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On the spectrum of Wenger graphs

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ABSTRACT

Let $q = p^e$, where p is a prime and $e \geqslant 1$ is an integer. For $m \geqslant 1$, let P and L be two copies of the $(m+1)$-dimensional vector spaces over the finite field $\mathbb{F}_q$. Consider the bipartite graph $W_m(q)$ with partite sets P and L defined as follows: a point $(p) = (p_1, p_2, \ldots, p_{m+1}) \in P$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{m+1}] \in L$ if and only if the following m equalities hold: $l_{i+1} + p_{i+1} = l_i p_1$ for $i = 1, \ldots, m$. We call the graphs $W_m(q)$ Wenger graphs. In this paper, we determine all distinct eigenvalues of the adjacency matrix of $W_m(q)$ and their multiplicities. We also survey results on Wenger graphs.

1. Introduction

All graph theory notions can be found in Bollobás [2]. Let $q = p^e$, where p is a prime and $e \geqslant 1$ is an integer. For $m \geqslant 1$, let P and L be two copies of the $(m+1)$-dimensional vector spaces over the finite field $\mathbb{F}_q$. We call the elements of P *points* and the elements of L *lines*. If $a \in \mathbb{F}_q^{m+1}$, then we write $(a) \in P$ and $[a] \in L$. Consider the bipartite graph $W_m(q)$ with partite sets P and L defined as follows: a point $(p) = (p_1, p_2, \ldots, p_{m+1}) \in P$ is adjacent to a line $[l] = [l_1, l_2, \ldots, l_{m+1}] \in L$ if and only if the following m equalities hold:

$$l_2 + p_2 = l_1 p_1,$$

$$l_3 + p_3 = l_2 p_1,$$

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$$\vdots$$

$$l_{m+1} + p_{m+1} = l_m p_1.$$

The graph $W_m(q)$ has $2q^{m+1}$ vertices, is q-regular and has q^{m+2} edges.

In [25], Wenger introduced a family of p-regular bipartite graphs $H_k(p)$ as follows. For every $k \geqslant 2$, and every prime p, the partite sets of $H_k(p)$ are two copies of integer sequences $\{0, 1, \ldots, p-1\}^k$, with vertices $a = (a_0, a_1, \ldots, a_{k-1})$ and $b = (b_0, b_1, \ldots, b_{k-1})$ forming an edge if

$$b_j \equiv a_j + a_{j+1} b_{k-1} \pmod{p} \text{ for all } j = 0, \ldots, k-2.$$

The introduction and study of these graphs were motivated by an extremal graph theory problem of determining the largest number of edges in a graph of order n containing no cycle of length $2k$. This parameter also known as the Turán number of the cycle C_{2k}, is denoted by $\mathrm{ex}(n, C_{2k})$. Bondy and Simonovits [3] showed that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$, $n \to \infty$. Lower bounds of magnitude $n^{1+1/k}$ were known (and still are) for $k = 2, 3, 5$ only, and the graphs $H_k(p)$, $k = 2, 3, 5$, provided new and simpler examples of such magnitude extremal graphs. For many results on $\mathrm{ex}(n, C_{2k})$, see Verstraëte [21], Pikhurko [19] and references therein.

In [9], Lazebnik and Ustimenko, using a construction based on a certain Lie algebra, arrived at a family of bipartite graphs $H'_n(q)$, $n \geqslant 3$, q is a prime power, whose partite sets were two copies of $\mathbb{F}_q^{n-1}$, with vertices $(p) = (p_2, p_3, \ldots, p_n)$ and $[l] = [l_1, l_3, \ldots, l_n]$ forming an edge if

$$l_k - p_k = l_1 p_{k-1} \quad \text{for all } k = 3, \ldots, n.$$

It is easy to see that for all $k \geqslant 2$ and prime p, graphs $H_k(p)$ and $H'_{k+1}(p)$ are isomorphic, and the map

$$\phi : (a_0, a_1, \ldots, a_{k-1}) \mapsto (a_{k-1}, a_{k-2}, \ldots, a_0),$$
$$(b_0, b_1, \ldots, b_{k-1}) \mapsto [b_{k-1}, b_{k-2}, \ldots, b_0],$$

provides an isomorphism from $H_k(p)$ to $H'_{k+1}(p)$. Hence, graphs $H'_n(q)$ can be viewed as generalizations of graphs $H_k(p)$. It is also easy to show that graphs $H'_{m+2}(q)$ and $W_m(q)$ are isomorphic: the function

$$\psi : (p_2, p_3, \ldots, p_{m+2}) \mapsto [p_2, p_3, \ldots, p_{m+2}],$$
$$[l_1, l_3, \ldots, l_{m+2}] \mapsto (-l_1, -l_3, \ldots, -l_{m+1}),$$

mapping points to lines and lines to points, is an isomorphism of $H'_{m+2}(q)$ to $W_m(q)$. Combining this isomorphism with the results in [9], we obtain that the graph $W_1(q)$

is isomorphic to an induced subgraph of the point-line incidence graph of the projective plane $PG(2,q)$, the graph $W_2(q)$ is isomorphic to an induced subgraph of the point-line incidence graph of the generalized quadrangle $Q(4,q)$, and $W_3(q)$ is a homomorphic image of an induced subgraph of the point-line incidence graph of the generalized hexagon $H(q)$.

We call the graphs $W_m(q)$ *Wenger graphs*. The representation of Wenger graphs as $W_m(q)$ graphs first appeared in Lazebnik and Viglione [11]. These authors suggested another useful representation of these graphs, where the right-hand sides of equations are represented as monomials of p_1 and l_1 only, see [22]. For this, define a bipartite graph $W'_m(q)$ with the same partite sets as $W_m(q)$, where $(p) = (p_1, p_2, \dots, p_{m+1})$ and $[l] = [l_1, l_2, \dots, l_{m+1}]$ are adjacent if

$$l_k + p_k = l_1 p_1^{k-1} \quad \text{for all } k = 2, \dots, m+1. \tag{1}$$

The map

$$\omega : (p) \mapsto \big(p_1, p_2, p'_3, \dots, p'_{m+1}\big), \quad \text{where } p'_k = p_k + \sum_{i=2}^{k-1} p_i p_1^{k-i},\ k = 3, \dots, m+1,$$
$$[l] \mapsto [l_1, l_2, \dots, l_{m+1}],$$

defines an isomorphism from $W_m(q)$ and $W'_m(q)$.

It was shown in [9] that the automorphism group of $W_m(q)$ acts transitively on each of P and L, and on the set of edges of $W_m(q)$. In other words, the graphs $W_m(q)$ are point-, line-, and edge-transitive. A more detailed study, see [11], also showed that $W_1(q)$ is vertex-transitive for all q, and that $W_2(q)$ is vertex-transitive for even q. For all $m \geqslant 3$ and $q \geqslant 3$, and for $m = 2$ and all odd q, the graphs $W_m(q)$ are not vertex-transitive. Another result of [11] is that $W_m(q)$ is connected when $1 \leqslant m \leqslant q-1$, and disconnected when $m \geqslant q$, in which case it has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. In [23], Viglione proved that when $1 \leqslant m \leqslant q-1$, the diameter of $W_m(q)$ is $2m+2$. We wish to note that the statement about the number of components of $W_m(q)$ becomes apparent from the representation (1). Indeed, as $l_1 p_1^i = l_1 p_1^{i+q-1}$, all points and lines in a component have the property that their coordinates i and j, where $i \equiv j \bmod (q-1)$, are equal. Hence, points (p), having $p_1 = \dots = p_q = 0$, and at least one distinct coordinate p_i, $q+1 \leqslant i \leqslant m+1$, belong to different components. This shows that the number of components is at least q^{m-q+1}. As $W_{q-1}(q)$ is connected and $W_m(q)$ is edge-transitive, all components are isomorphic to $W_{q-1}(q)$. Hence, there are exactly q^{m-q+1} of them. A result of Mader [16] also obtained independently by Watkins [24], and the edge-transitivity of $W_m(q)$ imply that the vertex connectivity (and consequently the edge connectivity) of $W_m(q)$ equals the degree of regularity q, for any $1 \leqslant m \leqslant q-1$.

Shao, He and Shan [20] proved that in $W_m(q)$, $q = p^e$, p prime, for $m \geqslant 2$, for any integer $l \neq 5, 4 \leqslant l \leqslant 2p$ and any vertex v, there is a cycle of length $2l$ passing through the vertex v. We wish to remark that the edge-transitivity of $W_m(q)$ implies the existence of a $2l$ cycle through any edge, a stronger statement. Li and Lih [12] used the Wenger graphs to determine the asymptotic behavior of the Ramsey number $r_n(C_{2k}) = \Theta(n^{k/(k-1)})$ when $k \in \{2, 3, 5\}$ and $n \to \infty$; the Ramsey number $r_n(G)$ equals the minimum integer N such that in any edge-coloring of the complete graph K_N with n colors, there is a monochromatic G. Representation (1) points to a relation of Wenger graphs with the moment curve $t \mapsto (1, t, t^2, t^3, \ldots, t^m)$, and, hence, with the Vandermonde's determinant, which was explicitly used in [25]. This is also in the background of some geometric constructions by Mellinger and Mubayi [17] of magnitude extremal graphs without short even cycles.

In Section 2, we determine the spectrum of the graphs $W_m(q)$, defined as the multiset of the eigenvalues of the adjacency matrix of $W_m(q)$. Futorny and Ustimenko [6] considered applications of Wenger graphs in cryptography and coding theory, as well as some generalizations. They also conjectured that the second largest eigenvalue λ_2 of the adjacency matrix of Wenger graphs $W_m(q)$ is bounded from above by $2\sqrt{q}$. The results of this paper confirm the conjecture for $m = 1$ and 2, or $m = 3$ and $q \geqslant 4$, and refute it in other cases. We wish to point out that for $m = 1$ and 2, or $m = 3$ and $q \geqslant 4$, the upper bound $2\sqrt{q}$ also follows from the known values of λ_2 for the point-line $(q + 1)$-regular incidence graphs of the generalized polygons $PG(2, q)$, $Q(4, q)$ and $H(q)$ and eigenvalue interlacing (see Brouwer, Cohen and Neumaier [4]). In [13], Li, Lu and Wang showed that the graphs $W_m(q)$, $m = 1, 2$, are Ramanujan, by computing the eigenvalues of another family of graph described by systems of linear equations in [10], $D(k, q)$, for $k = 2, 3$. Their result follows from the fact that $W_1(q) \simeq D(2, q)$, and $W_2(q) \simeq D(3, q)$. For more on Ramanujan graphs, see Lubotzky, Phillips and Sarnak [15], or Murty [18]. Our results also imply that for fixed m and large q, the Wenger graph $W_m(q)$ are expanders. For more details on expanders and their applications, see Hoory, Linial and Wigderson [7], and references therein.

2. Main results

Theorem 2.1. *For all prime power q and $1 \leqslant m \leqslant q - 1$, the distinct eigenvalues of $W_m(q)$ are*

$$\pm q,\ \pm\sqrt{mq},\ \pm\sqrt{(m-1)q},\ \ldots,\ \pm\sqrt{2q},\ \pm\sqrt{q},\ 0. \tag{2}$$

The multiplicity of the eigenvalue $\pm\sqrt{iq}$ of $W_m(q)$, $0 \leqslant i \leqslant m$, is

$$(q-1)\binom{q}{i}\sum_{d=i}^{m}\sum_{k=0}^{d-i}(-1)^k\binom{q-i}{k}q^{d-i-k}. \tag{3}$$

Proof. As the graph $W_m(q)$ is bipartite with partitions L and P, we can arrange the rows and the columns of an adjacency matrix A of $W_m(q)$ such that A has the following form:

$$A = \begin{array}{c} \\ L \\ P \end{array} \begin{array}{c} \begin{array}{cc} L & P \end{array} \\ \begin{pmatrix} 0 & N^T \\ N & 0 \end{pmatrix} \end{array} \tag{4}$$

which implies that

$$A^2 = \begin{pmatrix} N^T N & 0 \\ 0 & NN^T \end{pmatrix}. \tag{5}$$

As the matrices $N^T N$ and NN^T have the same spectrum, we just need to compute the spectrum for one of these matrices. To determine the spectrum of $N^T N$, let H denote the point-graph of $W_m(q)$ on L. This means that the vertex set of H is L, and two distinct lines $[l]$ and $[l']$ of $W_m(q)$ are adjacent in H if there exists a point $(p) \in P$, such that $[l] \sim (p) \sim [l']$ in $W_m(q)$. More precisely, $[l]$ and $[l']$ are adjacent in H, if there exists $p_1 \in \mathbb{F}_q$ such that for all $i = 1, \ldots, m$, we have

$$\begin{aligned} l_1 \neq l_1' \quad \text{and} \quad l_{i+1} - l_{i+1}' = p_1\big(l_i - l_i'\big) \quad &\Leftrightarrow \\ l_1 \neq l_1' \quad \text{and} \quad l_{i+1} - l_{i+1}' = p_1^i\big(l_1 - l_1'\big). \end{aligned}$$

This implies that H is actually the Cayley graph of the additive group of the vector space $\mathbb{F}_q^{m+1}$ with a generating set

$$S = \big\{\big(t, tu, \ldots, tu^m\big) \bigm| t \in \mathbb{F}_q^*,\ u \in \mathbb{F}_q\big\}. \tag{6}$$

Let ω be a complex p-th root of unity. For $x \in \mathbb{F}_q$, the trace of x is defined as $tr(x) = \sum_{i=0}^{e-1} x^{p^i}$. The eigenvalues of H are indexed after the $(m+1)$-tuples $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, and can be represented in the following form (see Babai [1] and Lovász [14] for more details):

$$\begin{aligned} \lambda_{(w_1,\ldots,w_{m+1})} &= \sum_{(t,tu,\ldots,tu^m)\in S} \omega^{tr(tw_1)} \cdot \omega^{tr(tuw_2)} \cdot \ldots \cdot \omega^{tr(tu^m w_{m+1})} \\ &= \sum_{t\in\mathbb{F}_q^*,\, u\in\mathbb{F}_q} \omega^{tr(tw_1+tuw_2+\cdots+tu^m w_{m+1})} \\ &= \sum_{t\in\mathbb{F}_q^*,\, u\in\mathbb{F}_q} \omega^{tr(t(f(u)))} \quad \big(\text{where } f(u) := w_1 + w_2 u + \cdots + w_{m+1}u^m\big) \\ &= \sum_{t\in\mathbb{F}_q^*,\, f(u)=0} \omega^{tr(t(f(u)))} + \sum_{t\in\mathbb{F}_q^*,\, f(u)\neq 0} \omega^{tr(t(f(u)))}. \end{aligned}$$

As $\sum_{t\in\mathbb{F}_q^*} \omega^{tr(tx)} = q - 1$ for $x = 0$, and $\sum_{t\in\mathbb{F}_q^*} \omega^{tr(tx)} = -1$ for every $x \in \mathbb{F}_q^*$, we obtain that

$$\lambda_{(w_1,\ldots,w_{m+1})} = \big|\big\{u \in \mathbb{F}_q \bigm| f(u) = 0\big\}\big|(q-1) - \big|\big\{u \in \mathbb{F}_q \bigm| f(u) \neq 0\big\}\big|. \tag{7}$$

Let B be the adjacency matrix of H. Then $N^T N = B + qI$; this fact can be seen easily by examining the on- and off-diagonal entries of both sides of the equation. Therefore, the eigenvalues of $W_m(q)$ can be written in the form

$$\pm\sqrt{\lambda_{(w_1,\ldots,w_{m+1})} + q},$$

where $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$. Let $f(X) = w_1 + w_2X + \cdots + w_{m+1}X^m \in \mathbb{F}_q[X]$. We consider two cases.

1. $f = 0$. In this case, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = q$, and $\lambda_{(w_1,\ldots,w_{m+1})} = q(q-1)$. Thus, $W_m(q)$ has $\pm q$ as its eigenvalues.
2. $f \neq 0$. In this case, let $i = |\{u \in \mathbb{F}_q \mid f(u) = 0\}| \leqslant m$ as $1 \leqslant m \leqslant q-1$. This shows that $\lambda_{(w_1,\ldots,w_{m+1})} = i(q-1) - (q-i) = iq - q$ and implies that $\pm\sqrt{\lambda_{(w_1,\ldots,w_{m+1})} + q} = \pm\sqrt{iq}$ are eigenvalues of $W_m(q)$. Note that for any $0 \leqslant i \leqslant m$, there exists a polynomial f over $\mathbb{F}_q$ of degree at most $m \leqslant q-1$, which has exactly i distinct roots in $\mathbb{F}_q$. For such f, $|\{u \in \mathbb{F}_q \mid f(u) = 0\}| = i$, and, hence, there exists $(w_1, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, such that $\lambda_{(w_1,\ldots,w_{m+1})} = iq - q$. Thus, $W_m(q)$ has $\pm\sqrt{iq}$ as its eigenvalues, for any $0 \leqslant i \leqslant m$, and the first statement of the theorem is proven.

The arguments above imply that the multiplicity of the eigenvalue $\pm\sqrt{iq}$ of $W_m(q)$ equals the number of polynomials of degree at most m (not necessarily monic) having exactly i distinct roots in $\mathbb{F}_q$. To calculate these multiplicities, we need the following lemma. Particular cases of the lemma were considered in Zsigmondy [26], and in Cohen [5]. The complete result appears in A. Knopfmacher and J. Knopfmacher [8].

Lemma 2.2. *(See [8].) Let q be a prime power, and let d and i be integers such that $0 \leqslant i \leqslant d \leqslant q-1$. Then the number $b(q,d,i)$ of monic polynomials in $\mathbb{F}_q[X]$ of degree d, having exactly i distinct roots in $\mathbb{F}_q$ is given by*

$$b(q,d,i) = \binom{q}{i} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k}. \tag{8}$$

By Lemma 2.2, the number of polynomials of degree at most m in $\mathbb{F}_q[X]$ (not necessarily monic) having exactly i distinct roots in $\mathbb{F}_q$ is

$$\sum_{d=i}^{m} (q-1)\, b(q,d,i) = (q-1)\binom{q}{i} \sum_{d=i}^{m} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k}. \tag{9}$$

This concludes the proof the theorem. □

The previous result shows that $W_m(q)$ is connected and has $2m+3$ distinct eigenvalues, for any $1 \leqslant m \leqslant q-1$. As the diameter of a graph is strictly less than the number of distinct eigenvalues (see [4, Section 4.1] for example), this implies that the diameter of Wenger graph is less or equal to $2m+2$. This is actually the exact value of the diameter of the Wenger graph as shown by Viglione [23].

Since the sum of multiplicities of all eigenvalues of the graph $W_m(q)$ is equal to its order, and remembering that the multiplicity of $\pm q$ is one when $1 \leqslant m \leqslant q-1$, we have a combinatorial proof of the following identity.

Corollary 2.3. *For every prime power q, and every m, $1 \leqslant m \leqslant q-1$,*

$$\sum_{i=0}^{m} \binom{q}{i} \sum_{d=i}^{m} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k} = \frac{q^{m+1}-1}{q-1}. \tag{10}$$

The identity (10) seems to hold for all integers $q \geqslant 3$, so a direct proof is desirable. Other identities can be obtained by taking the higher moments of the eigenvalues of $W_m(q)$.

As we discussed in the introduction, for $m \geqslant q$, the graph $W_m(q)$ has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. This, together with Theorem 2.1, immediately implies the following.

Proposition 2.4. *For $m \geqslant q$, the distinct eigenvalues of $W_m(q)$ are*

$$\pm q,\ \pm\sqrt{(q-1)q},\ \pm\sqrt{(q-2)q},\ \ldots,\ \pm\sqrt{2q},\ \pm\sqrt{q},\ 0,$$

and the multiplicity of the eigenvalue $\pm\sqrt{iq}$, $0 \leqslant i \leqslant q-1$, is

$$(q-1)q^{m+1-q} \binom{q}{i} \sum_{d=i}^{q} \sum_{k=0}^{d-i} (-1)^k \binom{q-i}{k} q^{d-i-k}.$$

3. Open questions

There are several open questions about the Wenger graphs $W_m(q)$ that we think are worth investigating: deciding whether these graphs are Hamiltonian, finding the lengths of all their cycles, determining their automorphism group,[1] or determining the parameters of the linear codes whose Tanner graphs are the Wenger graphs.

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