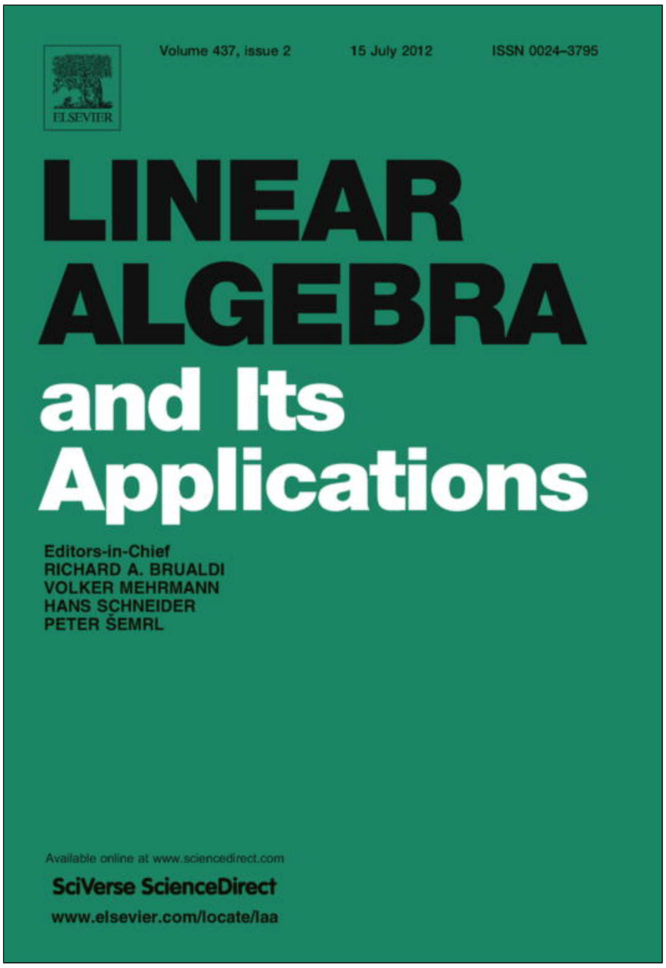
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Edge-disjoint spanning trees and eigenvalues of regular graphs

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ABSTRACT

Partially answering a question of Paul Seymour, we obtain a sufficient eigenvalue condition for the existence of k edge-disjoint spanning trees in a regular graph, when $k \in \{2, 3\}$. More precisely, we show that if the second largest eigenvalue of a d-regular graph G is less than $d - \frac{2k-1}{d+1}$, then G contains at least k edge-disjoint spanning trees, when $k \in \{2, 3\}$. We construct examples of graphs that show our bounds are essentially best possible. We conjecture that the above statement is true for any $k < d/2$.

1. Introduction

Our graph notation is standard (see West [21] for undefined terms). The adjacency matrix of a graph G with n vertices has its rows and columns indexed after the vertices of G and the (u, v)-entry of A is 1 if $uv = \{u, v\}$ is an edge of G and 0 otherwise. If G is undirected, then A is symmetric. Therefore, its eigenvalues are real numbers, and we order them as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The Laplacian matrix L of G equals $D - A$, where D is the diagonal degree matrix of G. The Laplacian matrix is positive semidefinite and we order its eigenvalues as $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$. It is well known that if G is connected and d-regular, then $\mu_i = d - \lambda_i$ for each $1 \leqslant i \leqslant n$, $\lambda_1 = d$ and $\lambda_i < d$ for any $i \neq 1$ (see [3,9]).

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Kirchhoff Matrix Tree Theorem [13] (see [3, Section 1.3.5] or [9, Section 13.2] for short proofs) is one of the classical results of combinatorics. It states that the number of spanning trees of a graph G with n vertices is the principal minor of the Laplacian matrix L of the graph and consequently, equals $\frac{\prod_{i=2}^{n} \mu_i}{n}$. In particular, if G is a d-regular graph, then the number of spanning trees of G is $\frac{\prod_{i=2}^{n}(d-\lambda_i)}{n}$.

Motivated by these facts and by a question of Seymour [19], in this paper, we find relations between the maximum number of edge-disjoint spanning trees (also called the spanning tree packing number or tree packing number; see Palmer [18] for a survey of this parameter) and the eigenvalues of a regular graph. Let $\sigma(G)$ denote the maximum number of edge-disjoint spanning trees of G. Obviously, G is connected if and only if $\sigma(G) \geqslant 1$.

A classical result, due to Nash-Williams [16] and independently, Tutte [20] (see [12] for a recent short constructive proof), states that a graph G contains k edge-disjoint spanning trees if and only if for any partition of its vertex set $V(G) = X_1 \cup \cdots \cup X_t$ into t non-empty subsets, the following condition is satisfied:

$$\sum_{1 \leqslant i < j \leqslant t} e(X_i, X_j) \geqslant k(t-1) \tag{1}$$

A simple consequence of Nash-Williams/Tutte Theorem is that if G is a $2k$-edge-connected graph, then $\sigma(G) \geqslant k$ (see Kundu [15]). Catlin [4] (see also [5]) improved this result and showed that a graph G is $2k$-edge-connected if and only if the graph obtained from removing any k edges from G contains at least k edge-disjoint spanning trees.

An obvious attempt to find relations between $\sigma(G)$ and the eigenvalues of G is by using the relations between eigenvalues and edge-connectivity of a regular graph as well as the previous observations relating the edge-connectivity to $\sigma(G)$. Cioabă [7] has proven that if G is a d-regular graph and $2 \leqslant r \leqslant d$ is an integer such that $\lambda_2 < d - \frac{2(r-1)}{d+1}$, then G is r-edge-connected. While not mentioned in [7], it can be shown that the upper bound above is essentially best possible. An obvious consequence of these facts is that if G is a d-regular graph with $\lambda_2 < d - \frac{2(2k-1)}{d+1}$ for some integer k, $2 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$, then G is $2k$-edge-connected and consequently, G contains k-edge-disjoint spanning trees.

In this paper, we improve the bound above as follows.

Theorem 1.1. *If $d \geqslant 4$ is an integer and G is a d-regular graph such that $\lambda_2(G) < d - \frac{3}{d+1}$, then G contains at least 2 edge-disjoint spanning trees.*

We remark that the existence of 2 edge-disjoint spanning trees in a graph implies some good properties (cf. [17]); for example, every graph G with $\sigma(G) \geqslant 2$ has a cycle double cover (see [17] for more details). The proof of Theorem 1.1 is contained in Section 2. In Section 2, we will also show that Theorem 1.1 is essentially best possible by constructing examples of d-regular graphs $\mathcal{G}_d$ such that $\sigma(\mathcal{G}_d) < 2$ and $\lambda_2(\mathcal{G}_d) \in \left(d - \frac{3}{d+2}, d - \frac{3}{d+3}\right)$. In Section 2, we will answer a question of Palmer [18, Section 3.7, p. 19] by proving that the minimum number of vertices of a d-regular graph with edge-connectivity 2 and spanning tree number 1 is $3(d+1)$.

Theorem 1.2. *If $d \geqslant 6$ is an integer and G is a d-regular graph such that $\lambda_2(G) < d - \frac{5}{d+1}$, then G contains at least 3 edge-disjoint spanning trees.*

The proof of this result is contained in Section 3. In Section 3, we will also show that Theorem 1.2 is essentially best possible by constructing examples of d-regular graphs $\mathcal{H}_d$ such that $\sigma(\mathcal{H}_d) < 3$ and $\lambda_2(\mathcal{H}_d) \in \left[d - \frac{5}{d+1}, d - \frac{5}{d+3}\right)$. We conclude the paper with some final remarks and open problems.

The main tools in our paper are Nash-Williams/Tutte Theorem stated above and eigenvalue interlacing described below (see also [3,9–11]).

Theorem 1.3. *Let $\lambda_j(M)$ be the j-th largest eigenvalue of a matrix M. If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A with order $m \times m$, then for $1 \leqslant i \leqslant m$,*

$$\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{n-m+i}(A). \quad (2)$$

This theorem implies that if H is an induced subgraph of a graph G, then the eigenvalues of H interlace the eigenvalues of G.

If S and T are disjoint subsets of the vertex set of G, then we denote by $E(S,T)$ the set of edges with one endpoint in S and another endpoint in T. Also, let $e(S,T) = |E(S,T)|$. If S is a subset of vertices of G, let $G[S]$ denote the subgraph of G induced by S. The previous interlacing result implies that if A and B are two disjoint subsets of a graph G such that $e(A,B) = 0$, then the eigenvalues of $G[A \cup B]$ interlace the eigenvalues of G. As the spectrum of $G[A \cup B]$ is the union of the spectrum of $G[A]$ and the spectrum of $G[B]$ (this follows from $e(A,B) = 0$), it follows that

$$\lambda_2(G) \geqslant \lambda_2(G[A \cup B]) \geqslant \min(\lambda_1(G[A]), \lambda_1(G[B])) \geqslant \min(\overline{d}(A), \overline{d}(B)), \quad (3)$$

where $\overline{d}(S)$ denotes the average degree of $G[S]$.

Consider a partition $V(G) = V_1 \cup \ldots V_s$ of the vertex set of G into s non-empty subsets. For $1 \leqslant i,j \leqslant s$, let $b_{i,j}$ denote the average number of neighbors in V_j of the vertices in V_i. The quotient matrix of this partition is the $s \times s$ matrix whose (i,j)-th entry equals $b_{i,j}$. A theorem of Haemers (see [10] and also, [3,9]) states that the eigenvalues of the quotient matrix interlace the eigenvalues of G. The previous partition is called equitable if for each $1 \leqslant i,j \leqslant s$, any vertex $v \in V_i$ has exactly $b_{i,j}$ neighbors in V_j. In this case, the eigenvalues of the quotient matrix are eigenvalues of G and the spectral radius of the quotient matrix equals the spectral radius of G (see [3,9,10] for more details).

2. Eigenvalue condition for 2 edge-disjoint spanning trees

In this section, we give a proof of Theorem 1.1 showing that if G is a d-regular graph such that $\lambda_2(G) < d - \frac{3}{d+1}$, then G contains at least 2 edge-disjoint spanning trees. We show that the bound $d - \frac{3}{d+1}$ is essentially best possible by constructing examples of d-regular graphs $\mathcal{G}_d$ having $\sigma(\mathcal{G}_d) < 2$ and $d - \frac{3}{d+2} < \lambda_2(\mathcal{G}_d) < d - \frac{3}{d+3}$.

Proof of Theorem 1.1. We prove the contrapositive. Assume that G does not contain 2-edge-disjoint spanning trees. We will show that $\lambda_2(G) \geqslant d - \frac{3}{d+1}$.

By Nash-Williams/Tutte Theorem, there exists a partition of the vertex set of G into t subsets $X_1, \ldots, X_t$ such that

$$\sum_{1 \leqslant i < j \leqslant t} e(X_i, X_j) \leqslant 2(t-1) - 1 = 2t - 3. \quad (4)$$

It follows that

$$\sum_{i=1}^{t} r_i \leqslant 4t - 6 \quad (5)$$

where $r_i = e(X_i, V \setminus X_i)$.

Let $n_i = |X_i|$ for $1 \leqslant i \leqslant t$. It is easy to see that $r_i \leqslant d-1$ implies $n_i \geqslant d+1$ for each $1 \leqslant i \leqslant 3$.

If $t = 2$, then $e(X_1, V \setminus X_1) = 1$. By results of [7], it follows that $\lambda_2(G) > d - \frac{2}{d+4} > d - \frac{3}{d+1}$ and this finishes the proof of this case. Actually, we may assume $r_i \geqslant 2$ for every $1 \leqslant i \leqslant t$ since $r_i = 1$ and results of [7] would imply $\lambda_2(G) > d - \frac{2}{d+4} > d - \frac{3}{d+1}$.

If $t = 3$, then $r_1 + r_2 + r_3 \leqslant 6$ which implies $r_1 = r_2 = r_3 = 2$. The only way this can happen is if $e(X_i, X_j) = 1$ for every $1 \leqslant i < j \leqslant 3$. Consider the partition of G into X_1, X_2 and X_3. The quotient matrix of this partition is

$$A_3 = \begin{bmatrix} d - \frac{2}{n_1} & \frac{1}{n_1} & \frac{1}{n_1} \\ \frac{1}{n_2} & d - \frac{2}{n_2} & \frac{1}{n_2} \\ \frac{1}{n_3} & \frac{1}{n_3} & d - \frac{2}{n_3} \end{bmatrix}.$$

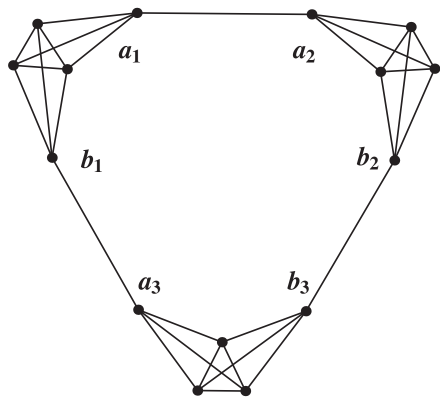

Fig. 1. The 4-regular graph $\mathcal{G}_4$ with $\sigma(\mathcal{G}_4) = 1$ and $3.5 = 4 - \frac{3}{4+2} < \lambda_2(\mathcal{G}_4) \approx 3.569 < 4 - \frac{3}{4+3} \approx 3.571$.

The largest eigenvalue of A_3 is d and the second eigenvalue of A_3 equals

$$d - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} + \sqrt{\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} - \frac{1}{n_1 n_2} - \frac{1}{n_2 n_3} - \frac{1}{n_3 n_1}},$$

which is greater than $d - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}$. Thus, eigenvalue interlacing and $n_i \geqslant d + 1$ for $1 \leqslant i \leqslant 3$ imply $\lambda_2(G) \geqslant \lambda_2(A_3) \geqslant d - \frac{3}{d+1}$. This finishes the proof of the case $t = 3$.

Assume $t \geqslant 4$ from now on. Let a denote the number of r_i's that equal 2 and b denote the number of r_j's that equal 3. Using Eq. (5), we get

$$4t - 6 \geqslant \sum_{i=1}^{t} r_i \geqslant 2a + 3b + 4(t - a - b) = 4t - 2a - b,$$

which implies $2a + b \geqslant 6$.

Recall that $\overline{d}(A)$ denotes the average degree of the subgraph of G induced by the subset $A \subset V(G)$.

If $a = 0$, then $b \geqslant 6$. This implies that there exist two indices $1 \leqslant i < j \leqslant t$ such that $r_i = r_j = 3$ and $e(X_i, X_j) = 0$. Eigenvalue interlacing (3) implies $\lambda_2(G) \geqslant \lambda_2(G[X_i \cup X_j]) \geqslant \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geqslant \min(\overline{d}(X_i), \overline{d}(X_j) \geqslant \min(d - \frac{3}{n_i}, d - \frac{3}{n_j}) \geqslant d - \frac{3}{d+1}$.

If $a = 1$, then $b \geqslant 4$. This implies there exist two indices $1 \leqslant i < j \leqslant t$ such that $r_i = 2, r_j = 3$ and $e(X_i, X_j) = 0$. Eigenvalue interlacing (3) implies $\lambda_2(G) \geqslant \lambda_2(G[X_i \cup X_j]) \geqslant \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geqslant \min(\overline{d}(X_i), \overline{d}(X_j)) \geqslant \min(d - \frac{2}{n_i}, d - \frac{3}{n_j}) \geqslant d - \frac{3}{d+1}$.

If $a = 2$, then $b \geqslant 2$. If there exist two indices $1 \leqslant i < j \leqslant t$ such that $r_i = r_j = 2$ and $e(X_i, X_j) = 0$, then eigenvalue interlacing (3) implies $\lambda_2(G) \geqslant \lambda_2(G[(X_i \cup X_j]) \geqslant \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geqslant \min(\overline{d}(X_i), \overline{d}(X_j)) \geqslant \min(d - \frac{2}{n_i}, d - \frac{2}{n_j}) \geqslant d - \frac{2}{d+1} > d - \frac{3}{d+1}$. Otherwise, there exist two indices $1 \leqslant p < q \leqslant t$ such that $r_p = 2$, $r_q = 3$ and $e(X_p, X_q) = 0$. By a similar eigenvalue interlacing argument, we get $\lambda_2(G) \geqslant d - \frac{3}{d+1}$ in this case as well.

If $a = 3$, then if there exist two indices $1 \leqslant i < j \leqslant t$ such that $r_i = r_j = 2$ and $e(X_i, X_j) = 0$, then as before, eigenvalue interlacing (3) implies $\lambda_2(G) \geqslant d - \frac{2}{d+1} > d - \frac{3}{d+1}$. This finishes the proof of Theorem 1.1. □

We show that our bound is essentially best possible by presenting a family of d-regular graphs $\mathcal{G}_d$ with $d - \frac{3}{d+2} < \lambda_2(\mathcal{G}_d) < d - \frac{3}{d+3}$ and $\sigma(\mathcal{G}_d) = 1$, for every $d \geqslant 4$.

For $d \geqslant 4$, consider three vertex disjoint copies G_1, G_2, G_3 of K_{d+1} minus one edge. Let a_i and b_i be the two non adjacent vertices in G_i for $1 \leqslant i \leqslant 3$. Let $\mathcal{G}_d$ be the d-regular graph obtained by joining a_1 with a_2, b_2 and b_3 and a_3 and b_1. The graph $\mathcal{G}_d$ has $3(d + 1)$ vertices and is d-regular. Fig. 1 depicts this external graph in the case $d = 4$. The partition of the vertex set of $\mathcal{G}_d$ into $V(G_1), V(G_2), V(G_3)$ has the property that the number of edges between the parts equals 3. By Nash-Williams/Tutte Theorem, this implies $\sigma(\mathcal{G}_d) < 2$.

For $d \geqslant 4$, denote by θ_d the largest root of the cubic polynomial

$$P_3(x) = x^3 + (2-d)x^2 + (1-2d)x + 2d - 3. \tag{6}$$

Lemma 2.1. *For every integer $d \geqslant 4$, the second largest eigenvalue of $\mathcal{G}_d$ is θ_d.*

Proof. Consider the following partition of the vertex set of $\mathcal{G}_d$ into nine parts: $V(G_1) \setminus \{a_1, b_1\}$, $V(G_2) \setminus \{a_2, b_2\}$, $V(G_3) \setminus \{a_3, b_3\}$, $\{a_1\}$, $\{b_1\}$, $\{a_2\}$, $\{b_2\}$, $\{a_3\}$, $\{b_3\}$. This is an equitable partition whose quotient matrix is the following

$$A_9 = \begin{bmatrix} d-2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & d-2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & d-2 & 0 & 0 & 0 & 0 & 1 & 1 \\ d-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ d-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & d-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & d-1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & d-1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

The characteristic polynomial of A_9 is

$$P_9(x) = (x-d)(x+1)^2[x^3 + (2-d)x^2 + (1-2d)x + 2d - 3]^2. \tag{8}$$

Let $\lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ denote the solutions of the equation $x^3 + (2-d)x^2 + (1-2d)x + 2d - 3 = 0$. Because the above partition is equitable, it follows that $d, \lambda_2, \lambda_3, \lambda_4$ and -1 are eigenvalues of $\mathcal{G}_d$, and the multiplicity of each of them as an eigenvalue of $\mathcal{G}_d$ is at least 2.

We claim the spectrum of $\mathcal{G}_d$ is

$$d^{(1)}, \lambda_2^{(2)}, \lambda_3^{(2)}, \lambda_4^{(2)}, (-1)^{(3d-4)}. \tag{9}$$

It suffices to obtain $3d-4$ linearly independent eigenvectors corresponding to -1. Consider two distinct vertices u_1 and u_2 in $V(G_1) \setminus \{a_1, b_1\}$. Define an eigenvector where the entry corresponding to u_1 is 1, the entry corresponding to u_2 is -1, and all the other entries are 0. We create $d-2$ eigenvectors by letting u_2 to be each of the $d-2$ vertices in $V(G_1) \setminus \{a_1, b_1, u_1\}$. This can also be done to two vertices $u_1', u_2' \in V(G_2) \setminus \{a_2, b_2\}$ or two vertices $u_1'', u_2'' \in V(G_3) \setminus \{a_3, b_3\}$. This way, we obtain a total of $3d-6$ linearly independent eigenvectors corresponding to -1. Furthermore, define an vector with entries at three fixed vertices $u_1 \in V(G_1) \setminus \{a_1, b_1\}$, $u_1' \in V(G_2) \setminus \{a_2, b_2\}$, $u_1'' \in V(G_3) \setminus \{a_3, b_3\}$ equal to -1, with entries at a_1, b_2, a_3 equal to 1 and with entries 0 everywhere else. It is easy to check this is an eigenvector corresponding to 0. To obtain the final eigenvector, define a new vector by setting the entries at three fixed vertices $u_1 \in V(G_1) \setminus \{a_1, b_1\}$, $u_1' \in V(G_2) \setminus \{a_2, b_2\}$, $u_1'' \in V(G_3) \setminus \{a_3, b_3\}$ to be -1, the entries at b_1, a_2, and b_3 to be 1 and the remaining entries to be 0. It is easy to check all these $3d-4$ vectors are linearly independent eigenvectors corresponding to eigenvalue -1. Having obtained the entire spectrum of $\mathcal{G}_d$, the second largest eigenvalue of $\mathcal{G}_d$ must be θ_d. $\square$

Lemma 2.2. *For every integer $d \geqslant 4$,*

$$d - \frac{3}{d+2} < \theta_d < d - \frac{3}{d+3}.$$

Proof. We find that for $d \geq 4$,

$$P_3\left(d - \frac{3}{d+2}\right) = -\frac{3\left(9 + d\left(-2 + d + d^2\right)\right)}{(2+d)^3} < 0,$$

$$P_3\left(d - \frac{3}{d+3}\right) = \frac{-81 + 6d^2}{(3+d)^3} > 0,$$

and $P_3'(x) > 0$ beyond $x = \frac{1}{3}(-1 + 2d) < d - \frac{3}{d+3}$. Hence,

$$d - \frac{3}{d+2} < \theta_d < d - \frac{3}{d+3} \tag{10}$$

for every $d \geqslant 4$. $\square$

Palmer [18] asked whether or not the graph $\mathcal{G}_4$ has the smallest number of vertices among all 4-regular graphs with edge-connectivity 2 and spanning tree number 1. We answer this question affirmatively below.

Proposition 2.3. *Let $d \geqslant 4$ be an integer. If G is a d-regular graph such that $\kappa'(G) = 2$ and $\sigma(G) = 1$, then G has at least $3(d+1)$ vertices. The only graph with these properties and $3(d+1)$ vertices is $\mathcal{G}_d$.*

Proof. As $\sigma(G) = 1 < 2$, by Nash-Williams/Tutte Theorem, there exists a partition $V(G) = X_1 \cup \cdots \cup X_t$ such that $e(X_1, \ldots, X_t) \leqslant 2t - 3$. This implies $r_1 + \cdots + r_t \leqslant 4t - 6$. As $\kappa'(G) = 2$, it means that $r_i \geqslant 2$ for each $1 \leqslant i \leqslant t$ which implies $4t - 6 \geqslant 2t$ and thus, $t \geqslant 3$.

If $t = 3$, then $r_i = 2$ for each $1 \leqslant i \leqslant 3$ and thus, $e(X_i, X_j) = 1$ for each $1 \leqslant i \neq j \leqslant 3$. As $d \geqslant 4$ and $r_i = 2$, we deduce that $|X_i| \geqslant d + 1$. Equality happens if and only if X_i induces a K_{d+1} without one edge. Thus, we obtain that $|V(G)| = |X_1| + |X_2| + |X_3| \geqslant 3(d+1)$ with equality if and only if $G = \mathcal{G}_d$.

If $t \geqslant 4$, then let α denote the number of X_i's such that $|X_i| \geqslant d+1$. If $\alpha \geqslant 3$, then $|V(G)| > 3(d+1)$ and we are done. Otherwise, $\alpha \leqslant 2$. Note that if $|X_i| \leqslant d$, then $r_i \geqslant d$. Thus,

$$4t - 6 \geqslant r_1 + \cdots + r_t \geqslant 2\alpha + d(t - \alpha) = dt - (d-2)\alpha$$

which implies $(d-2)\alpha \geqslant (d-4)t + 6$. As $\alpha \leqslant 2$ and $t \geqslant 4$, we obtain $2(d-2) \geqslant (d-4)4 + 6$ which is equivalent to $2d \leqslant 6$, contradiction. This finishes our proof. $\square$

3. Eigenvalue condition for 3 edge-disjoint spanning trees

In this section, we give a proof of Theorem 1.2 showing that if G is a d-regular graph such that $\lambda_2(G) < d - \frac{5}{d+1}$, then G contains at least 3 edge-disjoint spanning trees. We show that the bound $d - \frac{5}{d+1}$ is essentially best possible by constructing examples of d-regular graphs $\mathcal{H}_d$ having $\sigma(\mathcal{H}_d) < 3$ and $d - \frac{5}{d+1} \leq \lambda_2(\mathcal{H}_d) < d - \frac{5}{d+3}$.

Proof of Theorem 1.2. We prove the contrapositive. We assume that G does not contain 3-edge-disjoint spanning trees and we prove that $\lambda_2(G) \geqslant d - \frac{5}{d+1}$.

By Nash-Williams/Tutte Theorem, there exists a partition of the vertex set of G into t subsets $X_1, \ldots, X_t$ such that

$$\sum_{1 \leqslant i < j \leqslant t} e(X_i, X_j) \leqslant 3(t-1) - 1 = 3t - 4.$$

It follows that $\sum_{i=1}^{t} r_i \leqslant 6t - 8$, where $r_i = e(X_i, V \setminus X_i)$.

If $r_i \leqslant 2$ for some i between 1 and t, then by results of [7], it follows that $\lambda_2(G) \geqslant d - \frac{4}{d+3} > d - \frac{5}{d+1}$.

Assume $r_i \geqslant 3$ for each $1 \leqslant i \leqslant t$ from now on. Let $a = |\{i : 1 \leqslant i \leqslant t, r_i = 3\}|$, $b = |\{i : 1 \leqslant i \leqslant t, r_i = 4\}|$ and $c = |\{i : 1 \leqslant i \leqslant t, r_i = 5\}|$. We get that

$$6t - 8 \geqslant r_1 + \cdots + r_t \geqslant 3a + 4b + 5c + 6(t - a - b - c)$$

which implies

$$3a + 2b + c \geqslant 8. \tag{11}$$

If for some $1 \leqslant i < j \leqslant t$, we have $e(X_i, X_j) = 0$ and $\max(r_i, r_j) \leqslant 5$, then eigenvalue interlacing (3) implies $\lambda_2(G) \geqslant \lambda_2(G[X_i \cup X_j]) \geqslant \min(\lambda_2(G[X_i]), \lambda_2(G[X_j])) \geqslant \min(\overline{d}(X_i), \overline{d}(X_j)) \geqslant d - \frac{5}{d+1}$ and we would be done. Thus, we may assume that

$$e(X_i, X_j) \geqslant 1 \tag{12}$$

for every $1 \leqslant i < j \leqslant t$ with $\max(r_i, r_j) \leqslant 5$. Similar arguments imply for example that

$$a + b + c \leqslant 6, \quad a + b \leqslant 5, \quad a \leqslant 4. \tag{13}$$

For the rest of the proof, we have to consider the following cases:

Case 1. $a \geqslant 2$.
The inequality $\sum_{1 \leqslant i < j \leqslant t} e(X_i, X_j) \leqslant 3t - 4$ implies $t \geqslant 3$.
As $a = |\{i : r_i = 3\}|$, assume without loss of generality that $r_1 = r_2 = 3$. Because G is connected, this implies $e(X_1, X_2) < 3$. Otherwise, $e(X_1 \cup X_2, V(G) \setminus (X_1 \cup X_2)) = 0$, contradiction.
If $e(X_1, X_2) = 2$, then $e(X_1 \cup X_2, V(G) \setminus (X_1 \cup X_2)) = 2$. Using the results in [7], this implies $\lambda_2(G) \geqslant d - \frac{4}{d+2} > d - \frac{5}{d+1}$ and finishes the proof.
Thus, we may assume $e(X_1, X_2) = 1$. Let $Y_3 = V(G) \setminus (X_1 \cup X_2)$. As $r_1 = r_2 = 3$, we deduce that $e(X_1, Y_3) = e(X_2, Y_3) = 2$. This means $e(Y_3, V(G) \setminus Y_3) = 4$ and since $d \geqslant 6$, this implies $n'_3 := |Y_3| \geqslant d + 1$.
Consider the partition of the vertex set of G into three parts: X_1, X_2 and Y_3. The quotient matrix of this partition is

$$B_3 = \begin{bmatrix} d - \frac{3}{n_1} & \frac{1}{n_1} & \frac{2}{n_1} \\ \frac{1}{n_2} & d - \frac{3}{n_2} & \frac{2}{n_2} \\ \frac{2}{n'_3} & \frac{2}{n'_3} & d - \frac{4}{n'_3} \end{bmatrix}.$$

The largest eigenvalue of B_3 is d. Eigenvalue interlacing and $n_1, n_2, n'_3 \geqslant d + 1$ imply

$$\lambda_2(G) \geqslant \lambda_2(B_3) \geqslant \frac{tr(B_3) - d}{2} \geqslant d - \frac{3}{2n_1} - \frac{3}{2n_2} - \frac{2}{n'_3}$$
$$\geqslant d - \frac{3}{2(d+1)} - \frac{3}{2(d+1)} - \frac{2}{d+1} = d - \frac{5}{d+1}.$$

This finishes the proof of this case.

Case 2. $a = 1$.
Inequalities (11) and (13) imply $2b+c \geqslant 5 \geqslant b+c$. Actually, because we assumed that $e(X_i, X_j) \geqslant 1$ for every $1 \leqslant i \neq j \leqslant t$ with $\max(r_i, r_j) \leqslant 5$, we deduce that $b + c \leqslant 3$. Otherwise, if $b + c \geqslant 4$, then there exists $i \neq j$ such that $r_i = 3$, $r_j \in \{4, 5\}$ and $e(X_i, X_j) = 0$.
The only solution of the previous inequalities is $b = 2$ and $c = 1$. Without loss of generality, we may assume $r_1 = 3$, $r_2 = r_3 = 4$ and $r_4 = 5$. Using the facts of the previous paragraph, we deduce that $e(X_1, X_j) = 1$ for each $2 \leqslant j \leqslant 4$ and $e(X_i, X_j) \geqslant 1$ for each $2 \leqslant i \neq j \leqslant 4$.
If $e(X_2, X_3) \geqslant 3$, then $e(X_2, X_4) = 0$ which is a contradiction with the first paragraph of this subcase.

If $e(X_2, X_3) = 2$, then $t \geqslant 5$ and $e(X_1 \cup X_2 \cup X_3 \cup X_4, V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)) = 2$. Using results from [7], it follows that $\lambda_2(G) \geqslant d - \frac{4}{d+2} > d - \frac{5}{d+1}$ which finishes the proof of this subcase.

If $e(X_2, X_3) = 1$, then there are some subcases to consider:

(1) If $e(X_2, X_4) = e(X_3, X_4) = 1$, then $t \geqslant 5$. If $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$, then $e(X_4, Y_5) = 2$, $e(X_3, Y_5) = e(X_2, Y_5) = 1$. These facts imply $e(Y_5, V(G) \setminus Y_5) = 4$ and $e(X_1, Y_5) = 0$. As $d \geqslant 6$, it follows that $n_5' := |Y_5| \geqslant d + 1$. Eigenvalue interlacing (3) implies

$$\lambda_2(G) \geqslant \lambda_2(G[X_1 \cup Y_5]) \geqslant \min(\lambda_1(G[X_1]), \lambda_1(G[Y_5])) \geqslant \min(\bar{d}(X_1), \bar{d}(Y_5))$$
$$\geqslant \min\left(d - \frac{3}{n_1}, d - \frac{4}{n_5'}\right) \geqslant d - \frac{4}{d+1} > d - \frac{5}{d+1}$$

which finishes the proof of this subcase.

(2) If $e(X_2, X_4) = 2$ and $e(X_3, X_4) = 1$, then $t \geqslant 5$. If $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$, then $e(X_4, Y_5) = e(X_3, Y_5) = 1$. These facts imply $e(Y_5, V(G) \setminus Y_5) = 2$. Using results in [7], we obtain $\lambda_2(G) > d - \frac{4}{d+2} > d - \frac{5}{d+1}$ which finishes the proof of this subcase.

(3) If $e(X_2, X_4) = 1$ and $e(X_3, X_4) = 2$, then the proof is similar to the previous case and we omit the details.

(4) If $e(X_2, X_4) = e(X_3, X_4) = 2$, then $t = 4$. Consider the partition of the vertex set of G into three parts: $X_1, X_2, X_3 \cup X_4$. The quotient matrix of this partition is

$$C_3 = \begin{bmatrix} d - \frac{3}{n_1} & \frac{1}{n_1} & \frac{2}{n_1} \\ \frac{1}{n_2} & d - \frac{4}{n_2} & \frac{3}{n_2} \\ \frac{3}{n_3'} & \frac{2}{n_3'} & d - \frac{5}{n_3'} \end{bmatrix}$$

where $n_3' = |X_3 \cup X_4| = |X_3| + |X_4| \geqslant 2(d+1)$.

The largest eigenvalue of C_3 is d. Eigenvalue interlacing and $n_1, n_2 \geq d + 1$, $n_3' \geq 2(d+1)$ imply

$$\lambda_2(G) \geqslant \lambda_2(C_3) \geqslant \frac{tr(C_3) - d}{2} \geqslant d - \frac{3}{2n_1} - \frac{2}{n_2} - \frac{5}{2n_3'}$$
$$\geqslant d - \frac{3}{2(d+1)} - \frac{2}{d+1} - \frac{5}{4(d+1)} \geq d - \frac{4.75}{d+1} > d - \frac{5}{d+1}.$$

Case 3. $a = 0$.

Inequalities (11) and (13) imply $2b + c \geqslant 8$, $b + c \leqslant 6$, $b \leqslant 5$.

If $b = 0$, then $c \geqslant 8$ and $c \leqslant 6$ which is a contradiction that finishes the proof of this subcase.

If $b = 1$, then $c \geqslant 6$ and $c \leqslant 5$ which is a contradiction that finishes the proof of our subcase.

If $b = 2$, then $c \geqslant 4$ which implies that there exists $i \neq j$ such that $e(X_i, X_j) = 0$ and $r_i = 4$ and $r_j \in \{4, 5\}$. This contradicts (12) and finishes the proof.

If $b = 3$, then $c \geqslant 2$. Assume that $c = 2$ first. Without loss of generality, assume $r_1 = r_2 = r_3 = 4$ and $r_4 = r_5 = 5$. Eq. (12) implies that $e(X_i, X_j) = 1$ for each $1 \leqslant i < j \leqslant 5$ except when $i = 4$ and $j = 5$ where $e(X_4, X_5) = 2$.

Consider the partition of the vertex set of G into three parts: X_1, $X_2 \cup X_3$, and $X_4 \cup X_5$. The quotient matrix of this partition is

$$D_3 = \begin{bmatrix} d - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n_2'} & d - \frac{6}{n_2'} & \frac{4}{n_2'} \\ \frac{2}{n_3'} & \frac{4}{n_3'} & d - \frac{6}{n_3'} \end{bmatrix}$$

where $n_2' = |X_2 \cup X_3| = |X_2| + |X_3| \geqslant 2(d+1)$ and $n_3' = |X_4 \cup X_5| = |X_4| + |X_5| \geqslant 2(d+1)$.

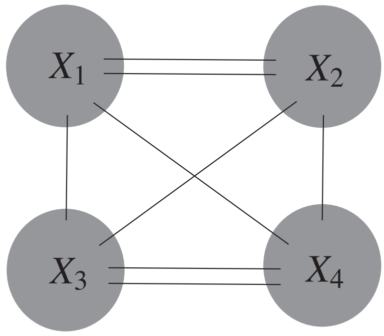

Fig. 2. The structure of G when $a = 0, b = 4, c = 0$, and $t = 4$.

The largest eigenvalue of D_3 is d. Eigenvalue interlacing and $n_1 \geq d+1, n_2', n_3' \geq 2(d+1)$ imply

$$\lambda_2(G) \geqslant \lambda_2(D_3) \geqslant \frac{tr(D_3) - d}{2} \geqslant d - \frac{2}{n_1} - \frac{3}{n_2'} - \frac{3}{n_3'}$$
$$\geqslant d - \frac{2}{d+1} - \frac{3}{2(d+1)} - \frac{3}{2(d+1)} = d - \frac{5}{d+1}.$$

This finishes the proof of this subcase.

If $c \geqslant 3$, then since $b = 3$, it follows that there exists $i \neq j$ such that $e(X_i, X_j) = 0$ and $r_i = 4$ and $r_j \in \{4, 5\}$. This contradicts (12) and finish the proof of this subcase.

If $b = 4$, we have inequality (13) implies $c \leqslant 2$. If $c = 2$, then there exist $i \neq j$ such that $e(X_i, X_j) = 0$, $r_i = 4$ and $r_j \in \{4, 5\}$. This contradicts (12) and finishes the proof of this subcase.

Suppose $c = 0$. Without loss of generality, assume that $r_i = 4$ for $1 \leqslant i \leqslant 4$. If $t = 4$, then (12) implies that the graph G is necessarily of the form shown in Fig. 2.

Consider the partition of the vertex set of G into three parts: $X_1, X_2, X_3 \cup X_4$. The quotient matrix of this partition is

$$E_3 = \begin{bmatrix} d - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n_2} & d - \frac{4}{n_2} & \frac{2}{n_2} \\ \frac{2}{n_3'} & \frac{2}{n_3'} & d - \frac{4}{n_3'} \end{bmatrix}$$

where $n_3' = |X_3 \cup X_4| = |X_3| + |X_4| \geqslant 2(d+1)$.

The largest eigenvalue of E_3 is d. Eigenvalue interlacing and $n_1, n_2 \geq d+1, n_3' \geq 2(d+1)$ imply

$$\lambda_2(G) \geqslant \lambda_2(E_3) \geqslant \frac{tr(E_3) - d}{2} \geqslant d - \frac{2}{n_1} - \frac{2}{n_2} - \frac{2}{n_3'}$$
$$\geqslant d - \frac{2}{d+1} - \frac{2}{d+1} - \frac{2}{2(d+1)} = d - \frac{5}{d+1}.$$

If $t \geqslant 5$, then there are two possibilities: either $e(X_i, X_j) = 1$ for each $1 \leqslant i < j \leqslant 4$ or without loss of generality, $e(X_i, X_j) = 1$ for each $1 \leqslant i < j \leqslant 4$ except for $i = 1$ and $j = 2$ where $e(X_1, X_2) = 2$.

In the first situation, if $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$, then $e(X_i, Y_5) = 1$ for each $1 \leqslant i \leqslant 4$ and thus, $e(Y_5, V(G) \setminus Y_5) = 4$. This implies $|Y_5| \geqslant d+1$. Consider the partition of $V(G)$ into three parts $X_1, X_2 \cup X_3, X_4 \cup Y_5$. The quotient matrix of this partition is

$$F_3 = \begin{bmatrix} d - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n_2'} & d - \frac{6}{n_2'} & \frac{4}{n_2'} \\ \frac{2}{n_3'} & \frac{4}{n_3'} & d - \frac{6}{n_3'} \end{bmatrix}$$

where $n_2' = |X_2 \cup X_3| = |X_2| + |X_3| \geqslant 2(d+1)$ and $n_3' = |X_4 \cup Y_5| = |X_4| + |Y_5| \geqslant 2(d+1)$.

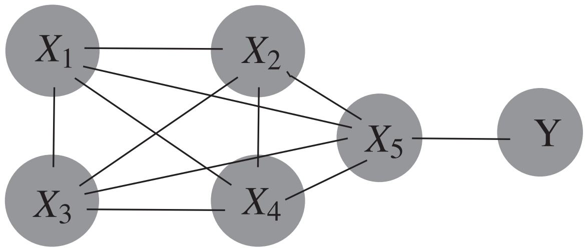

Fig. 3. The structure of G when $a = 0, b = 4, c = 1$, and $t \geq 5$.

The largest eigenvalue of F_3 is d. Eigenvalue interlacing and $n_1 \geq d + 1, n'_2, n'_3 \geq 2(d + 1)$ imply

$$\lambda_2(G) \geqslant \lambda_2(F_3) \geqslant \frac{tr(F_3) - d}{2} \geqslant d - \frac{2}{n_1} - \frac{3}{n'_2} - \frac{3}{n'_3}$$
$$\geqslant d - \frac{2}{d+1} - \frac{3}{2(d+1)} - \frac{3}{2(d+1)} = d - \frac{5}{d+1},$$

which finishes the proof of this subcase.

In the second situation, if $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$ then $e(X_1, Y_5) = e(X_2, Y_5) = 0$ and $e(X_3, Y_5) = e(X_4, Y_5) = 1$. This implies $e(Y_5, V(G) \setminus Y_5) = 2$. By results of [7], we deduce that $\lambda_2(G) \geqslant d - \frac{4}{d+2} > d - \frac{5}{d+1}$ which finishes the proof of this subcase.

Assume that $c = 1$. Without loss of generality, assume that $r_i = 4$ for $1 \leq i \leq 4$, and $r_5 = 5$. Our assumption (12) implies that the graph is necessarily of the form shown in Fig. 3, where Y is a component that necessarily joins to X_5. By results of [7], it follows that $\lambda_2(G) > d - \frac{2}{d+4} > d - \frac{5}{d+1}$ and this finishes the proof of this case.

If $b = 5$, then $c = 0$ by (12). Also, by (12), it follows that $t = 5$ and $e(X_i, X_j) = 1$ for each $1 \leqslant i < j \leqslant 5$. Consider the partition of the vertex set of G into three parts: $X_1, X_2 \cup X_3, X_4 \cup X_5$. The quotient matrix of this partition is

$$G_3 = \begin{bmatrix} d - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\ \frac{2}{n'_2} & d - \frac{6}{n'_2} & \frac{4}{n'_2} \\ \frac{2}{n'_3} & \frac{4}{n'_3} & d - \frac{6}{n'_3} \end{bmatrix},$$

which is identical to the quotient matrix F_3 in a previous case, which yields $\lambda_2(G) \geq d - \frac{5}{d+1}$.

If $b > 5$, then (12) will yield a contradiction. This finishes the proof of Theorem 1.2. $\square$

We show that our bound is essentially best possible by presenting a family of d-regular graphs $\mathcal{H}_d$ with $d - \frac{5}{d+1} \leq \lambda_2(\mathcal{H}_d) < d - \frac{5}{d+3}$ and $\sigma(\mathcal{H}_d) = 2$, for every $d \geqslant 6$.

For $d \geqslant 6$, consider the graph obtained from K_{d+1} by removing two disjoint edges. Consider now 5 vertex disjoint copies H_1, H_2, H_3, H_4, H_5 of this graph. For each copy H_i, $1 \leqslant i \leqslant 5$, denote the two pairs of non-adjacent vertices in H_i by a_i, c_i and b_i, d_i. Let $\mathcal{H}_d$ be the d-regular graph whose vertex set is $\cup_{i=1}^5 V(H_i)$ and whose edge set is the union $\cup_{i=1}^5 E(H_i)$ with the following set of 10 edges:

$$\{b_1a_2, b_2a_3, b_3a_4, b_4a_5, b_5a_1, c_1d_3, c_3d_5, c_5d_2, c_2d_4, c_4d_1\}.$$

The graph $\mathcal{H}_d$ is d-regular and has $5(d + 1)$ vertices. Fig. 4 depicts this external graph when $d = 10$. The partition of the vertex set of $\mathcal{H}_d$ into the five parts: $V(H_1), V(H_2), V(H_3), V(H_4), V(H_5)$ has the property that the number of edges between the parts equals $10 < 12 = 3(5 - 1)$. By Nash-Williams/Tutte Theorem, this implies $\sigma(\mathcal{H}_d) < 3$.

For $d \geqslant 6$, denote by γ_d the largest root of the polynomial

$$x^{10} + (8 - 2d)x^9 + (d^2 - 16d + 30)x^8 + (8d^2 - 50d + 58)x^7 + (20d^2 - 66d + 36)x^6$$
$$+ (8d^2 + 18d - 70)x^5 + (-29d^2 + 140d - 146)x^4 + (-20d^2 + 57d - 21)x^3$$
$$+ (14d^2 - 83d + 109)x^2 + (4d^2 - 13d + 5)x - d^2 + 5d - 5.$$

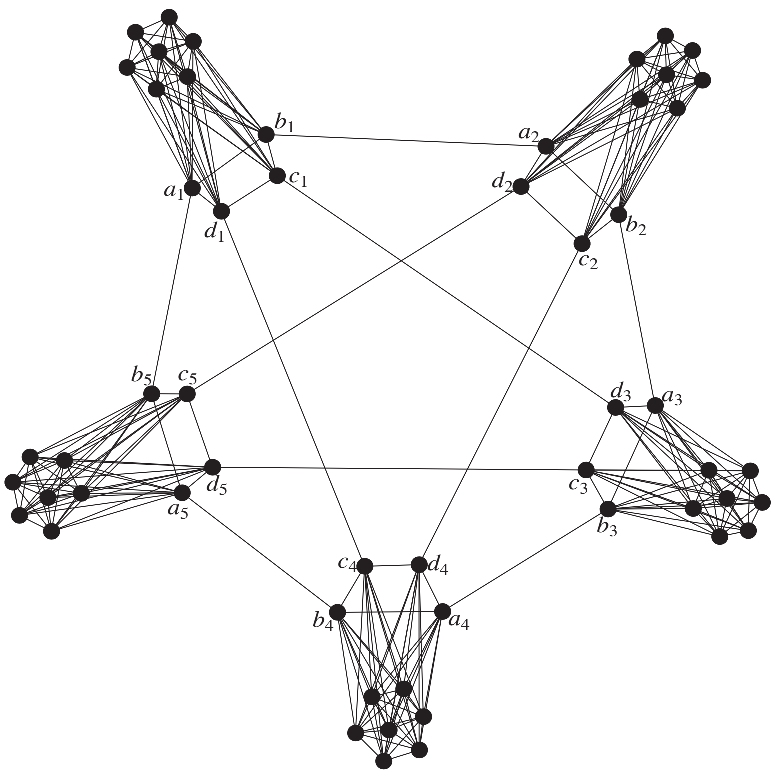

Fig. 4. The 10-regular graph $\mathcal{H}_{10}$ with $\sigma\,(\mathcal{H}_{10}) = 2$ and $9.545 \approx 10 - \frac{5}{10+1} < \lambda_2\,(H_{10}) \approx 9.609 < 10 - \frac{5}{10+3} \approx 9.615$.

Lemma 3.1. *For every integer $d \geqslant 6$, the second largest eigenvalue of $\mathcal{H}_d$ is γ_d.*

Proof. Consider the following partition of the vertex set of $\mathcal{H}_d$ into 25 parts: 5 parts of the form $V(H_i) \setminus \{a_i, b_i, c_i, d_i\}$, $i = 1, 2, 3, 4, 5$. The remaining 20 parts consist of the 20 individual vertices $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, i = 1, 2, 3, 4, 5$. This partition is equitable and the characteristic polynomial of its quotient matrix (which is described in Section 3) is

$$\begin{aligned} P_{25}(x) = {} & (x-d)(x-1)(x+1)^2(x+3)[x^{10} + (8-2d)x^9 + (d^2-16d+30)x^8 \\ & + (8d^2-50d+58)x^7 + (20d^2-66d+36)x^6 + (8d^2+18d-70)x^5 \\ & + (-29d^2+140d-146)x^4 + (-20d^2+57d-21)x^3 + (14d^2-83d+109)x^2 \\ & + (4d^2-13d+5)x - d^2 + 5d - 5]^2. \end{aligned}$$

Let $\lambda_2 \geqslant \lambda_3 \geqslant \cdots \geq \lambda_{11}$ denote the solutions of the degree 10 polynomial $P_{10}(x)$. Because the partition is equitable, it follows that these 10 solutions, d, 1, -1, and -3 are eigenvalues of $\mathcal{H}_d$, including multiplicity.

We claim the spectrum of $\mathcal{H}_d$ is

$$d^{(1)}, 1^{(1)}, -3^{(1)}, -1^{(5d-18)}, \lambda_i^{(2)} \quad \text{for } i = 2, 3, \ldots, 11. \tag{14}$$

It suffices to obtain $5d - 18$ linearly independent eigenvectors corresponding to -1. Consider two distinct vertices u_1^1 and u_2^1 in $V(H_1) \setminus \{a_1, b_1, c_1, d_1\}$. Define a vector where the entry corresponding to u_1^1 is 1, the entry corresponding to u_2^1 is -1, and all other entries are 0. This is an eigenvector corresponding to the eigenvalue -1. We can create $d - 4$ eigenvectors by letting u_2^1 to be each of the $d - 4$ vertices in $V(H_1) \setminus \{a_1, b_1, c_1, d_1, u_1^1\}$. This can also be applied to 2 vertices u_1^i, u_2^i in $V(H_i) \setminus \{a_i, b_i, c_i, d_i\}$, for $i = 2, 3, 4, 5$. This way, we obtain a total of $5d - 20$ linearly independent eigenvectors corresponding to the eigenvalue -1.

Furthermore, define a vector whose entry at some fixed vertex $u_1^i \in V(H_i) \setminus \{a_i, b_i, c_i, d_i\}$ is -2, whose entries at a_i and d_i are 1, for each $1 \leqslant i \leqslant 5$ and whose remaining entries are 0. Define another vector whose entries at a fixed vertex $u_1^i \in V(H_i) \setminus \{a_i, b_i, c_i, d_i\}$ is -2, whose entries at b_i and c_i are 1, for each $1 \leqslant i \leqslant 5$ and whose remaining entries are 0. These last two vectors are also eigenvectors corresponding to the eigenvalue -1. It is easy to check that all these $5d - 18$ vectors we have constructed are linearly independent eigenvectors corresponding to the eigenvalue -1. By obtaining the entire spectrum of $\mathcal{H}_d$, we conclude that the second largest eigenvalue of $\mathcal{H}_d$ is γ_d. □

Lemma 3.2. *For every integer $d \geqslant 6$,*

$$d - \frac{5}{d+1} \leq \gamma_d < d - \frac{5}{d+3}.$$

Proof. The lower bound follows directly from Theorem 1.2 as $\sigma(\mathcal{H}_d) < 3$. Moreover, by some technical calculations (done in Mathematica and included in Section 3)

$$P_{10}^{(n)}\left(d - \frac{5}{d+3}\right) > 0, \quad \text{for } n = 0, 1, \ldots, 10.$$

Descartes' Rule of Signs implies $\gamma_d < d - \frac{5}{d+3}$. Hence,

$$d - \frac{5}{d+1} \leq \gamma_d < d - \frac{5}{d+3} \tag{15}$$

for every $d \geqslant 6$. □

4. Final remarks

In this paper, we studied the relations between the eigenvalues of a regular graph and its spanning tree packing number. Based on the results contained in this paper, we make the following conjecture.

Conjecture 4.1. *Let $d \geqslant 8$ and $4 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$ be two integers. If G is a d-regular graph such that $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then G contains at least k edge-disjoint spanning trees.*

Let $\omega(H)$ denote the number of components of the graph H. The vertex-toughness of G is defined as $\min \frac{|S|}{\omega(G \setminus S)}$, where the minimum is taken over all subsets of vertices S whose removal disconnects G. Alon [1] and independently, Brouwer [2] have found close relations between the eigenvalues of a regular graph and its vertex-toughness. These connections were used by Alon in [1] to disprove a conjecture of Chvátal that a graph with sufficiently large vertex-toughness is pancyclic. For $c \geqslant 1$, the higher order edge-toughness $\tau_c(G)$ is defined as

$$\tau_c(G) := \min \frac{|X|}{\omega(G \setminus X) - c}$$

where the minimum is taken over all subsets X of edges of G with the property $\omega(G \setminus X) > c$ (see Chen et al. [6] or Catlin et al. [5] for more details). The Nash-Williams/Tutte Theorem states that $\sigma(G) = \lfloor \tau_1(G) \rfloor$. Cunningham [8] generalized this result and showed that if $\tau_1(G) \geqslant \frac{p}{q}$ for some natural numbers p and q, then G contains p spanning trees (repetitions allowed) such that each edge of G lies in at most q of the p trees. Chen et al. [6] proved that $\tau_c(G) \geqslant k$ if and only if G contains at least c edge-disjoint forests with exactly c components. It would be interesting to find connections between the eigenvalues of the adjacency matrix (or of the Laplacian) of a graph G and $\tau_c(G)$.

Another question of interest is to determine sufficient eigenvalue condition for the existence of *nice* spanning trees in pseudorandom graphs. A lot of work has been done on this problem in the case of random graphs (see Krivelevich [14] for example).

Acknowledgment

We thank the referee for some useful remarks.

Appendix A. Calculations for Lemma 3.2

A.1. Justify characteristic polynomial in 25 parts

The following is the characteristic polynomial of the equitable partition in 25 parts:

$$\text{Factor}\left[\text{Characteristic Polynomial}\left[\begin{pmatrix}
d-4 & 0 & 0 & 0 & 0 & 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0 & d-4 & 0 & 0 & 0 & 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0 & 0 & d-4 & 0 & 0 & 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0 & 0 & 0 & d-4 & 0 & 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0 & 0 & 0 & 0 & d-4 & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
d-3 & 0 & 0 & 0 & 0 & 0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
d-3 & 0 & 0 & 0 & 0 & 1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
d-3 & 0 & 0 & 0 & 0 & 0&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
d-3 & 0 & 0 & 0 & 0 & 1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0 & d-3 & 0 & 0 & 0 & 0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0 & d-3 & 0 & 0 & 0 & 0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0 & d-3 & 0 & 0 & 0 & 0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0 & d-3 & 0 & 0 & 0 & 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0 & 0 & d-3 & 0 & 0 & 0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0 & 0 & d-3 & 0 & 0 & 0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0 & 0 & d-3 & 0 & 0 & 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1\\
0 & 0 & d-3 & 0 & 0 & 0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0 & 0 & 0 & d-3 & 0 & 0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0 & 0 & 0 & d-3 & 0 & 0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\
0 & 0 & 0 & d-3 & 0 & 0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0 & 0 & 0 & d-3 & 0 & 0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0 & 0 & 0 & 0 & d-3 & 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1\\
0 & 0 & 0 & 0 & d-3 & 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0 & 0 & 0 & 0 & d-3 & 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1\\
0 & 0 & 0 & 0 & d-3 & 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0
\end{pmatrix}, x\right]\right]$$

$$(d-x)(-1+x)(1+x)^2(3+x)(-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2$$
$$+14d^2x^2-21x^3+57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5$$
$$+8d^2x^5+36x^6-66x^6+20d^2x^6+58x^7-50dx^7+8d^2x^7$$
$$+30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10})^2$$

A.2. Justify $P_{10}^{(n)}\left(d-\frac{5}{d+3}\right) > 0$, *for* $n = 0, 1, \ldots, 10$.

A.2.1. $n = 0$

$$\text{Factor}\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+\\ 57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+\\ 8d^2x^5+36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+\\ 30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10}\end{array} /.x \to d-5/(d+3)\right]$$

$$\frac{5\left(\begin{array}{c}209081+2789848d+4225996d^2-7988400d^3-2586890d^4+3149694d^5+1156227d^6\\-317856d^7-185275d^8-9630d^9+7239d^{10}+1412d^{11}+79d^{12}\end{array}\right)}{(3+d)^{10}}$$

Looking at the numerator,

$$\begin{aligned}&209081+2789848d+4225996d^2-7988400d^3-2586890d^4+3149694d^5+1156227d^6\\&\quad-317856d^7-185275d^8-9630d^9+7239d^{10}+1412d^{11}+79d^{12}\\&\geqslant 209081+2789848d+4225996d^2-7988400d^3-2586890d^4+3149694\left(6^2\right)d^3\\&\quad+1156227\left(6^2\right)d^4-317856d^7-185275d^8-9630d^9+7239\left(6^3\right)d^7+1412\left(6^3\right)d^8\\&\quad+79\left(6^3\right)d^9\\&=209081+2789848d+4225996d^2+105400584d^3+39037282d^4+1245768d^7\\&\quad+119717d^8+7434d^9>0.\end{aligned}$$

A.2.2. $n=1$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+57dx^3-\\20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+36x^6-\\66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+\\8x^9-2dx^9+x^{10}\end{array},x\right]/.x\to d-5/(d+3)\right]\right]$$

$$-154125-6265d+9235d^2-1605d^3-80d^4+40d^5-\frac{19531250}{(3+d)^9}-\frac{56250000}{(3+d)^8}$$
$$-\frac{43125000}{(3+d)^7}+\frac{14000000}{(3+d)^6}+\frac{26231250}{(3+d)^5}+\frac{250000}{(3+d)^4}-\frac{6723000}{(3+d)^3}-\frac{224000}{(3+d)^2}+\frac{981525}{3+d}$$

Looking at the fraction terms,

$$\text{Together}\left[-\frac{19531250}{(3+d)^9}-\frac{56250000}{(3+d)^8}-\frac{43125000}{(3+d)^7}+\frac{14000000}{(3+d)^6}+\frac{26231250}{(3+d)^5}\right.$$
$$\left.+\frac{250000}{(3+d)^4}-\frac{6723000}{(3+d)^3}-\frac{224000}{(3+d)^2}+\frac{981525}{3+d}\right]$$

$$\frac{25\left(121436221+368991216d+491609352d^2+377696288d^3+179037720d^4+52838632d^5+9436692d^6+933304d^7+39261d^8\right)}{(3+d)^9}$$

The expression is positive. The only concern now are the terms $-154125-6265d+9235d^2-1605d^3-80d^4+40d^5$. Direct calculations for $d=6$ and 7 yield the values 1425 and 184220, respectively. For $d\geq 8$,

$$\begin{aligned}&(-154125-6265d+9235d^2)-1605d^3-80d^4+40d^5\\&\quad=9235d+(d-1)(9235)d-6265d-154125+80d^4+(d-2)(40)d^4-80d^4-1605d^3\\&\quad\geq 9235d+(7)(9235)(8)-6265d-154125+80d^4+(6)(40)(8)d^3-80d^4-1605d^3\\&\quad=(1920-1605)d^3+(9235-6265)d+(517160-154125)>0.\end{aligned}$$

A.2.3. $n=2$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+\\14d^2x^2-21x^3+57dx^3-20d^2x^3-146x^4+\\140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+\\36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+\\30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10}\end{array},\{x,2\}\right]/.x\to d-5/(d+3)\right]\right]$$

$$-501172+218908d-37582d^2-2480d^3+2472d^4-344d^5-60d^6+16d^7+2d^8+\frac{35156250}{(3+d)^8}$$
$$+\frac{90000000}{(3+d)^7}+\frac{54750000}{(3+d)^6}-\frac{30800000}{(3+d)^5}-\frac{34412500}{(3+d)^4}+\frac{4300000}{(3+d)^3}+\frac{8668800}{(3+d)^2}-\frac{574400}{3+d}$$

Looking at the fraction terms and $2d^8$,

$$\text{Together}\left[\tfrac{35156250}{(3+d)^8}+\tfrac{90000000}{(3+d)^7}+\tfrac{54750000}{(3+d)^6}-\tfrac{30800000}{(3+d)^5}-\tfrac{34412500}{(3+d)^4}+\tfrac{4300000}{(3+d)^3}+\tfrac{8668800}{(3+d)^2}-\tfrac{574400}{3+d}+2d^8\right]$$

$$\frac{1}{(3+d)^8}2\begin{pmatrix} 1643568075+3659898600d+3340851900d^2+1497989000d^3+328783750d^4 \\ +25888400d^5-1696800d^6-287200d^7+6561d^8+17496d^9+20412d^{10} \\ +13608d^{11}+5670d^{12}+1512d^{13}+252d^{14}+24d^{15}+d^{16}\end{pmatrix}$$

By comparing terms, the expression is positive. The only concern now are the terms $-501172+218908d-37582d^2-2480d^3+2472d^4-344d^5-60d^6+16d^7$. Direct calculations for $d=6$ and 7 yield the values 1132028 and 4610438, respectively. Clearly we have for the first two terms that $-501172+218908d>0$. Now assume $d\geq 8$. Looking at the next three terms,

$$-37582d^2-2480d^3+2472d^4=4944d^3+(d-2)(2472)d^3-2480d^3-37583d^2$$

$$\geqslant 4944d^3+(6)(2472)(8)d^2-2480d^3-37583d^2>0$$

For the final three terms,

$$\begin{aligned}-344d^5-60d^6+16d^7&=64d^6+16(d-4)d^6-60d^6-344d^5\\&\geqslant 64d^6+16(4)(8)d^5-60d^6-344d^5>0.\end{aligned}$$

A.2.4. $n=3$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{matrix}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+ \\ 14d^2x^2-21x^3+57dx^3-20d^2x^3-146x^4+ \\ 140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+ \\ 36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+ \\ 30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10}\end{matrix}, \{x,3\}\right]/.x\to d-5/(d+3)\right]\right]$$

$$2377554-293322d-71280d^2+40944d^3-5340d^4-1380d^5+336d^6+48d^7-\tfrac{56250000}{(3+d)^7}-\tfrac{126000000}{(3+d)^6}-\tfrac{56700000}{(3+d)^5}+\tfrac{50400000}{(3+d)^4}+\tfrac{35947500}{(3+d)^3}-\tfrac{10020000}{(3+d)^2}-\tfrac{8360520}{3+d}$$

Looking at the fraction terms and $48d^7$,

$$\text{Together}\left[-\tfrac{56250000}{(3+d)^7}-\tfrac{126000000}{(3+d)^6}-\tfrac{56700000}{(3+d)^5}+\tfrac{50400000}{(3+d)^4}+\tfrac{35947500}{(3+d)^3}-\tfrac{10020000}{(3+d)^2}-\tfrac{8360520}{3+d}+48d^7\right]$$

$$\frac{1}{(3+d)^7}12\begin{pmatrix} -433473465-955900680d-877113900d^2-411225900d^3-103585225d^4 \\ -13375780d^5-696710d^6+8748d^7+20412d^8+20412d^9+11340d^{10} \\ +3780d^{11}+756d^{12}+84d^{13}+4d^{14}\end{pmatrix}$$

Looking at the numerator,

$$-433473465-955900680d-877113900d^2-411225900d^3-103585225d^4-13375780d^5$$
$$-696710d^6+8748d^7+20412d^8+20412d^9+11340d^{10}+3780d^{11}+756d^{12}+84d^{13}+4d^{14}$$

$$
\begin{aligned}
&\geq -433473465 - 955900680d - 877113900d^2 - 411225900d^3 - 103585225d^4 \\
&\quad -13375780d^5 - 696710d^6 + 8748d^7 + 20412\left(6^8\right) \\
&\quad +20412\left(6^8\right)d + 11340\left(6^8\right)d^2 + 3780\left(6^8\right)d^3 + 756\left(6^8\right)d^4 + 84\left(6^8\right)d^5 + 4\left(6^8\right)d^6 \\
&= 33850848327 + 33328421112d + 18169731540d^2 + 5937722580d^3 + 1166204471d^4 \\
&\quad +127711964d^5 + 6021754d^6 + 8748d^7 > 0.
\end{aligned}
$$

The only concern now are the terms $2377554-293322d-71280d^2+40944d^3-5340d^4-1380d^5+336d^6$. Direct calculations for $d = 6$ and 7 yield the values 4920342 and 14390436, respectively. We ignore the first positive constant, and assume $d \geq 8$. Looking at the next three terms,

$$
\begin{aligned}
-293322d - 71280d^2 + 40944d^3 &= 81888d^2 + (d-2)(40944)d^2 - 71280d^2 - 293322d \\
&\geq 81888d^2 + (6)(40944)(8)d - 71280d^2 - 293322d > 0
\end{aligned}
$$

For the final three terms,

$$
\begin{aligned}
-5340d^4 - 1380d^5 + 336d^6 &= 1680d^5 + (d-5)336d^5 - 1380d^5 - 5340d^4 \\
&\geq 1680d^5 + (3)336(8)d^4 - 1380d^5 - 5340d^4 > 0.
\end{aligned}
$$

A.2.5. $n = 4$

$$
\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c} -5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+ \\ 57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+ \\ 36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+ \\ 8x^9-2dx^9+x^{10} \end{array}, \{x,4\}\right] /.x \to d-5/(d+3)\right]\right]
$$

$$
-285504-1017840d+396024d^2-41280d^3-18000d^4+4032d^5+672d^6+\tfrac{78750000}{(3+d)^6}+\tfrac{151200000}{(3+d)^5} +\tfrac{44100000}{(3+d)^4}-\tfrac{63840000}{(3+d)^3}-\tfrac{28026000}{(3+d)^2}+\tfrac{13488000}{3+d}
$$

Looking at the fraction terms and $672d^6$,

$$
\text{Together}\left[672d^6+\tfrac{78750000}{(3+d)^6}+\tfrac{151200000}{(3+d)^5}+\tfrac{44100000}{(3+d)^4}-\tfrac{63840000}{(3+d)^3}-\tfrac{28026000}{(3+d)^2}+\tfrac{13488000}{3+d}\right]
$$

$$
\frac{48\left(\begin{array}{c} 4438500+23499000d+33289500d^2+16953500d^3+3631125d^4 \\ +281000d^5+10206d^6+20412d^7+17010d^8+7560d^9+1890d^{10}+252d^{11}+14d^{12} \end{array}\right)}{(3+d)^6}
$$

This expression is positive. The only concern now are the terms $-285504-1017840d+396024d^2-41280d^3-18000d^4+4032d^5$. Direct calculations for $d = 6$ and 7 yield the values 6972672 and 22383576, respectively. Now assume $d \geq 8$. Looking at the first 3 terms,

$$
\begin{aligned}
-285504 - 1017840d + 396024d^2 &= 1188072d + (d-3)396024d - 1017840d - 285504 \\
&\geqslant 1188072d + (5)396024(8) - 1017840d - 285504 > 0.
\end{aligned}
$$

For the final three terms,

$$
\begin{aligned}
-41280d^3 - 18000d^4 + 4032d^5 &= 20160d^4 + (d-5)4032d^4 - 18000d^4 - 41280d^3 \\
&\geqslant 20160d^4 + (3)4032(8)d^3 - 18000d^4 - 41280d^3 > 0.
\end{aligned}
$$

A.2.6. $n = 5$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+\\57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+\\36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+\\8x^9-2dx^9+x^{10}\end{array},\{x,5\}\right]/.x\to d-5/(d+3)\right]\right]$$

$$-8576400+2476080d-152400d^2-162000d^3+33600d^4+6720d^5-\frac{94500000}{(3+d)^5}-\frac{151200000}{(3+d)^4}-\frac{20160000}{(3+d)^3}+\frac{61824000}{(3+d)^2}+\frac{14024400}{3+d}$$

Looking at the fraction terms,

$$\text{Together}\left[-\frac{94500000}{(3+d)^5}-\frac{151200000}{(3+d)^4}-\frac{20160000}{(3+d)^3}+\frac{61824000}{(3+d)^2}+\frac{14024400}{3+d}\right]$$

$$\frac{1200\left(1729737+2426436d+1077978d^2+191764d^3+11687d^4\right)}{(3+d)^5}$$

This expression is positive. The only concern now are the terms $-8576400+2476080d-152400d^2-162000d^3+33600d^4+6720d^5$. Clearly for the first two terms we have $-8576400+2476080d > 0$ for $d \geq 6$. Looking at the four remaining terms,

$$\begin{aligned}&-152400d^2-162000d^3+33600d^4+6720d^5\\&\geqslant -152400d^2-162000d^3+33600(36)d^2+6720(36)d^3\\&=-152400d^2-162000d^3+1209600d^2+241920d^3>0.\end{aligned}$$

A.2.7. $n = 6$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+\\57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+\\36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+\\8x^9-2dx^9+x^{10}\end{array},\{x,6\}\right]/.x\to d-5/(d+3)\right]\right]$$

$$9349920+244800d-1044000d^2+201600d^3+50400d^4+\frac{94500000}{(3+d)^4}+\frac{120960000}{(3+d)^3}-\frac{3024000}{(3+d)^2}-\frac{43545600}{3+d}$$

Looking at the fraction terms and $50400d^4+9349920+244800d$,

$$\text{Together}\left[\frac{94500000}{(3+d)^4}+\frac{120960000}{(3+d)^3}-\frac{3024000}{(3+d)^2}-\frac{43545600}{3+d}+50400d^4+9349920+244800d\right]$$

$$\frac{1440(8178-30066d+94722d^2+56856d^3+11368d^4+3950d^5+1890d^6+420d^7+35d^8)}{(3+d)^4}$$

This expression is clearly positive for $d \geq 6$. The only terms left are $-1044000d^2+201600d^3$, and we get

$$-1044000d^2+201600d^3 \geq -1044000d^2+201600(6)d^2 \geq -1044000d^2+1209600d^2 > 0.$$

A.2.8. $n = 7$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c}-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+\\57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+\\36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+\\8x^9-2dx^9+x^{10}\end{array},\{x,7\}\right]/.x\to d-5/(d+3)\right]\right]$$

$$5937120-4687200d+846720d^2+282240d^3-\frac{75600000}{(3+d)^3}-\frac{72576000}{(3+d)^2}+\frac{13305600}{3+d}$$

Looking at the fraction terms and $8282240d^3$, Together $\left[8282240d^3 - \frac{75600000}{(3+d)^3} - \frac{72576000}{(3+d)^2} + \frac{13305600}{3+d}\right]$

$$\frac{640(-271215+11340d+20790d^2+349407d^3+349407d^4+116469d^5+12941d^6)}{(3+d)^3}$$

This expression is clearly positive for $d \geq 6$. The only remaining terms are $5937120 - 4687200d + 846720d^2$. We have

$$\begin{aligned} 5937120 - 4687200d + 846720d^2 &\geq 5937120 - 4687200d + 846720(6)d \\ &= 5937120 - 4687200d + 5080320d > 0. \end{aligned}$$

A.2.9. $n = 8$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c} -5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+ \\ 57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+ \\ 36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+ \\ 8x^9-2dx^9+x^{10} \end{array}, \{x, 8\}\right] /.x \to d - 5/(d+3)\right]\right]$$

$$-13305600 + 2257920d + 1128960d^2 + \frac{45360000}{(3+d)^2} + \frac{29030400}{3+d}$$

At $d = 6$, the value is 44670080. Clearly the expression is increasing for $d \geq 6$, and hence always positive for $d \geq 6$.

A.2.10. $n = 9$

$$\text{Apart}\left[\text{FullSimplify}\left[D\left[\begin{array}{c} -5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+ \\ 57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+ \\ 36x^6-66dx^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+ \\ 8x^9-2dx^9+x^{10} \end{array}, \{x, 9\}\right] /.x \to d - 5/(d+3)\right]\right]$$

$$2903040 + 2903040d - \frac{18144000}{3+d}$$

At $d = 6$, the value is 18305280. Clearly the expression is increasing for $d \geq 6$, and hence always positive for $d \geq 6$.

A.2.11. $n = 10$

The value will be $10! > 0$.

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