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On the connectedness of the complement of a ball in distance-regular graphs

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Abstract An important property of strongly regular graphs is that the second subconstituent of any primitive strongly regular graph is always connected. Brouwer asked to what extent this statement can be generalized to distance-regular graphs. In this paper, we show that if γ is any vertex of a distance-regular graph Γ and t is the index where the standard sequence corresponding to the second largest eigenvalue of Γ changes sign, then the subgraph induced by the vertices at distance at least t from γ, is connected.

Keywords Distance-regular graph · Strongly regular graph · Subconstituent · Connectivity · Eigenvalue · Standard sequence

1 Introduction

Let Γ be a distance-regular graph of diameter d. (For notation and definitions related to distance-regular graphs, see [4].) For primitive strongly regular graphs (the case $d = 2$) it is known that $\Gamma_2(\gamma)$, the subgraph of Γ induced by the vertices at distance 2 from a given vertex γ (also called the second subconstituent of γ), is connected. See [4, p. 86] or [3, Proposition 9.3.1] for an eigenvalue proof due to Haemers or [6] for a combinatorial proof of this fact. The properties of the second subconstituents of strongly regular graphs have been studied by many authors [3–6, 8] and their connectedness is a very important property of strongly regular graphs (cf. [3, Sect. 12.6]).

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During the GAC5 conference [2] (see also [1]), Andries Brouwer asked whether this could be generalized to a statement that for general Γ and suitable t, the subgraph $\Gamma_{\geq t}(\gamma)$ is connected, where $\Gamma_{\geq t}(\gamma)$ is the subgraph of Γ induced by the vertices of distance at least t to γ. In this paper, we show that one can take for t the position where the standard sequence corresponding to the second largest eigenvalue changes sign.

2 Results

Let the distance-regular graph Γ have intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$ and put $k = b_0$, $a_i = k - b_i - c_i$ as usual. Define the tridiagonal matrices

$$L_i = \begin{bmatrix} 0 & b_0 & & & & & \\ c_1 & a_1 & b_1 & & & & \\ & c_2 & a_2 & b_2 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & c_{i-1} & a_{i-1} & b_{i-1} \\ & & & & & c_i & a_i \end{bmatrix}$$

and

$$M_i = \begin{bmatrix} a_i & b_i & & & & \\ c_{i+1} & a_{i+1} & b_{i+1} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{bmatrix}.$$

Let ρ_i be the largest eigenvalue of L_i, and let σ_i be the largest eigenvalue of M_i. For $\gamma \in V(\Gamma)$, $\Gamma_{\leq i}(\gamma)$ denotes the subgraph of Γ induced by the vertices of distance at most i from γ and $\Gamma_{\geq i}(\gamma)$ denotes the subgraph of Γ induced by the vertices at distance at least i from γ. Then ρ_i is the largest eigenvalue of the subgraphs $\Gamma_{\leq i}(\gamma)$ of Γ, and σ_i is the largest eigenvalue of the subgraphs $\Gamma_{\geq i}(\gamma)$ of Γ.

Proposition 1 (i) *The ρ_i are increasing*: $\rho_i < \rho_j$ *when* $0 \leq i < j \leq d$.
(ii) *The σ_i are decreasing*: $\sigma_i > \sigma_j$ *when* $0 \leq i < j \leq d$.
(iii) $\rho_i \leq \sigma_{d-i}$ *for* $0 \leq i \leq d$.

Proof The Perron–Frobenius Theorem (see [4, Sect. 3.1], [7, Sect. 2.6] or [8, Sect. 8.8]) tells us that the largest eigenvalue of a graph is at least as large as the largest eigenvalue of a subgraph, and this gives (iii) because $\Gamma_{\leq i}(\delta)$ is a subgraph of $\Gamma_{\geq d-i}(\gamma)$ when $d(\gamma, \delta) = d$. That theorem moreover says that the inequality is strict for a proper subgraph of a connected graph, and this gives (i) because all $\Gamma_{\leq i}(\delta)$ are connected, and also (ii) because each connected component of $\Gamma_{\geq j}(\gamma)$ is a proper subgraph of some connected component of $\Gamma_{\geq i}(\gamma)$. □

Let $L = L_d = M_0$. Let Γ have the $d + 1$ distinct eigenvalues $k = \theta_0 > \theta_1 > \dots > \theta_d$. Then the θ_i are the eigenvalues of the matrix L. The *standard sequence*

$u = (u_0, \ldots, u_d)^\top$ of an eigenvalue θ of Γ is the corresponding right eigenvector of L, so that $Lu = \theta u$, normalized such that $u_0 = 1$. From now on, let u denote the standard sequence of the second largest eigenvalue θ_1 of Γ. It is known that it has precisely one sign change (see [4, Chap. 4]). In particular, $u_d < 0$.

Proposition 2 *Let* $1 \le t \le d-1$.

(i) *If* $u_t = 0$, *then* $\rho_{t-1} = \theta_1 = \sigma_{t+1}$.
(ii) *If* $u_t > 0$, *then* $\rho_{t-1} < \theta_1 < \sigma_{t+1}$.
(iii) *If* $u_t < 0$, *then* $\rho_{t-1} > \theta_1 > \sigma_{t+1}$.'

Proof Since the ρ_i are increasing, and the σ_i are decreasing, when proving (ii) we may assume that t is the largest index for which $u_t > 0$, and when proving (iii) we may assume that t is the smallest index for which $u_t < 0$.

Let $x = (u_0, \ldots, u_{t-1})^\top$ and $y = (u_{t+1}, \ldots, u_d)^\top$. Now $v := L_{t-1}x - \theta_1 x = (0, \ldots, 0, -b_{t-1}u_t)^\top$, and the last component of v is zero, negative, resp. positive in the three cases of the proposition. Let $z^\top$ be a positive left eigenvector of L_{t-1} for its largest eigenvalue ρ_{t-1}. We see that $(\rho_{t-1} - \theta_1)z^\top x = z^\top v$ is zero, negative, positive in the three cases, while $z^\top x > 0$. This proves the left-hand inequalities.

Similarly, $w := M_{t+1}y - \theta_1 y = (-c_{t+1}u_t, 0, \ldots, 0)^\top$, and the first component of w is zero, negative, resp. positive in the three cases of the proposition. Let $z^\top$ be a positive left eigenvector of M_{t+1} for its largest eigenvalue σ_{t+1}. We see that $(\sigma_{t+1} - \theta_1)z^\top y = z^\top w$ is zero, negative, positive in the three cases, while $z^\top y < 0$. This proves the right-hand inequalities. □

Since $\rho_i \le \sigma_j$ when $i + j \le d$, and $\rho_i < \sigma_j$ when $i + j < d$, it follows that $t > d/2$ when $u_t < 0$, and $t \ge d/2$ when $u_t \le 0$. See also [10] for some related results.

Theorem 3 *Let* Γ *be a distance-regular graph of diameter* $d > 0$. *Let* $u = (u_0, \ldots, u_d)^\top$ *be the standard sequence corresponding to the second largest eigenvalue of* Γ. *If* $u_{t-1} > 0$, *then* $\Gamma_{\ge t}(\gamma)$ *is connected for each vertex* γ *of* Γ.

Proof Let $H := \Gamma_{\ge t}(\gamma)$, and let A_H be its adjacency matrix. Let C be a connected component of H. Let z be a positive eigenvector of M_t (indexed by $\{t, \ldots, d\}$) for its largest eigenvalue σ_t. The vector y defined by $y_\delta = z_{d(\gamma,\delta)}$ for $\delta \in C$, and $y_\delta = 0$ for $\delta \in V(H) \setminus C$ is an eigenvector of H with eigenvalue $\sigma_t > \theta_1$. By eigenvalue interlacing (see [4, Sect. 3.3], [3, Sect. 2.5], [7, Chap. 5] or [8, Chap. 9]), H can have at most a single eigenvalue larger than θ_1, so H has only one connected component. □

3 Final remarks and open problems

The result stated in the introduction that the second subconstituent of a primitive strongly regular graph is connected, can be deduced from our work. The standard sequence $u = (u_0, u_1, u_2)$ corresponding to the second largest eigenvalue θ_1 of connected strongly regular graph has a sign change at u_1 (as $u_0 = 1, u_1 = \theta_1/k$ and u_2

is negative) unless $\theta_1 \leq 0$. This happens if and only if the connected strongly regular graph is a complete multipartite graph.

We remark that there are many distance-regular graphs Γ of diameter $d \geq 3$ with the property that the subgraph induced by $\Gamma_d(\gamma)$ is not connected for any $\gamma \in V(\Gamma)$. If Γ is a distance-regular graph with $k_d > 1$ and $a_d = 0$, then $\Gamma_d(\gamma)$ will be disconnected for any $\gamma \in V(\Gamma)$. Distance-regular graphs having this property include antipodal r-covers with $r \geq 3$ and bipartite distance-regular graphs with $k_d \neq 1$, but there are also primitive examples. The Patterson graph [4, p. 410] (for $d = 4$), the Livingstone graph [4, p. 406] (for $d = 4$) and the Biggs–Smith graph [4, p. 403] (for $d = 7$) all have $k_d > 1$ and $a_d = 0$. The Coxeter graph [4, p. 382] has $d = 4$, $k_d = 6$ and $a_d = 1$ (and thus $\Gamma_4(\gamma)$ is isomorphic to $3K_2$). The Sylvester graph [4, p. 394] has $d = 3$, $k_d = 10$ and $a_d = 1$ (and thus $\Gamma_3(\gamma)$ is isomorphic to $5K_2$). The Odd graph $\Gamma = O_{d+1}$ [4, p. 259] is the graph whose vertex set is formed by all d-subsets of a set with $2d + 1$ elements, where two d-subsets are adjacent if and only if they are disjoint. This graph is distance-regular with degree $d + 1$, diameter d and $a_d = \lceil \frac{d+1}{2} \rceil$. For $d \geq 3$, one can check easily that the subgraph induced by $\Gamma_d(\gamma)$ is disconnected for any $\gamma \in V(\Gamma)$.

If $d \in \{3, 4\}$, then our results imply that the induced subgraph on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected for any $\gamma \in V(\Gamma)$, unless $d = 4$ and Γ is an antipodal r-cover for $r \geq 3$. All the known primitive distance-regular graphs that we have checked have $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ connected. Is this statement true for all primitive distance-regular graphs? We showed it for diameter up to 4. If $d = 2s + 1$, then it is not clear when $\Gamma_{\geq s+1}(\gamma)$ is disconnected.

The Johnson graph $J(n, d)$ has as vertices the subsets of order d of a set with n elements with two such subsets being adjacent if their intersection has size $d - 1$. It is well known that $J(n, d)$ is a distance-regular graph of diameter d and degree $d(n - d)$ having $b_j = (d - j)(n - d - j)$ and $c_j = j^2$ for each $0 \leq j \leq d$ (see [4, Sect. 9.1]). For every $\gamma \in V(J(n, d))$, the induced subgraph $\Gamma_d(\gamma)$ is connected as is isomorphic to $J(n - d, d)$. When $n > d^2$, we can deduce that $\Gamma_d(\gamma)$ is connected using our previous results. The standard sequence corresponding to $\theta_1 = (d - 1)(n - d - 1) - 1 = b_1 - 1$ is $u_i = 1 - \frac{in}{d(n-d)}$ for $0 \leq i \leq d$ (see [4, Proposition 4.4.9]). It follows easily that if n divides $d(n - d)$, then the standard sequence has a zero u_t, where $t = \frac{d(n-d)}{n}$. If n does not divide $d(n - d)$, then $u_{t-1} > 0$ and $u_t < 0$, when $t = \lceil \frac{d(n-d)}{n} \rceil$. Thus, when $n > d^2$, then $u_{d-1} > 0$ and $u_d < 0$ which implies $\Gamma_d(\gamma)$ is connected for every $\gamma \in V(J(n, d))$. However, when $n = 4t - 3$, $d = 2t - 2$, we obtain $u_{t-1} > 0$ and $u_t < 0$. Also, when $n = 4t - 1$, $d = 2t - 1$, we obtain $u_{t-1} > 0$ and $u_t < 0$.

The Hamming graph $H(d, q)$ has as vertex set the Cartesian product Q^d, where Q is a set of q elements. Two vertices of $H(d, q)$ are adjacent if they differ in exactly one position. It is well known that $H(d, q)$ is a distance-regular graph of diameter d and degree $d(q - 1)$ having $b_j = (d - j)(q - 1)$, $c_j = j$ for each $0 \leq j \leq d$ (see [4, Sect. 9.2]). For every $\gamma \in V(H(d, q))$, the induced subgraph $\Gamma_d(\gamma)$ is connected as is isomorphic to $H(d, q - 1)$. When $q > d$, we can use our previous results to deduce that $\Gamma_d(\gamma)$ is connected. The standard sequence corresponding to $\theta_1 = dq - d - q = b_1 - 1$ is $u_i = 1 - \frac{iq}{d(q-1)}$ for $0 \leq i \leq d$ (see [4, Proposition 4.4.9]). It follows that if q divides d, then the standard sequence has a zero u_t, where $t = \frac{d(q-1)}{q}$. If q does not divide d, then $u_{t-1} > 0$ and $u_t < 0$, when $t = \lceil \frac{d(q-1)}{q} \rceil$. Thus, when $q > d$, then $u_{d-1} > 0$ and $u_d < 0$ which implies $\Gamma_d(\gamma)$ is connected for every $\gamma \in V(H(d,q))$. However, when $q = 2$, $d = 2t - 1$, then $u_{t-1} > 0$ and $u_t < 0$.

Motivated by the previous results, we call a distance-regular graph Γ of diameter d a *generalized Shilla graph* if the standard sequence corresponding to the second largest eigenvalue of Γ contains a zero. When $d = 3$, this notion is the same as the notion of Shilla graph introduced by Koolen and Park in [9]. The previous examples show that many distance-regular graphs such as Hamming graphs or Johnson graphs are generalized Shilla graphs. It would be interesting to determine what other distance-regular graphs are generalized Shilla graphs. A related problem would be to classify all the distance-regular graphs of diameter $d = 3$ such that $\Gamma_d(\gamma)$ is disconnected for some $\gamma \in V(\Gamma)$. Such graphs would have to satisfy the condition $u_2 \leq 0$. The case $u_2 = 0$ is particularly interesting and these graphs will have $\theta_1 = a_3 = \frac{a_1 + \sqrt{a_1^2 + 4k}}{2}$ (see Koolen and Park [9] for more details on such graphs). Another problem worth investigating is the relation between the index t, where the standard sequence corresponding to θ_1 changes its sign, and the index s such that $k_s = \max_{0 \leq i \leq d} k_i$. If $\theta_1 < k/2$, then one can show that $u_{t-1} \geq 0$ and $u_t < 0$ imply that $b_{t-2} \geq k - \theta_1 > k/2$ and $c_{t+1} > k - \theta_1 > k/2$. This shows that $b_{t-3} > c_{t-2}$ and $c_{t+2} > b_{t+1}$ and hence $k_{t-3} < k_{t-2}$ and $k_{t+1} > k_{t+2}$. Thus, when $\theta_1 < k/2$, the parameter s satisfies $t - 2 \leq s \leq t + 1$. It would be interesting to determine how far apart can these parameters be in general.

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References

1. Brouwer, A.E.: Connectivity and spectrum of graphs. CWI Q. **9**, 37–40 (1996)
2. Brouwer, A.E.: Private communication to the authors during GAC5, Oisterwijk, The Netherlands, August 2011
3. Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Universitext. Springer, Berlin (2012). 250 pp.
4. Brouwer, A.E., Cohen, A., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin (1989)
5. Cameron, P.J., Goethals, J.-M., Seidel, J.J.: Strongly regular graphs having strongly regular subconstituents. J. Algebra, 257–280 (1978)
6. Gardiner, A., Godsil, C., Hensel, A.D., Royle, G.F.: Second neighbourhoods of strongly regular graphs. Discrete Math. **103**(2), 161–170 (1992)
7. Godsil, C.: Algebraic Combinatorics. Chapman and Hall Mathematics Series. Chapman & Hall, New York (1993). xvi+362 pp. ISBN 0-412-04131-6
8. Godsil, C., Royle, G.: Algebraic Graph Theory. Graduate Texts in Mathematics, vol. 207. Springer, New York (2001)
9. Koolen, J.H., Park, J.: Shilla distance-regular graphs. Eur. J. Comb. **31**, 2064–2073 (2010)
10. Park, J., Koolen, J.H., Markowsky, G.: There are finitely many distance-regular graphs with valency at least three, fixed ratio $\frac{k_2}{k}$ and large diameter. arXiv:1012.2632