of the complete r-uniform hypergraph on n vertices K_n^r, and then using an optimal cover of K_n^r for $\mathcal{H}$. Thus, K_n^r is the hardest r-graph to cut and we let $s_r(n) = \text{cs}(K_n^r)$. In this paper, we are interested in determining $s_r(n)$ when $n \geq r \geq 3$.

In Sect. 2 we prove a crucial structural result and use it to determine $s_r(n)$ when $n \leq 4(r-1)$. In Sect. 4, we use known results for $s_2(n)$ to determine $s_3(n)$ exactly for all $n \geq 3$ and to describe all optimal covers. In fact, when $n \geq 17$ we show that the unique optimal cover consists of $\lfloor \frac{n-2}{2} \rfloor$ cuts $[X_i, V - X_i]$ with $|X_i| = 2$ and (when n is odd) one cut with $|X_i| = 1$. In Sect. 6 we show that for all $r > 3$ there is a number n_r such that if $n > n_r$ and $n = m(r-1) + j$ (with $1 \leq j \leq r-1$,) then $s_r(n)$ is uniquely achieved by a cover with $m-1$ cuts $[X_i, V - X_i]$ with $|X_i| = r-1$ and one with $|X_i| = j$, and that $c_1 r 2^r < n_r < c_2 r^4 2^r$. This yields that $s_r(n) = r\binom{n}{r} - O(n^{r-1})$, since almost every edge is covered exactly r times. We conclude the paper with some open problems.

2 Cutting in Groups

In this paper a cut $[X_i, V - X_i]_r$ consists of all r-tuples with at least one vertex each in X_i and $V - X_i$, and its size is its number of edges. We say that a collection of such cuts *covers* the complete r-uniform hypergraph on n vertices, K_n^r, if every edge of K_n^r is in at least one cut, and the total size of this cover is the sum of the sizes of the cuts. Our aim is to study the minimum possible total size of such a cover, $s_r(n)$. Every cut cover of total size $s_r(n)$ will be called an *optimal cover*.

Let $\{[X_i, V - X_i]_r : 1 \leq i \leq m\}$ be a cover of K_n^r with m cuts. If $|X_i| = x_i$, then the *type* of this cut-cover of K_n^r is $(x_1, x_2, \ldots, x_m)_n^r$. If k of the x_i's have value a, then we can abbreviate this to $a^{(k)}$ in this expression.

A *grouping* is a maximal collection of vertices $v_1, \ldots, v_k$ such that $b_{v_1} = \cdots = b_{v_k}$, i.e. the vertices are on the same side of each cut in the cover. Clearly no grouping can have more than $r-1$ vertices, since otherwise some edge would not be covered, and the groupings form a partition of $V(K_n^r)$. If $n = c(r-1) + j$ with $1 \leq j \leq r-1$ then a cover of K_n^r is called *efficient* if one of its groupings has size j and all other groupings have size $r-1$.

Lemma 1 *If $n \geq 2r-3$, then every optimal cut-cover of K_n^r is efficient.*

Proof Consider an optimal cut-cover $\mathcal{C}$. It suffices to show that if u, v are in different groupings of size $< r-1$, then one of u, v can be moved to the grouping of the other such that the total size decreases.

Observe that as long as each grouping is of size at most $r-1$ we have a valid cut-cover, so that moving u or v accordingly is permitted. Furthermore, such a move only affects how many times an edge is covered if the edge contains at least one of u, v. We only need to consider cuts in which u, v are in different parts, since all other cuts remain unaffected by a move. An edge that contains both u and v is already covered in each such cut, so a move cannot increase the number of times this edge is covered.

It remains to consider the effect of a move on the edges that contain exactly one of u, v. Note that there is a simple one-to-one correspondence between the set of edges containing u and not containing v and the set of edges containing v and not containing u, given by $e \mapsto (e \cup \{v\}) \setminus \{u\}$. Let c_u be the total number of times the edges containing u but not v are cut, and similarly for c_v. If u is moved to the grouping of v, then the edges only containing v are not affected, and the edges only containing u are now covered c_v times as well, instead of c_u times as before. So if c is the total size of the old cover, then the new cover has total size at most $c - c_u + c_v$. Moving v to the grouping of u instead results in a cover of total size at most $c - c_v + c_u$, and one of these quantities is at most c.

To obtain equality we must have $c_u = c_v$ (so that it does not matter if we move u or v), and none of the edges containing u, v must be covered fewer times than before. To this end consider a cut with $u \in X_i$ and $v \in V - X_i$. Since $n \geq 2r - 3$ we may assume that $|X_i| \geq r - 1$ and thus moving v to X_i will cause an edge containing u, v not to be covered anymore in this cut that was covered before. □

3 Optimal Covers for $n \leq 4(r - 1)$

Proposition 2 shows that $s_r(n) = \binom{n}{r}$ for $n \leq 2r - 2$, and to avoid trivialities we will now assume that $n > 2(r - 1)$. By Lemma 1 it suffices to consider efficient covers. The following result is straight forward and its proof is omitted.

Lemma 2 *If $|X| = x$, then $|[X, V - X]_r| = \binom{n}{r} - \binom{x}{r} - \binom{n-x}{r}$. If $|Y| < |X| \leq n/2$, then $|[Y, V - Y]_r| < |[X, V - X]_r|$.*

A *simple cut* in a cut cover is a cut $[X, V - X]_r$ in which X is a grouping and thus $|X| \leq r - 1$. A cut-cover in which every cut is simple is called a *simple cover*, and we let $f_r(n)$ denote the minimum total size of a simple cover. The same proof as in Lemma 1 shows that $f_r(n)$ is achieved by an efficient cover, since moving u to the grouping of v does not change the fact that a cover is simple. Thus, for $n = c(r-1)+j$ with $1 \leq j \leq r - 1$, the simple cover of minimum size is of type $((r - 1)^{(c-1)}, j)^r_n$, and by Lemma 2

$$f_r(n) = c\binom{n}{r} - (c-1)\binom{n-r+1}{r} - \binom{n-j}{r} = r\binom{n}{r} - O(n^{r-1}). \qquad (2)$$

The second equality follows since in this cover every edge is covered r times, except those containing 2 vertices from the same grouping and there are at most $n \cdot (r-2) \cdot \binom{n-2}{r-2}$ such exceptional edges.

Proposition 3 *If $2(r - 1) < n \leq 3(r - 1)$, then $s_r(n) = f_r(n) = 2\binom{n}{r} - \binom{n-r+1}{r} - \binom{2r-2}{r}$, and the unique optimal cover is a simple cover.*

Proof Let $n = 2(r - 1) + j$ with $1 \leq j \leq r - 1$. We know that V breaks into 2 groupings X_1, X_2 of size $r - 1$, and one, X_3, of size j. The only possible cuts are simple cuts, so that $s_r(n) = f_r(n)$ and the optimal cover must be a simple cover. □

Simple covers are not optimal when we have 4 groupings:

Proposition 4 *If* $3(r-1) < n \le 4(r-1)$*, then* $s_r(n) = 2\binom{n}{r} - 2\binom{2r-2}{r} - 2\binom{n-2r+2}{r}$*, and equality is only achieved by a cover of type* $((2r-2)^{(2)})_n^r$.

Proof Let $n = 3(r-1) + j$ with $1 \le j \le r-1$. We know that V breaks into 3 groupings X_1, X_2, X_3 of size $r-1$, and one, X_4, of size j. Besides the simple cuts of the form $[X_i, V - X_i]_r$ for $1 \le i \le 4$, we also need to consider the large cuts of the form $[X_i \cup X_j, V - X_i \cup X_j]_r$. All these cuts have size $\binom{n}{r} - \binom{2r-2}{r} - \binom{n-2r+2}{r}$. If we use exactly one of these, then we also need two simple cuts, but this cut-cover has larger total size than just using 3 simple cuts to begin with. The only other reasonable option is to use 2 large cuts, resulting in a cover of total size

$$2\binom{n}{r} - 2\binom{2r-2}{r} - 2\binom{n-2r+2}{r},$$

A best-possible cover with 3 simple cuts has size exactly

$$3\binom{n}{r} - \binom{3r-3}{r} - 2\binom{n-r+1}{r},$$

and it suffices to compare these values. Subtracting the former from the latter, we get

$$\binom{3r-3+j}{r} - \binom{3r-3}{r} - 2\binom{2r-2+j}{r} + 2\binom{2r-2}{r} + 2\binom{r-1+j}{r}. \tag{3}$$

Since $\binom{m+k}{r} - \binom{m}{r} = \sum_{i=0}^{k-1} \binom{m+i}{r-1}$, this expression is at least $\sum_{i=0}^{j-1} \binom{3r-3+i}{r-1} - 2\binom{2r-2+i}{r-1}$. Now

$$\frac{\binom{3r-3+i}{r-1}}{\binom{2r-2+i}{r-1}} = \frac{(3r-3+i)(3r-3+i-1)\cdots(2r-1+i)}{(2r-2+i)(2r-2+i-1)\cdots(r+i)} \ge \left(\frac{3r-3+i}{2r-2+i}\right)^{r-1}$$
$$> \left(\frac{4}{3}\right)^{r-1}$$

for $0 \le i \le j-1 \le r-2$. For $r \ge 4$ we have that $(4/3)^{r-1} > 2$, so that (3) is a positive quantity, and an optimal cover must thus consist of two large cuts. For $r = 3$ we get that $i \le r-2 = 1$ and thus

$$\frac{(3r-3+i)(3r-3+i-1)\cdots(2r-1+i)}{(2r-2+i)(2r-2+i-1)\cdots(r+i)} = \frac{(6+i)(5+i)}{(4+i)(3+i)} \ge \frac{7\cdot 6}{5\cdot 4} > 2,$$

and the result follows. For $r = 2$ we have $r-1 = 1$, and the simple cut cover has size $3(1\cdot 3) = 9$ versus the large cover with size $2(2\cdot 2) = 8$. □

4 Determining $s_3(n)$

The propositions from the previous section allow us to easily compute $s_3(n)$ for small n. To be able to determine $s_3(n)$ for all values of n we begin with the observation, that if $|X| = x$, then

$$|[X, V - X]_3| = \binom{x}{2}(n - x) + \binom{n - x}{2}x = \frac{1}{2}x(n - x)(x - 1 + n - x - 1) \\ = \frac{n - 2}{2}x(n - x). \tag{4}$$

For $n = 2m + j$ (with $0 \leq j \leq 1$) a simple cover of minimum total size consists of $m-1$ $(2, n-2)$-cuts and (when $j = 1$) a $(1, n-1)$-cut. Thus, its type is $(1^{(j)}, 2^{(m-1)})^3_n$ and

$$f_3(n) = (m - 1)\frac{n - 2}{2}2(n - 2) + j\frac{n - 2}{2}1(n - 1) \\ = \begin{cases} \frac{1}{2}(n - 2)^3 & \text{for even } n \\ \frac{n-2}{2}(n^2 - 4n + 5) & \text{for odd } n \end{cases}. \tag{5}$$

The graph case $r = 2$ will be crucial for solving the case $r = 3$, so we recall it here.

Proposition 5 **(see [8,9,11,13])** *$s_2(m) = (m-1)^2 - \varepsilon = f_2(m) - \varepsilon$, where $\varepsilon = 1$ for $m = 4, 8$ and $\varepsilon = 0$ otherwise. The only covers of size at most $f_2(m)$ are the simple covers of type $(1^{(m-1)})$, as well as the optimal cut-covers of type $(2, 2)^2_4$, $(1, 2, 2)^2_5$, $(2, 2, 3)^2_6$, $(3, 3, 3)^2_7$ and $(4, 4, 4)^2_8$.*

So the optimal cut-cover is unique, except when $5 \leq n \leq 7$.

This result will enable us to prove

Theorem 1 *$s_3(n) = f_3(n) - (n - 2)\varepsilon_n$, where $\varepsilon_n = 0$, except $\varepsilon_8 = \varepsilon_{16} = 2$, and $\varepsilon_7 = \varepsilon_9 = \varepsilon_{15} = 1$. The optimal cut-cover is unique except when $10 \leq n \leq 14$.*

1. *If n is even, then the only efficient covers of size at most $f_3(n)$ are the simple covers as well as the optimal cut-covers of type* $(4, 4)^3_8$, $(2, 4, 4)^3_{10}$, $(4, 4, 6)^3_{12}$, $(6, 6, 6)^3_{14}$, *and* $(8, 8, 8)^3_{16}$.
2. *If n is odd, then an optimal cover is a simple cover (for $n \neq 7, 9, 15$) or one of* $(3, 3)^3_7$, $(1, 4, 4)^3_9$, $(3, 4, 5)^3_{11}$, $(5, 5, 6)^3_{13}$, *and* $(7, 7, 7)^3_{15}$.

Proof Since we are only interested in the case $r = 3$ and to improve readability, we will for each type leave out the superscript 3, and if we talk about a generic n we will also leave out the subscript n. We start with the case when $n = 2m$ is even. Any cut $C = [X, V - X]_3$ in an efficient cover of K^3_n will have $|X| = x$ being even, since it consists of groupings X_i of size 2. Let $X' = \{i : X_i \subset X\} \subset \{1, 2 \ldots m\} = V'$ and $C' = [X', V' - X']_2$. Observe that

$$|C| = \frac{n - 2}{2}x(n - x) = 2(n - 2)\frac{x}{2}\left(\frac{n}{2} - \frac{x}{2}\right) = 2(n - 2)|[X', V' - X']_2| \\ = 2(n - 2)|C'|.$$

It is also easy to see that a collection of cuts C_i is an efficient cover $\mathcal{C}$ of K_n^3 if and only if the corresponding cuts C_i' cover K_m^2, and that $\|\mathcal{C}\| = \sum |C_i| = 2(n-2)\sum |C_i'| = 2(n-2)\|\mathcal{C}'\|$.

Since $2(n-2)f_2(m) = 2(n-2)(m-1)^2 = \frac{1}{2}(n-2)^3 = f_3(n)$, it follows that every efficient cover $\mathcal{C}$ of K_n^3 of size at most $f_3(n)$ corresponds to a cover $\mathcal{C}'$ of K_m^2 of size at most $f_2(m)$ and the result follows from Proposition 5.

In order to prove the second part of the theorem, consider an optimal cut cover $\mathcal{C}$ for the case when n is odd. The case $n = 3$ is trivial, $n = 5$ follows from Proposition 3, and $n = 7$ follows from Proposition 4, so we may assume that $n > 7$. Let w be the singleton grouping, and each other grouping consists of two *mates* v_1, v_2. Removing a vertex v from every cut in $\mathcal{C}$ we obtain a cover $\mathcal{C}_v$ for K_{n-1}^3. If $v \neq w$, then this new cover is not efficient, and it can be improved by merging w with the mate of v, as in the proof of Lemma 1 to obtain a new efficient cover $\mathcal{C}_v'$ of smaller total size. We also let $\mathcal{C}_w' = \mathcal{C}_w$, so that each $\mathcal{C}_v'$ is an efficient cover of K_{n-1}^3. We break the analysis into four cases and show that in each case we either obtain a contradiction or that $\mathcal{C}$ is one of the covers listed in 2.

Case 1 Every $\mathcal{C}_v'$ has total size greater than $f_3(n-1)$.

Since $n-3$ is even we notice that $f_3(n-1) = \frac{1}{2}(n-3)^3$ is divisible by $2(n-3)$. By (4) the size of every cut in an efficient cover of K_{n-1}^3 is divisible by $2(n-3)$ as well (since $n-1$ and x are even), so that by hypothesis for every vertex v we get

$$\|\mathcal{C}_v\| \geq \|\mathcal{C}_v'\| \geq f_3(n-1) + 2(n-3) = \frac{n-3}{2}((n-3)^2+4) = \frac{n-3}{2}(n^2-6n+13).$$

Summing up the total sizes of the n covers $\mathcal{C}_v$ each edge is cut $(n-3)$ times as often in this sum as it was in $\mathcal{C}$:

$$(n-3)\|\mathcal{C}\| = \sum \|\mathcal{C}_v\| \geq n\frac{n-3}{2}(n^2-6n+13).$$

and thus we get that

$$\|\mathcal{C}\| \geq \frac{n^3-6n^2+13n}{2} = f_3(n) + 5.$$

which contradicts the optimality of $\mathcal{C}$.

Case 2 $\mathcal{C}_w$ is a simple cover for K_{n-1}^3, i.e. $\mathcal{C}_w$ has type $(2^{((n-3)/2)})$.

It suffices to show that in $\mathcal{C}$ the edges containing w are covered more than $s_3(n) - \|\mathcal{C}_w\| = f_3(n) - \varepsilon_n(n-2) - f_3(n-1) = \frac{1}{2}(3n^2-14n+17) - \varepsilon_n(n-2)$ times, except when $\mathcal{C}$ is a cover from our list. If w is on the larger side of a cut in $\mathcal{C}$, then this accounts for $2(n-3)+1 = 2n-5$ edges containing w, whereas when it is on the smaller side we have $2(n-3) + \binom{n-3}{2} = \frac{n^2-3n}{2}$ such edges. So if w is on the smaller side c

times, then the edges containing w are covered at least $c\frac{n^2-3n}{2}+(\frac{n-3}{2}-c)(2n-5)=\frac{2n^2-11n+15}{2}+c\frac{n^2-7n+10}{2}$ times. If c is 0 or 1, then w is on the same side as some grouping in every cut. Also, $\mathcal{C}$ must contain the additional cut $[w, V-w]_3$ of size $\binom{n-1}{2}$ for a total of at least $\frac{2n^2-11n+15}{2}+\binom{n-1}{2}=\frac{1}{2}(3n^2-14n+17)$ edges, and to achieve equality $\mathcal{C}$ must be a simple cover. For $c \geq 2$ we get that the number of edges cut is at least $\frac{2n^2-11n+15}{2}+2\frac{n^2-7n+10}{2}=\frac{4n^2-25n+35}{2}$, which is greater than $\frac{1}{2}(3n^2-14n+17)-\varepsilon_n(n-2)$ for $n \geq 9$.

Case 3 $\mathcal{C}_w$ is one of the remaining efficient covers for K^3_{n-1} listed in part a).

If $n=9, 17$, then $\mathcal{C}_w$ must be a cover of the form $(4,4)_8$ or $(8,8,8)_{16}$ and no matter how w is added back into this cover, there will be a triple that is not covered. Thus, for $n=9$, $\mathcal{C}$ must be of type $(1,4,4)_9$ for a total size of $\frac{9-2}{2}(1\cdot 8+2\cdot 4\cdot 5)=\frac{7}{2}\cdot 48=f_3(9)-7$ as was to be proved. Similarly, for $n=17$, $\mathcal{C}$ must be of type $(1,8,8,8)_{17}$ for a total of $\frac{17-2}{2}(1\cdot 16+3\cdot 8\cdot 9)=\frac{15}{2}\cdot 232=f_3(17)+45$ edges cut, not yielding an optimal cover.

For $n=11$ the only case for $\mathcal{C}_w$ to consider is $(2,4,4)_{10}$, which yields $(3,4,5)_{11}$ or $(1,2,4,4)_{11}$ for $\mathcal{C}$, which have total size $\frac{9}{2}82$ and $\frac{9}{2}84$ respectively, whereas $f(11)=\frac{9}{2}82$.

For $n=13$ the only case is $(4,4,6)_{12}$, which yields $(5,5,6)_{13}$ or $(1,4,4,6)_{13}$, which have total size $\frac{11}{2}122$ and $\frac{11}{2}126$ respectively, whereas $f(13)=\frac{11}{2}122$.

For $n=15$ the only case is $(6,6,6)_{14}$, which yields $(7,7,7)_{15}$ or $(1,6,6,6)_{15}$ which have total size $\frac{13}{2}168$ and $\frac{13}{2}176$ respectively, whereas $f(15)=\frac{13}{2}170$.

Case 4 Some $\mathcal{C}'_v$ for $v \neq w$ is an efficient cover of size at most $f_3(n-1)$ for K^3_{n-1}.

To recover $\mathcal{C}$ from $\mathcal{C}'_v$ we first have to find $\mathcal{C}_v$. $\mathcal{C}_v$ is obtained from $\mathcal{C}'_v$ by either moving the mate v' of v to the other part in some cuts, or by moving w to the other part in some cuts. But the latter case implies that $\mathcal{C}'_v$ is identical to $\mathcal{C}_w$ except that w is replaced by v. Thus, $\mathcal{C}_w$ is an efficient cover of size at most $f_3(n-1)$ as well, and we are back in case 2 or 3.

If $\mathcal{C}_v$ is obtained by moving v', then the edges involving v' must have been cut more often, than those involving w. Thus, reinserting v into $\mathcal{C}_v$ results in more edges being cut, than if v was made the mate of w. This means that $\mathcal{C}$ was not an optimal cover.

□

5 Optimal Covers for Small *n*

We showed in Sect. 4 that for $r=2, 3$ the optimal covers for $s_r(n)$ are simple covers when n gets large enough, a result we will extend to general r in the last section. However, when n is small there are cases when it is more efficient to make only few, but bigger, cuts, as illustrated by Proposition 4, and the exceptional cases $n=4, 8$ for $r=2$ and $n=7, 8, 9, 15, 16$ for $r=3$. When $n=c(r-1)$, then $\chi(K^r_n)=c$ and Proposition 1 implies that the smallest number of cuts in a cover of K^r_n is $\lceil \log_2 c \rceil$. A few simple calculations yield the following result, which can certainly be improved.

Proposition 6 *If* $c \geq 2$ *and* $n = c(r-1)$ *for* $r \geq \log(c-1)/\log(c/(c-1))$*, then* $s_r(n)$ *is achieved by a cut-cover with* $\lceil \log_2 c \rceil$ *cuts.*

Proof No cut-cover can have fewer cuts, so that it suffices to show that the fewer cuts we use, the smaller the total size. By Lemma 1 it is enough to consider efficient covers, and we observe that in such a cover all c groupings have size $(r-1)$. Now suppose that we have two efficient cut-covers $\mathcal{C}_1, \mathcal{C}_2$ with m_1, m_2 cuts respectively, where $m_1 < m_2 \leq c-1$. Combining the previous facts with Lemma 2, we deduce that every cut in $\mathcal{C}_1$ or $\mathcal{C}_2$ has between $\binom{n}{r} - \binom{n-r+1}{r}$ and $\binom{n}{r}$ edges. Subtracting the total size of $\mathcal{C}_1$ from the total size for $\mathcal{C}_2$ we get a difference of more than

$$m_2\left(\binom{n}{r} - \binom{n-(r-1)}{r}\right) - m_1\binom{n}{r} \geq \binom{n}{r} - (c-1)\binom{n-r+1}{r}. \quad (6)$$

Since

$$\frac{\binom{c(r-1)}{r}}{\binom{(c-1)(r-1)}{r}} = \frac{(c(r-1))(c(r-1)-1)\ldots(c(r-1)-(r-1))}{((c-1)(r-1))((c-1)(r-1)-1)\ldots((c-1)(r-1)-(r-1))}$$
$$\geq \left(\frac{c(r-1)}{(c-1)(r-1)}\right)^r = \left(\frac{c}{c-1}\right)^r,$$

the right-most expression in (6) is at least $((\frac{c}{c-1})^r - (c-1))\binom{n-r+1}{r}$. This is at least zero as long as $(\frac{c}{c-1})^r \geq (c-1)$, which is equivalent to the condition of our result. □

For example for $c = 4$, this result yields that when $r \geq \log 3/\log(4/3) = 3.81...$ then for $n = 4(r-1)$ we should use 2 cuts in an optimal cover. Also note that

$$\log(c/(c-1)) = \log c - \log(c-1) = \int_{c-1}^{c} 1/x\, dx \geq 1/c,$$

so that the condition can be relaxed to $r \geq c\log(c-1)$. This also allows us to reformulate as follows:

Proposition 7 *If* $n = c(r-1)$ *for* $c \leq r/\log r$*, then* $s_r(n)$ *is achieved by a cut-cover with* $\lceil \log_2 c \rceil$ *cuts.*

Proof Note that here $c\log(c-1) \leq c\log c \leq (r/\log r)(\log r - \log\log r) \leq r$. □

6 Determining $s_r(n)$ for n Sufficiently Large

We show now that for n sufficiently large, $s_r(n) = f_r(n)$, and thus $\lim_{n\to\infty} s_r(n)/\binom{n}{r} = r$ by (2).

Consider an optimal cut cover $\mathcal{C} = \{C_1, \ldots, C_m\}$ of K_n^r, where $C_i = [X_i, \{1, \ldots, n\} \setminus X_i]$ for $i \in \{1, \ldots, m\}$. Recall that for each vertex v of K_n^r, b_v is the m-dimensional binary vector with $b_v(i) = 1$ if and only if $v \in X_i$.

We start by showing that most of the edges of K_n^r are covered exactly r times in this cover.

If $b_1, \ldots, b_k \in \{0,1\}^m$ are the (not necessarily distinct) vectors corresponding to the vertices in a k-set $S \subseteq V(K_n^r)$, then we define the support of S by

$$\operatorname{supp} S = \operatorname{supp}\{b_1, \ldots, b_k\} = \{i : b_s(i) \neq b_t(i) \text{ for some } s, t \in [k]\}.$$

Observe that an edge E is covered exactly $|\operatorname{supp} E|$ times in $\mathcal{C}$. An edge E is called *typical* if $|\operatorname{supp} E| = r$, but $|\operatorname{supp}(E - v)| = r - 1$ for every $v \in E$.

Claim If $|\operatorname{supp} E| \leq r$, then either E is typical, or there is a $v \in E$ with $|\operatorname{supp} E| = |\operatorname{supp}(E - v)|$.

Proof Suppose that the vertices in E correspond to $b_1, \ldots, b_r \in \{0,1\}^m$ and

$$|\operatorname{supp}\{b_1, \ldots, b_r\}| > |\operatorname{supp}(\{b_1, \ldots, b_r\} \backslash \{b_s\})|$$

for each s with $1 \leq s \leq r$. Thus, for each such s there exists $i = i(s) \in \operatorname{supp}\{b_1, \ldots, b_r\}$ such that $b_s(i) = \alpha \in \{0,1\}$ and $b_t(i) = 1 - \alpha$ for each $t \in [r] \backslash \{s\}$.

If $r \geq 3$ then for all $s \neq t$, we must have $i(s) \neq i(t)$. Since $i(s) \in \operatorname{supp}\{b_1, \ldots, b_r\}$, it follows that

$$r = |\{i(s) : s \in [r]\}| \leq |\operatorname{supp}\{b_1, \ldots, b_r\}| \leq r.$$

Thus, $|\operatorname{supp} E| = |\{i(s) : 1 \leq s \leq r\}| = r$, and $|\operatorname{supp}(E - v)| = |\{i(s) : 1 \leq s \leq r, b_s \neq b_v\}| = r - 1$, and E is typical. This proves the claim. □

The following observation is also an immediate consequence of this proof:

For each typical edge E there is a unique vector $b(E) \in \{0,1\}^m$ such that the vectors corresponding to the vertices in E are of the form $b(E) + e_i$ for r distinct vectors e_i of weight 1 (i.e. standard basis vectors), where addition is modulo 2. Indeed, $b_s - e_{i(s)} = b_t - e_{i(t)}$ for all $s, t \in [r]$, and this vector is then $b(E)$.

Lemma 3 *All but $O(n^{r-1})$ edges in K_n^r are typical.*

Proof If $E^- = \{E \in K_n^r : |\operatorname{supp} E| < r\}$, $E^+ = \{E \in K_n^r : |\operatorname{supp} E| > r\}$, and $E^r = \{E \in K_n^r : |\operatorname{supp} E| = r\}$, then $\|\mathcal{C}\| = \sum |\operatorname{supp} E| \geq 1|E^-| + r|E^r| + (r+1)|E^+|$, and $\|\mathcal{C}\| \leq f_r(n) < r\binom{n}{r} = r|E^-| + r|E^r| + r|E^+|$, so that

$$|E^+| < (r-1)|E^-|.$$

Let E' denote the set of edges of K_n^r which are covered at most r times by $\mathcal{C}$, but which are not typical. We will prove that $|E'| \leq (r-1)2^r\binom{n}{r-1}$.

For each edge in E' whose corresponding set of vectors is $\{b_1, \ldots, b_{r-1}, b_r\}$, assume that $\operatorname{supp}\{b_1, \ldots, b_{r-1}, b_r\} = \operatorname{supp}\{b_1, \ldots, b_{r-1}\}$.

There are at most $\binom{n}{r-1}$ ways to choose $r-1$ vertices in K_n^r whose corresponding vectors are suitable $b_1, \ldots, b_{r-1}$ for an edge in E'.

Now we claim that the vertex whose vector is b_r can be chosen in at most $(r-1)2^r$ ways. This is because there are at most 2^r choices for b_r and at most $r-1$ choices for a vertex whose vector is b_r, as we show next.

The fact that there are at most 2^r choices for b_r follows because $|\text{supp}\{b_1, \ldots, b_{r-1}, b_r\}| = |\text{supp}\{b_1, \ldots, b_{r-1}\}| = p \le r$. The vectors $b_1, \ldots, b_{r-1}$ lie in a hypercube of dimension $p \le r$ in $\{0, 1\}^m$, since they completely agree on all but p coordinates. Since b_r must be in the same hypercube it can be chosen in at most $2^p \le 2^r$ ways.

The fact that there are at most $r-1$ choices for a vertex whose vector is b_r follows from the fact that any element in $\{0, 1\}^m$ can be the vector of at most $r-1$ vertices of K_n^r. Otherwise, an edge containing r such vertices would not be covered by $\mathcal{C}$.

Thus, we have proved that $|E'| \le (r-1)2^r\binom{n}{r-1}$. Since $E^- \subseteq E'$ it follows that $|E^+| < (r-1)|E^-| \le (r-1)^2 2^r\binom{n}{r-1}$, so that the number of edges that are not typical is at most

$$|E'| + |E^+| < r(r-1)2^r\binom{n}{r-1} = O(n^{r-1}).$$

□

Claim If E, F are typical edges with $|E \cap F| \ge 3$, then $b(E) = b(F)$.

Proof If b_1, b_2, b_3 are vectors corresponding to any three distinct vertices in $E \cap F$, then these vectors are distinct and all at distance 1 from $b(E)$ and $b(F)$. So since a hypercube does not contain $K_{2,3}$ as a subgraph it follows that $b(E) = b(F)$. □

Lemma 4 *Suppose $r \ge 4$. We can assume that for all but $O(1)$ vertices v we have b_v is a vector of weight* 1.

Proof Let $T(S)$ be the number of typical edges containing the set S. Summing $T(S)$ over all $(r-1)$-sets S, we count each typical edge exactly r times, so that

$$\sum_{|S|=r-1} T(S) = r\left(\binom{n}{r} - O(n^{r-1})\right) = r\binom{n}{r} - O(n^{r-1}).$$

Thus, there is some $(r-1)$-set S that is contained in at least

$$\frac{r\binom{n}{r} - O(n^{r-1})}{\binom{n}{r-1}} = n - O(1)$$

typical edges. Since all these edges E have an intersection of $|S| = r-1 \ge 3$ it follows by the claim that they all have the same $b(E)$, and we may assume without loss of generality that this $b(E)$ is the zero vector. Indeed, if some such $b(E)_i = 1$, then we simply interchange the role of X_i and $V - X_i$ in that cut, so that $(b_v)_i$ turns into $1 - (b_v)_i$ for each vertex v.

Now since each such $b(E)$ is the zero vector, each $v \in E$ must have vector b_v of weight 1. Thus, all but the $O(1)$ vertices not contained in any such edge E correspond to a vector of weight 1. □

Theorem 2 *For each r there is an n_r, so that if $n \geq n_r$ then $s_r(n) = f_r(n)$ and the unique optimal cut cover is a simple cover.*

Proof Observe that from the results stated in Sect. 4 it follows that $n_2 = 9$ (Proposition 5) and $n_3 = 17$ (Theorem 1), so that we may assume that $r \geq 4$. It suffices to show that in an optimal cover all vectors b_v have weight 1 or 0, so let S be the set of all vertices corresponding to vectors of weight at least 2. By the previous lemma $|S| \leq c_r$ for some constant c_r.

Let $v \in S$, and suppose without loss of generality that the first 2 coordinates of b_v are 1. The edges containing v are cut a total of $\sum_{E \ni v} |\text{supp}\, E|$ times.

If $v_1, \ldots, v_{r-1}$ correspond to distinct unit vectors e_i with the 1 in a position other than the first or second, then $|\text{supp}\,\{v, v_1, \ldots, v_{r-1}\}| \geq r+1$. Since each unit vector e_i (except for possibly one) corresponds to a grouping of $r-1$ vertices, we have at least $M = \lfloor \frac{n-c_r}{r-1} \rfloor - 3$ choices for the e_i's: there are $\lfloor \frac{n-|S|}{r-1} \rfloor$ groupings of size exactly $r-1$ and weight at most 1, one of which may have weight 0, and two of which may correspond to e_1 and e_2. Thus, there are at least $\binom{M}{r-1}(r-1)^{r-1}$ such $v_1, \ldots, v_{r-1}$ since for each of the $(r-1)$-groupings we can choose its representative in $r-1$ ways.

So the total number of times edges containing v, but no other vertices from S, are cut is at least $\binom{M}{r-1}(r-1)^{r-1}(r+1)$. Now if we replace each b_v for $v \in S$ by a unit vector, then we get that now no edge containing v is covered more than r times for a total of at most $\binom{n-1}{r-1}r$ edges cut. So the total decrease in the number of edges cut is at least $|S|\left(\binom{M}{r-1}(r-1)^{r-1}(r+1) - \binom{n-1}{r-1}r\right)$. If this number is positive, then we get the contradiction that the new cover has smaller total size. Indeed,

$$\begin{aligned}
\frac{\binom{M}{r-1}(r-1)^{r-1}}{\binom{n-1}{r-1}} &= \frac{M(M-1)\cdots(M-r+2)}{(n-1)(n-2)\cdots(n-r+1)}(r-1)^{r-1} \\
&\geq \left(\frac{M-r+2}{n-r+1}\right)^{r-1}(r-1)^{r-1} = \left(\frac{\lfloor \frac{n-c_r}{r-1} \rfloor - (r+1)}{n-r+1}\right)^{r-1}(r-1)^{r-1} \\
&> \left(\frac{n-c_r-(r+2)(r-1)}{n-r+1}\right)^{r-1}.
\end{aligned}$$

Since the last fraction converges to 1 for $n \to \infty$ it follows that for n sufficiently large this expression exceeds $r/(r+1)$, finishing the proof. □

Using the value $r(r-1)2^r\binom{n}{r-1}$ obtained for $O(n^{r-1})$ in Lemma 3, we see that the $O(1)$ in the middle of Lemma 4 is $(r-1)(r^2 2^r + 1)$ and we can use $c_r = r^3 2^r$ in the proof of Theorem 2. (With more care, using the fact that the first inequality in the proof of Lemma 3 is too crude, this can be slightly improved.) Letting $N = n - r + 1$ and setting the last expression in the proof of Theorem 2 $\geq r/(r+1)$ we get the inequality

$$n - r + 1 = N \geq \frac{c_r + r^2 - 1}{1 - (r/(r+1))^{1/(r-1)}}.$$

Since $\log\left(\frac{r}{r+1}\right) \le -\frac{1}{r+1}$ and $1-e^{-x} \le x$ we obtain

$$1-(r/(r+1))^{1/(r-1)} \ge 1-e^{-1/(r^2-1)} \ge \frac{1}{r^2-1}$$

which gives the inequality $n \ge (r-1)+(r^2-1)(c_r+r^2-1)$ so that we can choose $n_r = O(r^5 2^r)$. On the other hand it is easy to prove

Claim If $r \ge 7$, then $n_r > (r-1)2^{r-1}$.

Proof Let $n = (r-1)2^{r-1}$. For $r \ge 7$ we have $2^{r-1} \ge r^2$, and thus

$$r \ge \frac{(r-1)2^{r-1}}{2^{r-1}-r} = \frac{\frac{r-1}{2^{r-1}-r}}{\frac{1}{2^{r-1}}} \ge \frac{\int_{2^{r-1}-r}^{2^{r-1}-1} \frac{1}{x}\,dx}{\int_{2^{r-1}-1}^{2^{r-1}} \frac{1}{x}\,dx} = \frac{\log\left(\frac{2^{r-1}-1}{2^{r-1}-r}\right)}{\log\left(\frac{2^{r-1}}{2^{r-1}-1}\right)}.$$

It follows that

$$\frac{2^{r-1}-1}{2^{r-1}-r} \le \left(\frac{2^{r-1}}{2^{r-1}-1}\right)^r = \left(\frac{n}{n-(r-1)}\right)^r$$
$$\le \frac{n(n-1)(n-2)\cdots(n-r+1)}{(n-r+1)(n-r)\cdots(n-2r+2)} = \frac{\binom{n}{r}}{\binom{n-r+1}{r}}$$

and therefore $(2^{r-1}-r)\binom{n}{r} \ge (2^{r-1}-1)\binom{n-r+1}{r}$. Thus,

$$(2^{r-1}-1)\left(\binom{n}{r}-\binom{n-r+1}{r}\right) \ge (r-1)\binom{n}{r} > (r-1)\left(\binom{n}{r}-2\binom{n/2}{r}\right),$$

where the left-hand side is the total size of a smallest simple cover of K_n^r, and the right-hand side is the total size of a cover with $r-1$ cuts $[X_i, V-X_i]$ with $|X_i| = |V-X_i| = n/2$. □

7 Open Questions

Many open questions remain and we list some of them here.

1. Determine the asymptotics of n_r from Theorem 2 by improving the current bounds $(r-1)2^{r-1} < n_r < r^5 2^r$.
2. Generalize the results from [4,12] on 2-graphs to r-graphs. For example, determine $\mathrm{cs}(\mathcal{H})$ for random r-uniform hypergraphs $\mathcal{H}$.
3. For $r \ge 3$ other ways of generalizing cuts (and the corresponding covering questions) are worth considering. For example if $\mathcal{H} = (V, \mathcal{E})$ is an r-graph and if $X_1, \ldots, X_k$ is a partition of V, then the k-cut $[X_1, X_2, \ldots X_k]$ is the set of all edges of $\mathcal{H}$ that meet at least one vertex in each X_i.

Acknowledgments The research of the first author is supported by a startup grant from the Department of Mathematical Sciences of the University of Delaware. The authors are grateful to the anonymous referee for his or her thorough reading and constructive criticism. The authors also thank Fan Chung, Zoltan Füredi, Gyula Katona, Radhika Ramamurthi and Jacques Verstraëte for helpful discussions on the manuscript.

References

1. Bollobás, B., Scott, A.: Separating systems and oriented graphs of diameter two. J. Comb. Theory Ser B **97**, 193–203 (2007)
2. Bollobás, B., Scott, A.: On separating systems. Eur. J. Combinatorics **28**, 1068–1071 (2007)
3. Fredman, M.L., Komlós, J.: On the size of separating systems and families of perfect hash functions. SIAM J. Alg. Discr. Methods **5**, 61–68 (1984)
4. Füredi, Z., Kündgen, A.: Covering a graph with cuts of minimum size. Discrete Math. **237**, 129–148 (2001)
5. Garey, M.R., Johnson, D.S.: Computers and Intractability a guide to the theory of NP-completeness. W.H. Freeman and Company, New York (1978)
6. Garey, M.R., Johnson, D.S., So, H.C.: An application of graph coloring to printed circuit testing. IEEE Trans. Circuits Syst. **CAS-23**, 591–599 (1976)
7. Harary, F., Hsu, D., Miller, Z.: The biparticity of a graph. J. Graph Theory **1**, 131–133 (1977)
8. Jaeger, F., Khelladi, A., Mollard, M.: On the shortest cocycle covers of graphs. J. Combin. Theory Ser. B **39**, 153–163 (1985)
9. Jamshy, U., Tarsi, M.: Cycle coverings of binary matroids. J. Combin. Theory Ser. B **46**, 154–161 (1989)
10. Katona, G.: On separating systems of a finite set. J. Combin. Theory **1**, 174–194 (1966)
11. Klugerman, M., Russell, A., Sundaram, R.: A note on embedding complete graphs into hypercubes. Discrete Math. **186**, 289–293 (1998)
12. Kündgen, A., Spangler, M.: A bound on the total size of a cut cover. Discrete Math. **296**, 121–128 (2005)
13. Kündgen, A.: Covering cliques with spanning bicliques. J. Graph Theory **27**, 223–227 (1998)
14. Lovász, L.: Covering and coloring of hypergraphs. In: Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory and Computing, pp. 3–12. Utilitas Mathematica Publishing, Winnipeg (1973)
15. Matula, D.W.: k-components, clusters and slicings in graphs. SIAM J. Appl. Math. **22**, 459–480 (1972)
16. Miller, Z., Müller, H.: Chromatic numbers of hypergraphs and coverings of graphs. J. Graph Theory **5**, 299–305 (1981)
17. Radhakrishnan, J.: Entropy and Counting, IIT Kharagpur, Golden Jubilee Volume on Computational Mathematics, Modelling and Algorithms, Narosa Publishers, New Delhi (2001). Available online at http://www.tcs.tifr.res.in/~jaikumar/mypage.html
18. Rényi, A.: On random generating elements of a finite Boolean algebra. Acta Sci. Math. Szeged **22**, 75–81 (1961)